NO-GOOD-DEAL, LOCAL MEAN-VARIANCE AND AMBIGUITY RISK PRICING AND HEDGING FOR AN INSURANCE PAYMENT PROCESS

BY

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ABSTRACT

We study pricing and hedging for an insurance payment process. We investigate a Black-Scholes financial model with stochastic coefficients and a payment process with death, survival and annuity claims driven by a point process with a stochastic intensity. The dependence of the claims and the intensity on the financial market and on an additional background noise (correlated index, longevity risk) is allowed. We establish a general modeling framework for no-good-deal, local mean-variance and ambiguity risk pricing and hedging. We show that these three valuation approaches are equivalent under appropriate formulations. We characterize the price and the hedging strategy as a solution to a backward stochastic differential equation. The results could be applied to pricing and hedging of variable annuities, surrender options under an irrational lapse behavior and mortality derivatives.

KEYWORDS

Hansen-Jagannathan bound, instantaneous Sharpe ratio, equivalent martingale measure, probability priors, backward stochastic differential equation, variable annuities, longevity risk, irrational lapse behavior.

1. INTRODUCTION

Pricing and hedging in incomplete markets is the most important topic in the insurance and financial literature. Over the last years many approaches were proposed to deal with it. The most classical theoretical approaches include superhedging, utility maximization, mean-variance portfolio selection or quadratic loss minimization.

Superhedging is not very useful from the practical point of view as it results in too high prices which are incurred by the buyers (policyholders) and too high gains which are collected by the sellers (insurers). To overcome this problem no-good-deal pricing was introduced in Cochrane and Saá-Requejo (2000) and next extended in Björk and Slinko (2006). The idea of no-good-deal pricing

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is to exclude these prices which would allow for too high investment gains based on the assumption that too favorable trading deals cannot take place in a competitive market. According to Cochrane and Saá-Requejo (2000) and Björk and Slinko (2006) a deal is too favorable if the Sharpe ratio of its return is very high (over a certain threshold). Unfortunately, no-good deal pricing in its original formulation says nothing about hedging and probably this is the reason why it is not very popular in insurance and financial mathematics. However, the Sharpe ratio is well understood by investors. A reasonable pricing and hedging criterion based on the Sharpe ratio would be appreciated by the business.

As far as pricing and hedging is concerned utility maximization, mean-variance portfolio selection and quadratic loss minimization dominate the literature, see Carmona et al. (2009), Schweizer (2010) and the references therein. The advantage of these approaches consists in that the objective of the investor, the hedging strategy and the price are all clearly defined. These approaches (especially mean-variance portfolio selection) are accepted by the business.

Recently, risk measure minimization has become to play an important role in pricing and hedging, see Barrieu and El Karoui (2005). A simple but very interesting risk measure (we call it an ambiguity risk measure) arises if we allow in the expected value principle an ambiguity about the likelihood of possible events. It is reasonable to consider not one probability measure but a set of probability priors which represent different beliefs about the real-world evolution of the underlying dynamics, see Chen and Epstein (2002), and make decisions with respect to the worst prior. The developments in this area should interest insurers who should be prepared to cover claims under the worst-case scenario.

The motivation for this paper comes from Bayraktar and Young (2007), Young (2008), Bayraktar and Young (2008), Milevsky et al. (2005), Bayraktar et al. (2009), Leitner (2007) and Becherer (2009). In Bayraktar and Young (2007), Young (2008), Milevsky et al. (2005), Bayraktar et al. (2009), the authors consider pricing and hedging of life insurance contracts with fixed (deterministic) death, annuity, survival benefits in a financial market consisting of a bond and a bank account in the presence of a stochastic interest rate and a stochastic mortality intensity driven by two independent Brownian motions. The criterion is to locally minimize variance of a surplus process (a difference between a replicating wealth process and the price of a claim) which has the instantaneous Sharpe ratio at a target level. In Bayraktar and Young (2008) this criterion is applied to price and hedge a terminal claim contingent on an index correlated to a traded stock in a Black-Scholes model. The relation between the local variance minimization under the Sharpe ratio constraint and the no-good-deal pricing is noticed in their models. In Leitner (2007) and Becherer (2009) a Black-Scholes model is investigated with a claim contingent on an index correlated to a tradeable stock. Leitner (2007) solves the problem of local mean-variance risk minimization of a surplus process and global maximization of an ambiguity risk measure of a terminal surplus. The goal is to price and hedge a terminal claim with vanishing local and global risk measure. The equivalence between these two approaches is shown. Becherer (2009)

investigates no-good-deal pricing for a terminal claim and ambiguity risk pricing and hedging of a terminal surplus. By establishing the equivalence between the two techniques the author introduces a clear financial meaning of a no-good-deal hedging strategy by relating it to an ambiguity risk hedging strategy. This opens the way to talk not only about no-good-deal pricing but also about no-good-deal hedging.

The goal of this paper is to construct a fully-fledged general modeling framework for no-good-deal, local mean-variance and ambiguity risk pricing and hedging for an insurance payment process. As explained in the beginning, these three criteria are worth investigating. We consider a Black-Scholes model with stochastic coefficients and we deal with a stream of liabilities with death, annuity and survival claims driven by a point process with a stochastic intensity. The dependence of the claims and the intensity on the financial market and on an additional background noise is allowed. We work in a non-Markovian setting. We show how to solve no-good-deal, local mean-variance and ambiguity risk pricing and hedging problems in our general model and we establish the equivalence between these three valuation approaches. To the best of our knowledge such general problems are considered for the first time in the literature.

Our work provides a significant improvement over the results from the mentioned papers. Similarly to Cochrane and Saá-Requejo (2000) and Björk and Slinko (2006) we establish the Hansen-Jagannathan bound in our combined financial and insurance model with the payment process, which serves as a motivation for no-good-deal pricing. Compared to Leitner (2007) and Becherer (2009) we extend their models by incorporating a stream of claims into the study and we have to deal with a point process which generates the claims. We generalize the results from Bayraktar and Young (2007), Young (2008), Bayraktar and Young (2008), Milevsky et al. (2005), Bayraktar et al. (2009) by considering claims and an intensity contingent on the traded asset and on a background noise. As we work in a non-Markovian framework, path-dependent claims (like ratchet options common for variable annuities) are allowed in this paper. These extensions are very important from the point of view of insurance applications. We can investigate pricing and hedging of variable annuities and mortality derivatives with different payment profiles and dependency structures. In particular, we can deal with pricing and hedging of a surrender option in the case of an irrational lapse behavior linked to the financial market. We finally reformulate the criterion of local variance minimization from Bayraktar and Young (2007), Young (2008), Bayraktar and Young (2008), Milevsky et al. (2005), Bayraktar et al. (2009) to local mean-variance minimization so that the corresponding hedging strategy coincides with the hedging strategy derived from ambiguity risk minimization.

In our analysis we follow Leitner (2007) and Becherer (2009) and we apply backward stochastic differential equations (BSDEs). As we consider dynamics driven by a point process and Brownian motions, we face new mathematical problems. Fortunately, we are still able to characterize the price process and the hedging strategy as a solution to a BSDE. Our backward stochastic

differential equation can be solved analytically or numerically and we provide examples of both analytical and numerical solutions. An important advantage of our modeling approach is that it is possible to add more driving processes (more Brownian motions and point processes) and the structure of our solution would not be destroyed. This opens the way to constructing, based on the model from this paper, even more sophisticated models with many different risk factors. Importantly, these advanced models remain mathematically tractable. Such a flexibility is mainly due to the application of BSDEs in place of Hamilton-Jacobi-Bellman equations which are used in Bayraktar and Young (2007), Young (2008), Bayraktar and Young (2008), Milevsky et al. (2005), Bayraktar et al. (2009). We aim at showing advantages of using BSDEs in solving insurance and financial problems. Compared to Bayraktar and Young (2007), Young (2008), Bayraktar and Young (2008), Milevsky et al. (2005), Bayraktar et al. (2009), we do not have to deal with a complicated comparison theorem for partial differential equations and hard to check assumptions on the dynamics of the processes and the claims. Our assumptions are straightforward and much weaker. The comparison follows from the theory of BSDEs.

Very recently, Pelsser (2011) investigates cost-of-capital, no-good-deal and ambiguity pricing of liabilities. The author focuses only on a financial risk and deals with a Black-Scholes model. The equivalence of cost-capital, no-good deal and ambiguity pricing is shown. The solution from Pelsser (2011), who studies partial differential equations, arises as a special case of our model.

It is also worth mentioning the works by Murgoci (2008) and Donnelly (2010). In Murgoci (2008) no-good-deal pricing is considered in a Black-Scholes model under a counterparty risk where a default intensity depends on the financial market. The HJB equation is derived and the explicit solution is established in the case when the default intensity is not correlated to the financial market. In this paper we derive a solution for pricing a surrender option under a lapse intensity depending on the tradeable stock. In other words we complete the solution from Murgoci (2008) by finding a solution when the default intensity is perfectly correlated to the financial market. In Donnelly (2010) a regime-switching Black-Scholes model is investigated with regimes triggered by a Markov process with deterministic intensities. The HJB equation is derived and a solution is found by numerical experiments. It is possible to reformulate the results from Donnelly (2010) in our framework.

We believe that this paper could also serve as an important contribution to pricing and hedging of insurance liabilities from the practical point of view. Solvency II Directive forces insurance companies to build sophisticated internal models from which both market consistent prices and hedging strategies minimizing the mismatch between assets and liabilities must be derived. Such models must analyze many different financial and insurance risks in an integrated framework. Our paper proposes such a general model and the way how to find prices and hedging strategies. Moreover, we show that BSDEs could be a very helpful quantitative tool in integrated risk management. Our decision criteria take into account risk-reward considerations of the insurer,

formulated in terms of the Sharpe ratio of the arising surplus (the net asset wealth), which should be considered in all asset-liability modeling studies.

This paper is structured as follows. In Section 2 we introduce a financial and insurance model. We first investigate no-good-deal pricing in Section 3. Next, we consider local mean-variance hedging in Section 4. Finally, we discuss ambiguity risk pricing and hedging in Section 5. In Section 6 we study pricing and hedging of a surrender option in a unit-linked policy. We conclude with an example which shows how the BSDE characterizing the price and the hedging strategy could be solved numerically. The proofs are postponed till the Appendix. For information on BSDEs with jumps we refer to Barles et al. (1997), Becherer (2006), Royer (2006), Delong (2010).

2. Financial market and insurance payment process

Let us consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\mathcal{F} = (\mathcal{F}_t)_{0\le t\le T}$ and a finite time horizon $T < \infty$. We assume that $\mathcal{F}$ satisfies the usual hypotheses of completeness ($\mathcal{F}_0$ contains all sets of $\mathbb{P}$-measure zero) and right continuity ($\mathcal{F}_t = \mathcal{F}_{t+}$). Constants are denoted by K.

First, we introduce the financial model. We deal with a Black-Scholes model with stochastic coefficients. The dynamics of the bank account $S_0 := (S_0(t), 0 \le t \le T)$ is described by the equation

$$\frac{dS_0(t)}{S_0(t)} = r(t)\,dt,\ S_0(0) = s_0 > 0 \tag{2.1}$$

where $r := (r(t), 0 \le t \le T)$ denotes the risk-free rate. The dynamics of the stock price $S := (S(t), 0 \le t \le T)$ is given by the stochastic differential equation

$$\frac{dS(t)}{S(t)} = \mu(t)\,dt + \sigma(t)\,dW(t),\ S(0) = s > 0, \tag{2.2}$$

where $\mu := (\mu(t), 0 \le t \le T)$ denotes the expected return on the stock, $\sigma := (\sigma(t), 0 \le t \le T)$ denotes the price volatility and $W := (W(t), 0 \le t \le T)$ is an $\mathcal{F}$-adapted Brownian motion. We assume that

(A1) the processes r, μ, σ are predictable with respect to the natural filtration $\sigma(W(s), 0 \le s \le t)$ and they satisfy

$$\mu(t) \ge r(t) \ge 0, \quad \sigma(t) > 0, \quad 0 \le t \le T,$$

$$\int_0^T |\sigma(t)|^2\,dt < \infty,\ \mathbb{E}\left[\int_0^T |S(t)\sigma(t)|^2\,dt\right] < \infty,$$

$$|r(t)| + |\sigma^{-1}(t)| + \left|\frac{\mu(t) - r(t)}{\sigma(t)}\right| \le K, \quad 0 \le t \le T.$$

These are standard assumptions in financial models. Notice that (A1) implies that $\mathbb{E}[|S(t)|^2] < \infty$. The bound on r could be omitted but other assumptions would have to be made instead. Our financial model can include stochastic economic factors (like stochastic interest rate and stochastic volatility) which is desirable from the point of view of practical modeling.

Next, we introduce the insurance payment process. Let $N := (N(t), 0 \le t \le T)$ denote an $\mathcal{F}$-adapted point process. The process $N(t)$ counts the number of claims in the given period $(0, t]$. We assume that

(A2) the unique compensator $\vartheta(t)$ of the process N fulfills

$$\vartheta(t) = \int_0^t \eta(s)\,ds,\ 0 \le t \le T,$$

and the process $\eta : \Omega \times [0, T] \to [0, \infty)$ is an $\mathcal{F}$-predictable and satisfies

$$\mathbb{E}\left[\int_0^T \eta(s)\,ds\right] < \infty.$$

The process η defines the intensity of claims.

In this paper we investigate pricing and hedging of a stream of liabilities modeled by the payment process $P := (P(t), 0 \le t \le T)$ of the form

$$P(t) = \int_0^t H(s)\,ds + \int_0^t G(s)\,dN(s) + F\mathbf{1}_{t=T},\ 0 \le t \le T. \tag{2.3}$$

The process P contains payments H which occur continuously during the term of the contract (annuities); it contains claims G which occur at random times triggered by the jumps of the process N (death benefits) and a liability F which is settled at the end of the contract (a survival benefit). We assume that

(A3) the processes H, G are $\mathcal{F}$-predictable and the random variable F is $\mathcal{F}_T$-measurable. The processes H, G and the random variable F are non-negative and they satisfy

$$\mathbb{E}\left[\int_0^T |H(s)|^2\,ds\right] < \infty, \quad \mathbb{E}\left[\int_0^T |G(s)\,\eta(s)|^2\,ds\right] < \infty,$$
$$\mathbb{E}\left[\int_0^T |G(s)|^2\eta(s)\,ds\right] < \infty, \quad \mathbb{E}\left[|F|^2\right] < \infty.$$

Square integrability assumptions are standard in financial mathematics and are fulfilled in practice. Notice that (A3) imply that $(\int_0^t G(s)(dN(s) - \eta(s)\,ds))_{0 \le t \le T}$ is a square integrable martingale and $\mathbb{E}[|\int_0^t G(s)\,dN(s)|^2] < \infty$.

Finally, to enrich the model and extend the area of its applications we introduce a second $\mathcal{F}$-adapted Brownian motion $B := (B(t), 0 \le t \le T)$ which

is independent of W. The process B is a background driving noise and it models a third risk factor (correlated financial risk or longevity risk) next to the stock risk modeled by W and the claim risk modeled by N. The process B can affect both the claims' pay-offs and the intensity. The role of B is clarified in the next three examples.

We can investigate claims with interesting dependency structures between the financial, insurance and background risk factors. Most of the insurance and financial claims from the market fit into our framework.

Example 2.1. Take $H = G = 0$, $F = F(J(t), 0 \leq t \leq T)$, $\eta = 0$ and assume that the process J satisfies the stochastic differential equation

$$\frac{dJ(t)}{J(t)} = \mu^J(t)\,dt + \sigma_1^J(t)\,dW(t) + \sigma_2^J(t)\,dB(t).$$

We can consider pricing and hedging of a claim F paid at the terminal time the value of which depends on the path of an financial index J correlated to the stock S, see Bayraktar and Young (2008), Leitner (2007), Becherer (2009) and Pelsser (2011). □

Example 2.2. Take

$$\begin{aligned} H(t) &= (n - N(t-))\,h(t, S(t)), \quad G(t) = g(t, S(t)), \\ F &= (n - N(T))\,f(T, S(T)), \quad \eta(t) = (n - N(t-))\,\lambda(t), \end{aligned} \tag{2.4}$$

with continuous functions $h : [0,T] \times (0,\infty) \to [0,\infty)$, $g : [0,T] \times (0,\infty) \to [0,\infty)$, $f : (0,\infty) \to [0,\infty)$ and assume that the process λ satisfies the stochastic differential equation

$$d\lambda(t) = \mu^\lambda(t)\,dt + \sigma^\lambda(t)\,dB(t),$$

with μ^λ, σ^λ adapted to the natural filtration $\sigma(B(s), 0 \leq s \leq t)$. We can consider pricing and hedging of unit-linked endowments, annuities and death benefits arising from a portfolio consisting of n persons insured under an independent stochastic mortality intensity (longevity risk), see Bayraktar and Young (2007), Young (2008), Bayraktar and Young (2008), Milevsky et al. (2005), Bayraktar et al. (2009).

If we still take the characteristics (2.4) but we assume that the process λ satisfies

$$d\lambda(t) = \mu^\lambda(t)\,dt + \sigma^\lambda(t)\,dW(t),$$

with μ^λ, σ^λ adapted to the natural filtration $\sigma(W(s), 0 \leq s \leq t)$ then we can deal with unit-linked life insurance benefits under an irrational lapse behavior of

policyholders. The irrational lapse behavior is modeled by the point process N with the stochastic intensity λ depending on the financial market and the stock via the Brownian motion component W. It is now well-known that the lapse behavior is not independent of the financial market and such independence should not be assumed in actuarial models, see TP2.109-TP2.111 in European Commission QIS5 (2010). □

Example 2.3. Take

$$\eta(t) = (n - N(t-))\lambda(t),$$

$$d\lambda(t) = \mu^{\lambda}(t)dt + \sigma^{\lambda}(t)\,dB(t),$$

with μ^{λ}, σ^{λ} adapted to the natural filtration $\sigma(B(s),\ 0 \le s \le t)$ and link the benefits H, G, F to the number of deaths N in a given population or the death intensity λ in the population. We can deal with pay-offs from mortality derivatives, including survivor swaps from Dahl et al. (2008). □

To complete the description of our model we assume, see Chapter XII.2 in He et al. (1992), that under $(\mathbb{P}, \mathcal{F})$ the weak property of predictable representation holds:

(A4) every $(\mathbb{P}, \mathcal{F})$ local martingale $\mathcal{M}$ has a representation

$$\mathcal{M}(t) = \mathcal{M}(0) + \int_0^t \mathcal{Z}(s)dW(s) + \int_0^t \mathcal{U}(s)dB(s) + \int_0^t \mathcal{V}(s)d\tilde{N}(s)\ \ 0 \le t \le T,$$

with $\mathcal{F}$-predictable processes $(\mathcal{Z}, \mathcal{U}, \mathcal{V})$ integrable, in the sense of Itô calculus, with respect to the Brownian motions W, B and the compensated point process $d\tilde{N}(t) = dN(t) - \eta(t)dt$.

It is possible to construct the processes (W, B, N) and take the natural filtration $\mathcal{F}$ generated by these three processes to fulfill (A4), see Becherer (2006), He et al. (1992).

In the sequel we work with three sets $\mathcal{P}, \mathcal{Q}, \mathcal{Q}_M$ of measures. Let $L := (L(t),\ 0 \le t \le T)$ denote an $\mathcal{F}$-predictable process such that $0 \le L(t) \le K$, $0 \le t \le T$. We define the set

$$\begin{aligned}\mathcal{P} = \Big\{&\mathcal{F}\text{-predictablee processes } \alpha, \beta, \gamma \text{ such that}\\ &|\alpha(t)|^2 + |\beta(t)|^2 + |\gamma(t)|^2\eta(t) \le |L(t)|^2,\ \gamma(t) < 1,\ 0 \le t \le T\Big\},\end{aligned} \tag{2.5}$$

and we consider the process $M := (M(t),\ 0 \le t \le T)$ with the dynamics

$$\frac{dM(t)}{M(t-)} = -\alpha(t)dW(t) - \beta(t)dB(t) - \gamma(t)d\tilde{N}(t),\ M(0) = 1, \tag{2.6}$$

where $(\alpha, \beta, \gamma) \in \mathcal{P}$. One can show that M is a positive martingale and the process M is used to define an equivalent probability measure $\mathbb{Q} \sim \mathbb{P}$, see Theorem III.40 in Protter (2004) and Delong (2011). An important subset of $\mathcal{P}$ is the set of equivalent martingale measures under which the discounted prices process $e^{-\int_0^t r(s)ds} S(t)$ is a $\mathbb{Q}$-martingale. We define the set $\mathcal{Q} \subset \mathcal{P}$ as

$$\begin{aligned} \mathcal{Q} = \Big\{ & \mathcal{F}\text{-predictable processes } \alpha, \beta, \gamma \text{ such that} \\ & |\beta(t)|^2 + |\gamma(t)|^2 \eta(t) \leq |L(t)|^2 - \left| \frac{\mu(t) - r(t)}{\sigma(t)} \right|^2, \\ & \alpha(t) = \frac{\mu(t) - r(t)}{\sigma(t)}, \ \gamma(t) < 1, \ 0 \leq t \leq T \Big\}. \end{aligned} \tag{2.7}$$

We also need the set $\mathcal{Q}_M$ of all equivalent martingale measures which is defined as

$$\begin{aligned} \mathcal{Q}_M = \Big\{ & \mathcal{F}\text{-predictable processes } \alpha, \beta, \gamma; \\ & \text{the process } M \text{ defined in (2.6) is a } \mathbb{P}\text{-martingale}, \\ & \alpha(t) = \frac{\mu(t) - r(t)}{\sigma(t)}, \ \gamma(t) < 1, \ 0 \leq t \leq T \Big\}. \end{aligned} \tag{2.8}$$

Clearly, we have $\mathcal{Q} \subset \mathcal{Q}_M$. We denote $\theta(t) = \frac{\mu(t) - r(t)}{\sigma(t)}$.

In this paper we take the point of view of the insurer and we model a price $Y := (Y(t), 0 \leq t \leq T)$ of the contract which insures the payment process (2.3) as a solution to a backward stochastic differential equation with the dynamics

$$\begin{aligned} dY(t) = {} & \varphi(t)dt - H(t)dt - G(t)dN(t) \\ & + Z(t)dW(t) + U(t)dB(t) + V(t)d\tilde{N}(t), \ Y(T) = F, \end{aligned} \tag{2.9}$$

with an $\mathcal{F}$-predictable process φ. We treat the insurance contract as a derivative which can be traded in a liquid market in accordance with market-consistent valuation principles. By applying three different approaches which specify the notion of the price we arrive at the explicit and unique form of φ. This allows us to characterize the price Y and the hedging strategy in terms of (Z, U, V).

3. No-good-deal pricing

We consider the financial and insurance market (2.1)-(2.3) and we assume that

(A5) the combined financial and insurance market is arbitrage-free and there exists an equivalent martingale measure $\mathbb{Q} \in \mathcal{Q}_M$.

The non-arbitrage condition (A5) imposes a restriction on the form of φ in the price (2.9). We consider a gain process $\hat{Y} := \hat{Y}(t),\ 0 \le t \le T)$ arising from holding the insurance contract Y and receiving the benefits H, G, F to which a policyholder is entitled. We arrive at the dynamics

$$\begin{aligned} d\hat{Y}(t) &= dY(t) + H(t)\,dt + G(t)\,dN(t), \\ &= \varphi(t)\,dt + Z(t)\,dW(t) + U(t)\,dB(t) + V(t)\,d\tilde{N}(t),\ \ \hat{Y}(T) = F. \end{aligned} \tag{3.1}$$

We also conclude that under any martingale measure $\mathbb{Q} \in \mathcal{Q}_M$, constructed from $(\alpha, \beta, \gamma) \in \mathcal{Q}_M$, the discounted gain process

$$\hat{Y}^d(t) = e^{-\int_0^t r(s)ds}\,Y(t) + \int_0^t e^{-\int_0^s r(u)du}\,H(s)\,ds + \int_0^t e^{-\int_0^s r(u)du}\,G(s)\,dN(s),\ 0 \le t \le T,$$

satisfies the equation

$$\begin{aligned} d\hat{Y}^d(t) &= e^{-\int_0^t r(s)ds}\big(-Y(t-)r(t) + \eta(t) - Z(t)\theta(t) - U(t)\beta(t) - V(t)\gamma(t)\eta(t)\big)dt \\ &\quad + e^{-\int_0^t r(s)ds}Z(t)\,dW^{\mathbb{Q}}(t) + e^{-\int_0^t r(s)ds}U(t)\,dB^{\mathbb{Q}}(t) + e^{-\int_0^t r(s)ds}V(t)\,d\tilde{N}^{\mathbb{Q}}(t), \end{aligned}$$

and the discounted gain process $\hat{Y}^d$ is $\mathbb{Q}$-martingale provided that

$$\varphi(t) = Y(t-)r(t) + Z(t)\theta(t) + U(t)\beta(t) + V(t)\gamma(t)\eta(t),\ 0 \le t \le T. \tag{3.2}$$

The idea of no-good-deal pricing is to consider measures $\mathbb{Q}$ which belong to the subset $\mathcal{Q} \subset \mathcal{Q}_M$. The motivation for dealing with the set $\mathcal{Q}$ comes from investigating the Sharpe ratios for the investment opportunities in the market. Following Cochrane and Saá-Requejo (2000), Björk and Slinko (2006), Leitner (2007), for a given $0 \le t \le T$ we define the instantaneous Sharpe ratio of the investment into the stock S and the insurance contract Y by

$$\textit{Sharpe Ratio}\,(S) = \frac{\mathbb{E}[dS(t) - S(t)r(t)\,dt\,|\,\mathcal{F}_{t-}]/dt}{\sqrt{\mathbb{E}[d[S,S](t)\,|\,\mathcal{F}_{t-}]/dt}} = \theta(t),$$

$$\begin{aligned} \textit{Sharpe Ratio}\,(S) &= \frac{\mathbb{E}[d\hat{Y}(t) - Y(t-)r(t)\,dt\,|\,\mathcal{F}_{t-}]/dt}{\sqrt{\mathbb{E}[d[\hat{Y},\hat{Y}](t)\,|\,\mathcal{F}_{t-}]dt}} \\ &= \frac{\varphi(t) - Y(t-)r(t)}{\sqrt{|Z(t)|^2 + |U(t)|^2 + |V(t)|^2\eta(t)}}, \end{aligned}$$

where $t \mapsto [.,.]\,(t)$ denotes a quadratic variation process and we use the dynamics (3.1) for $\hat{Y}$. Notice that the Sharpe ratio for a long position in the insurance contract (for a policyholder) must compare a return earned by keeping the

contract (a change in the price and the benefits) with a risk-free return gained from selling the contract and investing in the bank account. The instantaneous Sharpe ratio for a short position in the insurance contract (for the insurer) takes the form of $-$ *Sharpe Ratio*(Y), as the liability positions must be compared.

The theory of finance claims that investment opportunities with high Sharpe ratios cannot survive in competitive markets, see Cochrane and Saá-Requejo (2000) and Björk and Slinko (2006). Empirical studies support the fact that Sharpe ratios take restricted range of values: Bayraktar et al. (2009) report that Sharpe ratios for equities are around 0.2 and Murgoci (2008) reports that Sharpe ratio above 2 is rare. The non-arbitrage requirement (3.2) on φ and Schwarz inequality lead to

$$\begin{aligned} &|\varphi(t) - Y(t)r(t)| = |Z(t)\theta(t) + U(t)\beta(t) + V(t)\gamma(t)\eta(t)| \\ &\leq \sqrt{|Z(t)|^2 + |U(t)|^2 + |V(t)|^2\eta(t)}\ \sqrt{|\theta(t)|^2 + |\beta(t)|^2 + |\gamma(t)|^2\eta(t)}, \end{aligned} \tag{3.3}$$

and an upper bound for the Sharpe ratio for the insurance contract could be derived

$$|\textit{Sharpe Ratio}(Y)| \leq \sqrt{|\theta(t)|^2 + |\beta(t)|^2 + |\gamma(t)|^2\eta(t)}.$$

Assuming that in our combined financial and insurance market the instantaneous Sharpe ratios should be bounded by a process L, which itself should be bounded to exclude too favorable deals, we put the restriction on the possible measure changes

$$|\theta(t)|^2 + |\beta(t)|^2 + |\gamma(t)|^2\eta(t) \leq |L(t)|^2 \leq K, \quad 0 \leq t \leq T. \tag{3.4}$$

The bound L on the Sharpe ratio for Y has to be no less than $\theta(t)$ as this risk premium is earned by investing in the stock. The insurance contract carries an additional risk and the investor requires to gain a risk premium strictly above $\theta(t)$. We can assume that

(A6) the process L is $\mathcal{F}$-predictable and satisfies $|\theta(t)|^2 + \varepsilon \leq |L(t)|^2 \leq K$, $0 \leq t \leq T$ with ε, $K > 0$.

The inequality (3.3) is called the Hansen-Jagannathan bound, see Cochrane and Saá-Requejo (2000), Björk and Slinko (2006), and we show that such a bound still holds in our general model with the payment process. The Hansen-Jagannathan bound (3.3) gives a sound financial meaning of the set $\boldsymbol{Q}$ which can now be used in pricing.

Let us now define the no-good-deal price $R := (R(t), 0 \leq t \leq T)$ for a contract insuring the payment process P. Choose the process L which represents a bound on possible gains in the market measured in terms of the instantaneous Sharpe ratios. We deal with

$$R(t) = \sup_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}^{\mathbb{Q}}\Big[e^{-\int_t^T r(s)s}F + \int_t^T e^{-\int_t^s r(u)du} G(s)\,dN(s) + \int_t^T e^{-\int_t^s r(u)du} H(s)ds \,|\, \mathcal{F}_t\Big],\ 0 \le t \le T. \tag{3.5}$$

In this section we work with two backward stochastic differential equations

$$\begin{aligned} dR^{\beta,\gamma}(t) = {} & R^{\beta,\gamma}(t-)r(t)dt - H(t)\,dt - G(t)dN(t) \\ & + Z^{\beta,\gamma}(t)\theta(t)dt + U^{\beta,\gamma}(t)\beta(t)dt + V^{\beta,\gamma}(t)\gamma(t)\eta(t)dt \\ & + Z^{\beta,\gamma}(t)\,dW(t) + U^{\beta,\gamma}(t)\,dB(t) + V^{\beta,\gamma}(t)d\tilde{N}(t),\ R^{\beta,\gamma}(T) = F, \end{aligned} \tag{3.6}$$

with $(\beta, \gamma) \in \mathcal{Q}$, and

$$\begin{aligned} dR^{*}(t) = {} & R^{*}(t-)r(t)dt - H(t)\,dt - G(t)dN(t) \\ & + Z^{*}(t)\theta(t)dt - \sqrt{|L(t)|^2 - |\theta(t)|^2}\,\sqrt{|U^{*}(t)|^2 + |V^{*}(t)|^2\eta(t)}\,dt \\ & + Z^{*}(t)dW(t) + U^{*}(t)dB(t) + V^{*}(t)d\tilde{N}(t),\ R^{*}(T) = F. \end{aligned} \tag{3.7}$$

Theorem 3.1. *Assume that (A1)-(A6) hold. Consider the BSDEs* (3.6) *and* (3.7). *For any* $(\beta,\gamma) \in \mathcal{Q}$ *we have that* $R^{\beta,\gamma}(t) \le R^{*}(t)$, $0 \le t \le T$. *If* $V^{*}(t) \ge 0$ *or* $|L(t)|^2 < \eta(t) + |\theta(t)|^2$ *on* $\eta(t) > 0$ *holds for* $0 \le t \le T$ *then* $R(t) = \sup_{(\beta,\,\gamma)\in\mathcal{Q}} R^{\beta,\gamma}(t) = R^{\beta^{*},\gamma^{*}}(t) = R^{*}(t)$, $0 \le t \le T$.

Notice that (A6): $|\theta(t)|^2 + \varepsilon \le |L(t)|^2$ and $|L(t)|^2 \le \eta(t) + |\theta(t)|^2$ on $\eta(t) > 0$ could hold only if $\eta(t) \ge \varepsilon > 0$ is satisfied on the set $\{\eta(t) > 0\}$. In Example 2.2 the requirement that $\eta(t) \ge \varepsilon > 0$ on $\eta(t) > 0$ is fulfilled if λ is uniformly bounded away from zero. A positive lower bound on λ could be interpreted as a lowest attainable mortality rate or lapse rate which remains after all causes of the decrement are eliminated, see Bayraktar and Young (2007), Young (2008), Milevsky et al. (2005), Bayraktar et al. (2009) for more explanation and motivation. We conclude that $\eta(t) \ge \varepsilon > 0$ on the set $\{\eta(t) > 0\}$ is a reasonable assumption in most insurance and financial applications and this assumption is indirectly imposed whenever $|\theta(t)|^2 + \varepsilon \le |L(t)|^2$ and $|L(t)|^2 \le \eta(t) + |\theta(t)|^2$ on $\eta(t) > 0$ are required to be fulfilled.

It is known that proving $\sup_{(\beta,\,\gamma)\in\mathcal{Q}} R^{\beta,\gamma}(t) = R^{*}(t)$ is more delicate in the case when jumps in the underlying dynamics are allowed, see Björk and Slinko (2006), Murgoci (2008), Donnelly (2010). It turns out that the inequality constraint for γ in the set $\mathcal{Q}$ in the optimization problem might be binding, see (A.3), (A.4), (A.5), which could lead to an arbitrage price R^{*} with $\gamma^{*} = 1$.

Additional conditions are needed to overcome this problem. We could introduce a tighter constraint $\gamma \leq 1-\varepsilon$, see Donnelly (2010). The other possibility, which we follow in this paper, is to guarantee that $\gamma^*(t) < 1$ holds at the optimum so that an arbitrage-free price R^* arises. One can easily show that this last requirement is satisfied if the control process of the equation (3.7) satisfies $V^*(t) \geq 0$ or if we impose the condition in the form $|L(t)|^2 < \eta(t) + |\theta(t)|^2$ holding on the set $\{\eta(t) > 0\}$. We comment on this in the next section.

We can state that the price process Y determined in Theorem 3.1 under the no-good-deal pricing principle satisfies the BSDE

$$\begin{aligned} dY(t) &= Y(t-)r(t)\,dt - H(t)\,dt - G(t)\,dN(t) \\ &\quad + Z(t)\theta(t)\,dt - \sqrt{|L(t)|^2 - |\theta(t)|^2}\,\sqrt{|U(t)|^2 + |V(t)|^2\,\eta(t)}\,dt \\ &\quad + Z(t)\,dW(t) + U(t)\,dB(t) + V(t)\,d\tilde{N}(t), \quad Y(T) = F. \end{aligned} \tag{3.8}$$

The arbitrage-free price (3.8) can be represented as the expected value of the discounted payments under the equivalent martingale measure $\mathbb{Q}^*$ arising from (2.6) with

$$\begin{aligned} \alpha^*(t) &= \theta(t), \\ \beta^*(t) &= -\frac{U^*(t)}{\sqrt{|U^*(t)|^2 + |V^*(t)|^2\,\eta(t)}}\,\sqrt{|L(t)|^2 - |\theta(t)|^2}\,\mathbf{1}\{U^*(t) \neq 0\}, \\ \gamma^*(t) &= -\frac{V^*(t)}{\sqrt{|U^*(t)|^2 + |V^*(t)|^2\,\eta(t)}}\,\sqrt{|L(t)|^2 - |\theta(t)|^2}\,\mathbf{1}\{V^*(t)\,\eta(t) \neq 0\}. \end{aligned} \tag{3.9}$$

Notice that the optimal change of measure (3.9) and the equivalent martingale measure $\mathbb{Q}^*$ depend on the payment process P.

We end up with a comparison principle for our price. The next lemma shows that the no-good-deal price is monotonic, as required, with respect to the benefits and the Sharpe ratio process L.

Lemma 3.1. *Assume that (A1)-(A6) hold. Consider the BSDE* (3.8). *Let* $F' \geq F$, $G(t)' \geq G(t)$, $H'(t) \geq H(t)$, $L'(t) \geq L(t)$, $0 \leq t \leq T$, *and denote the corresponding solutions to* (3.8) *by* Y *and* Y'. *If* $V(t) \geq 0$, $V'(t) \geq 0$ *or* $|L(t)|^2 < \eta(t) + |\theta(t)|^2$, $|L'(t)|^2 < \eta(t) + |\theta(t)|^2$ on $\eta(t) > 0$ *hold for* $0 \leq t \leq T$ *then the prices satisfy* $Y'(t) \geq Y(t)$, $0 \leq t \leq T$.

We point out that the bound on L or non-negativity of the process V is a crucial assumption in proving the comparison for our arbitrage-free no-good-deal price.

4. Local mean-variance hedging

In the previous section the assumption of no-arbitrage (A5) was our starting point. In this section we take a different point of view. We start with a mean-variance optimization criterion.

Let π denote an investment strategy. We define the usual set of admissible strategies

$$\Pi = \left\{ \pi \text{ is } \mathbb{F}\text{-predictable}, \mathbb{E}\left[\int_0^T \left| \sigma(t)\,\pi(t) \right|^2 dt \right] < \infty \right\}. \tag{4.1}$$

Let us consider the wealth process $X := (X(t),\, 0 \le t \le T)$ of the insurer who collects a premium $Y(0)$ for the contract, applies an investment strategy $\pi \in \Pi$ and covers the claims P. The process X represents the replicating portfolio for P. The dynamics of X is given by the stochastic differential equation

$$dX(t) = \pi(t)(\mu(t)dt + \sigma(t)\,dW(t)) + (X(t-) - \pi(t))r(t)dt - H(t)dt - G(t)dN(t),$$
$$X(0) = Y(0).$$

We define the surplus process $C := (C(t),\, 0 \le t \le T)$ as the difference between the wealth available to the insurer after covering the claims and the current price of the contract

$$C(t) = X(t) - Y(t),\ 0 \le t \le T.$$

Notice that the price at time $Y(t)$ could be understood as the reserve for the future liabilities. The process C models the net asset wealth of the insurer, the excess of the assets over the liabilities. We remark that the net asset wealth is an object of the key interest in Solvency II, see European Commission QIS5 (2010). Recalling (2.9) we obtain the dynamics of the surplus process

$$\begin{aligned} dC(t) &= X(t-)r(t)dt + \pi(t)(\mu(t) - r(t))dt - \varphi(t)dt \\ &\quad + (\sigma(t)\pi(t) - Z(t))dW(t) - U(t)dB(t) - V(t)d\tilde{N}(t),\ C(0) = 0. \end{aligned} \tag{4.2}$$

The processes X and C are square integrable under $\mathbb{P}$.

Recall the mean-variance risk measure

$$\rho(\xi) = L\sqrt{Var[\xi]} - \mathbb{E}[\xi], \tag{4.3}$$

where the parameter L measures the risk aversion against the standard deviation with respect to the expectation. The big advantage of the mean-variance risk measure is that it is well-understood by decision makers. Following Leitner (2007) we apply the risk measure (4.3) with the time-varying aversion coefficient

L to the infinitesimal change in the surplus process C. The local version of the risk measure takes the form

$$\begin{aligned}\rho(dC(t))/dt &= L(t)\sqrt{\mathbb{E}[d[C,C](t)\,|\,\mathcal{F}_{t-}]/dt} - \mathbb{E}[dC(t) - C(t-)\,r(t)\,dt\,|\,\mathcal{F}_{t-}]/dt \\ &= L(t)\sqrt{|\,\pi(t)\sigma(t) - Z(t)|^2 + |\,U(t)\,|^2 + |\,V(t)\,|^2\,\eta(t)} \\ &\quad - Y(t-)\,r(t) - \pi(t)(\mu(t) - r(t)) + \varphi(t). \end{aligned} \tag{4.4}$$

The objective is to find, for all $0 \le t \le T$, the hedging strategy π which minimizes the instantaneous risk measure $\rho(dC(t))$ and next, choose the process φ in the price dynamics Y which makes the risk measure vanish $\rho(dC(t)) = 0$.

Theorem 4.1. *Assume that (A1)-(A4), (A6) hold. The admissible investment strategy π which minimizes the local mean-variance risk measure of the surplus* (4.4) *and the price process Y which makes the risk measure* (4.4) *vanish for all $0 \le t \le T$ are of the form*

$$\pi(t) = \frac{1}{\sigma(t)}\left(Z(t) + \sqrt{\frac{|\theta(t)|^2}{|L(t)|^2 - |\theta(t)|^2}}\,\sqrt{|U(t)|^2 + |V(t)|^2\,\eta(t)}\right),\ 0 \le t \le T, \tag{4.5}$$

$$\begin{aligned} dY(t) &= Y(t-)r(t)dt - H(t)dt - G(t)dN(t) \\ &\quad + Z(t)\,\theta(t)\,dt - \sqrt{|L(t)|^2 - |\,\theta(t)\,|^2}\,\sqrt{|U(t)|^2 + |V(t)|^2\,\eta(t)}\,dt \\ &\quad + Z(t)\,dW(t) + U(t)\,dB(t) + V(t)\,d\tilde{N}(t),\ 0 \le t \le T, \\ Y(T) &= F. \end{aligned} \tag{4.6}$$

We remark that we can substitute $L(t) = \theta(t)$ in (3.8) or (4.6) and we arrive at a well-defined equation characterizing an arbitrage-free price. However, to end up with an admissible investment strategy which uniquely solves our optimization problem we have to assume (A6).

There is an obvious similarity between the prices defined in (3.8) and (4.6). This should not mislead us. The process Y in (3.8) arises from the optimization problem of the expected value of the discounted claims with respect to the set of equivalent martingale measures and must satisfy the non-arbitrage condition by our construction. The process Y in (4.6) arises from the local mean-variance optimization problem applied to the surplus process and is defined without the notion of no-arbitrage. The local mean-variance price from Theorem 4.1 exists but may violate the non-arbitrage condition and without assumptions on V or L, η cannot be related to the no-good-deal price from Theorem 3.1. We give two illustrating examples.

Example 4.1. Consider an endowment policy which pays 1 if the policyholder does not lapse the policy. Let N count if the individual surrenders the policy. The compensator of N takes the form $\eta(t) = (1 - N(t-))\,\lambda(t)$ and we assume that the intensity λ follows a stochastic process adapted to the natural filtration $\sigma(W(s),\, 0 \le s \le t)$, see Example 2.2 for details. For simplicity we consider a process λ which is uniformly bounded from above. The process L is assumed to be adapted to $\sigma(W(s),\, 0 \le s \le t)$ (its dynamics is linked to the financial market) which is a very reasonable assumption.

Our BSDE (4.6) takes the form

$$\begin{aligned} dY(t) \;=\; & Y(t-)r(t)\,dt + Z(t)\theta(t)\,dt - \sqrt{|L(t)|^2 - |\theta(t)|^2}\;|V(t)|\sqrt{(1-N(t-))\,\lambda(t)}\,dt \\ & + Z(t)\,dW(t) + V(t)\,d\tilde{N}(t),\; Y(T) = \mathbf{1}\{N(T) = 0\}. \end{aligned} \tag{4.7}$$

It is possible to find the unique square integrable solution (Y, Z, V) to (4.7), see Delong (2011). We change the measure to $\hat{\mathbb{Q}}$ with (2.6) and $\alpha(t) = \theta(t)$, $\beta(t) = \gamma(t) = 0$. The processes

$$\begin{aligned} Y(t) \;&=\; \mathbf{1}\{N(t) = 0\}\,\mathbb{E}^{\hat{\mathbb{Q}}}\Big[e^{-\int_t^T (r(s) + \lambda(s) - \sqrt{|L(s)|^2 - |\theta(s)|^2}\sqrt{\lambda(s)})\,ds}\,\Big|\,\mathcal{F}_t\Big],\; 0 \le t \le T, \\ Z(t) \;&=\; e^{\int_0^t (r(s) + \lambda(s) - \sqrt{|L(s)|^2 - |\theta(s)|^2}\sqrt{\lambda(s)})\,ds}(1 - N(t-))Z^A(t),\; 0 \le t \le T, \\ V(t) \;&=\; -Y(t-),\; 0 \le t \le T, \end{aligned} \tag{4.8}$$

solve the equation (4.7), where Z^A is a unique $\mathcal{F}$-predictable process satisfying

$$\begin{aligned} &\mathbb{E}^{\hat{\mathbb{Q}}}\Big[e^{-\int_0^T (r(s) + \lambda(s) - \sqrt{|L(s)|^2 - |\theta(s)|^2}\sqrt{\lambda(s)})\,ds}\,\Big|\,\mathcal{F}_t\Big] \\ &\quad = \mathbb{E}^{\hat{\mathbb{Q}}}\Big[e^{-\int_0^T (r(s) + \lambda(s) - \sqrt{|L(s)|^2 - |\theta(s)|^2}\sqrt{\lambda(s)})\,ds}\Big] + \int_0^t Z^A(s)\,dW^{\hat{\mathbb{Q}}}(s),\; 0 \le t \le T, \end{aligned}$$

We have found the unique solution to the local mean-variance minimization problem from Theorem 4.1 but without further assumptions the price process (4.8) may be larger than the price of the bond paying 1 at maturity and an arbitrage arises. To arrive at an arbitrage-free price we could introduce the condition $|L(t)|^2 < \lambda(t) + |\theta(t)|^2$. Under this additional requirement the local mean-variance price (4.8) coincides with the no-good-deal price from Theorem 3.1. □

Example 4.2. Consider now a surrender option which pays 1 at the terminal time if the policyholder lapses the policy. Our BSDE (4.6) takes the form

$$\begin{aligned} dY(t) \;=\; & Y(t-)r(t)\,dt + Z(t)\theta(t)\,dt - \sqrt{|L(t)|^2 - |\theta(t)|^2}\;|V(t)|\sqrt{(1-N(t-))\,\lambda(t)}\,dt \\ & + Z(t)\,dW(t) + V(t)\,d\tilde{N}(t),\; Y(T) = \mathbf{1}\{N(T) = 1\}. \end{aligned} \tag{4.9}$$

It is possible to find a unique square integrable solution (Y, Z, V) to (4.7), see Delong (2011) again. We change the measure to $\hat{\mathbb{Q}}$ with (2.6) and $\alpha(t) = \theta(t)$, $\beta(t) = 0$, $\gamma(t) = -\sqrt{|L(t)|^2 - |\theta(t)|^2}\frac{1}{\sqrt{(1-N(t-))\lambda(t)}}\mathbf{1}\{N(t-) = 0\}$. The processes

$$
\begin{aligned}
Y(t) &= \mathbb{E}^{\hat{\mathbb{Q}}}\left[e^{-\int_t^T r(s)ds}\,\middle|\,\mathcal{F}_t\right] \\
&\quad -\mathbf{1}\{N(t) = 0\}\,\mathbb{E}^{\hat{\mathbb{Q}}}\left[e^{-\int_t^T (r(s)+\lambda(s)+\sqrt{|L(s)|^2-|\theta(s)|^2}\sqrt{\lambda(s)})ds}\,\middle|\,\mathcal{F}_t\right],\ 0 \le t \le T, \\
Z(t) &= Z^D(t)\,e^{\int_0^t r(s)ds} - (1-N(t-))\,Z^E(t)\,e^{\int_0^t (r(s)+\lambda(s)+\sqrt{|L(s)|^2-|\theta(s)|^2}\sqrt{\lambda(s)})ds},\ 0 \le t \le T, \\
V(t) &= E(t)\,e^{\int_0^t (r(s)+\lambda(s)+\sqrt{|L(s)|^2-|\theta(s)|^2}\sqrt{\lambda(s)})ds},\ 0 \le t \le T, \qquad (4.10)
\end{aligned}
$$

solve the equation (4.9), where Z^D, Z^E are unique $\mathcal{F}$-predictable processes satisfying

$$
\begin{aligned}
&\mathbb{E}^{\hat{\mathbb{Q}}}\left[e^{-\int_0^T r(s)ds}\,\middle|\,\mathcal{F}_t\right] = \mathbb{E}^{\hat{\mathbb{Q}}}\left[e^{-\int_0^T r(s)ds}\right] + \int_0^t Z^D(t)\,dW^{\hat{\mathbb{Q}}}(t),\ 0 \le t \le T, \\
&\mathbb{E}^{\hat{\mathbb{Q}}}\left[e^{-\int_0^T (r(s)+\lambda(s)+\sqrt{|L(s)|^2-|\theta(s)|^2}\sqrt{\lambda(s)})ds}\,\middle|\,\mathcal{F}_t\right] \\
&\quad = \mathbb{E}^{\hat{\mathbb{Q}}}\left[e^{-\int_0^T (r(s)+\lambda(s)+\sqrt{|L(s)|^2-|\theta(s)|^2}\sqrt{\lambda(s)})ds}\right] + \int_0^t Z^E(t)\,dW^{\hat{\mathbb{Q}}}(t),\ 0 \le t \le T,
\end{aligned}
$$

and

$$
E(t) = \mathbb{E}^{\hat{\mathbb{Q}}}\left[e^{-\int_0^T (r(s)+\lambda(s)+\sqrt{|L(s)|^2-|\theta(s)|^2}\sqrt{\lambda(s)})ds}\,\middle|\,\mathcal{F}_t\right],\ 0 \le t \le T.
$$

Our local mean-variance price from Theorem 4.1 coincides with the no-good-deal price from Theorem 3.1 without additional requirements on L, η. □

Examples 4.1 and 4.2 are interesting from the analytic point of view as we provided explicit solutions (their representations) to our BSDE for two important insurance pricing and hedging problems. Example 4.2 also shows the significance of the condition $V^*(t) \ge 0$ in Theorem 3.1.

We state some important features of the solution derived in Theorem 4.1. First, notice that the criterion of vanishing local mean-variance risk measure $\rho(dC(t)) = 0$ in (4.4) is equivalent to requiring that the infinitesimal Sharpe ratio of the surplus process, which is earned by the insurer who sold the contract, equals exactly $L(t)$ under the optimal investment strategy. The derivation of the price process (4.6) could also be motivated by setting the objective under which the insurer determines the price of the contract so that the Sharpe ratio of the surplus equals the pre-specified target L, see Bayraktar and Young (2007), Young (2008), Bayraktar and Young (2008), Milevsky et al. (2005),

Bayraktar et al. (2009). We can interpret L as the process which controls the ratio of the expected earned surplus (the net asset wealth) to its deviation over time. The target level of L would be set by the insurer who performs asset-liability studies. This gives a link to the theory from the previous section and explain why we should still assume (A6). Secondly, the vanishing risk measure together with $L(t) > 0$ forces the infinitesimal change in the surplus arising from the optimal investment strategy to be positive (greater than the risk-free return) on average. We expect that the assets should cover the liabilities and a positive net asset wealth should arise. Our second intuitive argument is formalized by the following lemma.

Lemma 4.1. *Assume that (A1)-(A4), (A6) hold. Consider the surplus process arising from the investment strategy* (4.5) *and the price process* (4.6)*. The discounted surplus process* $e^{-\int_0^t r(s)ds}C(t)$ *is a* $\mathbb{P}$*-submartingale.*

Finally, we remark that the arbitrage-free mean-variance price from Theorem 4.1 clearly satisfies the comparison from Lemma 3.1.

5. Ambiguity risk measure

In the last two sections we assumed that we knew the true real-world probability measure $\mathbb{P}$ under which we calculated the Sharpe ratios or the local mean-variance risk. In real life an ambiguity about the true measure $\mathbb{P}$ or the true value of the estimated parameters arises which should be taken into account in the modeling process.

We deal with the following ambiguity risk measure

$$\rho(\xi) = -\inf_{\mathbb{Q}\in\mathcal{P}} \mathbb{E}^{\mathbb{Q}}[\xi] = \sup_{\mathbb{Q}\in\mathcal{P}} \mathbb{E}^{\mathbb{Q}}[-\xi]. \tag{5.1}$$

The set $\mathcal{P}$ denotes the set of prior probabilities (or probability laws) which represent different beliefs about the evolution of the dynamics in the model, see Chen and Epstein (2002). All prior probabilities are equivalent to the base measure $\mathbb{P}$ and "the difference" between the measures in $\mathcal{P}$ and the measure $\mathbb{P}$ is controlled by the process L which appears in the definition of the set $\mathcal{P}$.

Following Leitner (2007) and Becherer (2009) we apply the conditional version of the risk measure (5.1) to the discounted terminal surplus (the terminal net asset wealth) arising from managing the wealth X and paying the claims P. We deal with

$$\rho_t\left(e^{-\int_t^T r(s)ds}C(T)\right) = \rho_t\left(e^{-\int_t^T r(s)ds}X(T) - e^{-\int_t^T r(s)ds}F\right),\ 0 \le t \le T. \tag{5.2}$$

The objective is to find, for all $0 \le t \le T$, the hedging strategy π which minimizes the risk measure $\rho_t(e^{-\int_t^T r(s)ds}C(T))$ and the price $Y(t)$ which makes the

risk measure vanish $\rho_t(e^{-\int_t^T r(s)ds}C(T)) = 0$ under the condition that $X(t) = Y(t)$. The ambiguity risk price and hedging strategy follow from

$$Y(t) = \inf_{\pi}\left\{-\inf_{\mathbb{Q}\in\mathcal{P}}\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(s)ds}X(T) - X(t) - e^{-\int_t^T r(s)ds}F\,\middle|\,\mathcal{F}_t\right]\right\},\ 0\le t\le T. \quad (5.3)$$

The optimization problem of this section is to solve

$$\begin{aligned} J(t) = \inf_{\pi\in\Pi}\Bigg\{\sup_{\mathbb{Q}\in\mathcal{P}}\mathbb{E}^{\mathbb{Q}}\Bigg[&e^{-\int_t^T r(u)du}F + \int_t^T e^{-\int_t^s r(u)du}H(s)ds + \int_t^T e^{-\int_t^s r(u)du}G(s)\,dN(s) \\ &- \int_t^T e^{-\int_t^s r(u)du}\pi(s)\big((\mu(s)-r(s))ds + \sigma(s)dW(s)\big)\,\Bigg|\,\mathcal{F}_t\Bigg]\Bigg\},\ 0\le t\le T, \quad (5.4)\end{aligned}$$

where the set of admissible strategies Π is defined in (4.1). Under (5.3) and (5.4) we aim at finding the investment strategy which leads to the lowest expected terminal shortfall of the assets to cover the liabilities under the worst probability law. The ambiguity risk price covers the min-max expected shortfall. This is a sound optimization criterion especially for insurers who are forced by regulators to carry stress-tests on model parameters and hold a sufficient capital to withstand extreme scenarios. The process L define the range of possible scenarios or an uncertainty arising from the estimation process of the key parameters. Notice that the ambiguity risk pricing and hedging lead to a positive expected discounted terminal surplus (net asset wealth) in the case when the worst-law is not realized.

We work with three backward stochastic differential equations:

$$\begin{aligned} dJ^{\pi,\alpha,\beta,\gamma}(t) = {}& J^{\pi,\alpha,\beta,\gamma}(t-)r(t)dt - H(t)dt - G(t)dN(t) \\ &+ \pi(t)\big((\mu(t)-r(t))dt + \sigma(t)dW(t)\big) \\ &+ Z^{\pi,\alpha,\beta,\gamma}(t)\alpha(t)dt + U^{\pi,\alpha,\beta,\gamma}(t)\beta(t)dt + V^{\pi,\alpha,\beta,\gamma}(t)\gamma(t)\eta(t)dt \\ &+ Z^{\pi,\alpha,\beta,\gamma}(t)dW(t) + U^{\pi,\alpha,\beta,\gamma}(t)dB(t) + V^{\pi,\alpha,\beta,\gamma}(t)d\tilde{N}(t),\ J^{\pi,\alpha,\beta,\gamma}(T) = F, \end{aligned} \quad (5.5)$$

with processes $(\alpha,\beta,\gamma)\in\mathcal{P}$,

$$\begin{aligned} dJ^{\pi,*}(t) = {}& J^{\pi,*}(t-)r(t)dt - H(t)dt - G(t)dN(t) \\ &+ \pi(t)\big((\mu(t)-r(t))dt + \sigma(t)dW(t)\big) \\ &- L(t)\sqrt{|Z^{\pi,*}(t)|^2 + |U^{\pi,*}(t)|^2 + |V^{\pi,*}(t)|^2\eta(t)}\,dt \\ &+ Z^{\pi,*}(t)dW(t) + U^{\pi,*}(t)dB(t) + V^{\pi,*}(t)d\tilde{N}(t),\ J^{\pi,*}(T) = F, \end{aligned} \quad (5.6)$$

and

$$dJ^{**}(t) = J^{**}(t-)\,r(t)\,dt - H(t)\,dt - G(t)\,dN(t)$$
$$+ Z^{**}(t)\theta(t)\,dt - \sqrt{|L(t)|^2 - |\theta(t)|^2}\,\sqrt{|U^{**}(t)|^2 + |V^{**}(t)|^2\,\eta(t)}\,dt$$
$$+ Z^{**}(t)\,dW(t) + U^{**}(t)\,dB(t) + V^{**}(t)\,d\tilde{N}(t),\ J^{**}(T) = F. \qquad (5.7)$$

We give the key results of this section.

Theorem 5.1. *Assume that (A1)-(A4), (A6) hold. Consider the BSDEs* (5.5) *and* (5.6). *For any* $\pi \in \Pi$, $(\alpha, \beta, \gamma) \in \mathcal{P}$ *we have that* $J^{\pi,\alpha,\beta,\gamma}(t) \le J^{\pi,*}(t)$, $0 \le t \le T$. *For any* $\pi \in \Pi$ *under which*

$$-\frac{V^{\pi,*}(t)}{\sqrt{|Z^{\pi,*}(t)|^2 + |U^{\pi,*}(t)|^2 + |V^{\pi,*}(t)|^2\,\eta(t)}}\,L(t)\mathbf{1}\{V^{\pi,*}(t)\,\eta(t) \neq 0\} < 1,\ 0 \le t \le T,$$

we have that $\sup_{(\alpha,\beta,\gamma)\in\mathcal{P}} J^{\pi,\alpha,\beta,\gamma}(t) = J^{\pi,*}(t)$, $0 \le t \le T$.

Theorem 5.2. *Assume that (A1)-(A4), (A6) hold. Consider the BSDEs* (5.6) *and* (5.7). *Define the class of admissible strategies* $\mathcal{A}$ *consisting* of $\pi \in \Pi$ *under which*

$$-\frac{V^{\pi,*}(t)}{\sqrt{|Z^{\pi,*}(t)|^2 + |U^{\pi,*}(t)|^2 + |V^{\pi,*}(t)|^2\,\eta(t)}}\,L(t)\mathbf{1}\{V^{\pi,*}(t)\,\eta(t) \neq 0\} < 1,\ 0 \le t \le T,$$

$$-\frac{V^{*,*}(t)}{\sqrt{|Z^{\pi,*}(t)|^2 + |U^{\pi,*}(t)|^2 + |V^{*,*}(t)|^2\,\eta(t)}}\,L(t)\mathbf{1}\{V^{*,*}(t)\,\eta(t) \neq 0\} < 1,\ 0 \le t \le T,$$

For any admissible $\pi \in \mathcal{A}$ *we have that* $J^{\pi,*}(t) \ge J^{**}(t)$, $0 \le t \le T$. *If* $V^{*,*}(t) \ge 0$ *or* $|L(t)|^2 < \eta(t) + |\theta(t)|^2$ *on* $\eta(t) > 0$ *holds for* $0 \le t \le T$ *then* $\inf_{\pi\in\mathcal{A}} J^{\pi,*}(t) = J^{**}(t) = J(t)$, $0 \le t \le T$ *and the optimal admissible investment strategy is of the form*

$$\pi(t) = \frac{1}{\sigma(t)}\left(Z^{**}(t) + \sqrt{\frac{|\theta(t)|^2}{|L(t)|^2 - |\theta(t)|^2}}\,\sqrt{|U^{**}(t)|^2 + |V^{**}(t)|^2\,\eta(t)}\right),\ 0 \le t \le T. \qquad (5.8)$$

The restricted class $\mathcal{A} \subset \Pi$ arises due to the inherited difficulties in applying comparisons for processes with jumps, see Royer (2006). If the tighter condition: $|L(t)|^2 < \eta(t)$ on $\eta(t) > 0$ is imposed then the optimality of (5.8) holds in Π.

We can state that the price process Y determined in Theorems 5.1, 5.2 under the ambiguity risk measure optimization criterion satisfies the BSDE

$$\begin{aligned} dY(t) &= Y(t-)\,r(t)\,dt - H(t)\,dt - G(t)\,dN(t) \\ &\quad + Z(t)\,\theta(t)\,dt - \sqrt{|L(t)|^2 - |\theta(t)|^2}\,\sqrt{|U(t)|^2 + |V(t)|^2\,\eta(t)}\,dt \\ &\quad + Z(t)\,dW(t) + U(t)\,dB(t) + V(t)\,d\tilde{N}(t),\ Y(T) = F. \end{aligned} \tag{5.9}$$

The price (5.9) is arbitrage-free and coincides with the arbitrage-free prices Y defined in (3.8) and (4.6). The price (5.9) fulfills the comparison from Lemma 3.1. The hedging strategies (4.5) and (5.8) coincide. The submartingale property from Lemma 4.1 for the surplus under (5.8), (5.9) is satisfied. We conclude that the equivalence between ambiguity risk, local mean-variance and no-good-deal pricing and hedging holds.

We remark that the price derived under the local variance minimization criterion from Bayraktar and Young (2007), Young (2008), Bayraktar and Young (2008), Milevsky et al. (2005), Bayraktar et al. (2009) is equivalent to the no-good-deal price and the ambiguity risk price but the minimal local variance hedging strategy does not coincide with the ambiguity risk hedging strategy. For this reason we favor the local mean-variance minimization objective.

6. Pricing and hedging of a unit-linked policy with a surrender option – a numerical example

In this section we consider pricing and hedging of an unit-linked life insurance policy with a surrender option in the case when the lapse intensity depends on the evolution of the financial market, see Example 2.2 for a motivation. Pricing and hedging of a surrender option under an irrational lapse behavior linked to the financial market is an important practical and theoretical problem. We apply the results of the previous sections to price and hedge our contract in an arbitrage-free way. We present a scheme for solving the BSDE numerically.

We assume that the characteristics of the payment process P are given by (2.4) with $n = 1$ and

(A7) the lapse intensity process λ and the process L are adapted to the natural filtration $\sigma(W(s),\ 0 \le s \le t)$ and $\lambda(t) > 0,\ 0 \le t \le T$.

The processes λ and L depend only on the financial market. In the characteristics (2.4) of the payment process, the process h models partial surrenders (amounts which could be withdrawn from the policy without surrendering it), g denotes the surrender value of the policy if the policyholder lapses the policy and f is the survival benefit paid if the policyholder does not lapse the policy.

Our backward stochastic differential equation takes the form

$$\begin{aligned} dY(t) &= Y(t-)\,r(t)\,dt - (1 - N(t-))\,h(t,S(t))\,dt - g(t,S(t))\,dN(t) \\ &\quad + Z(t)\,\theta(t)\,dt - \sqrt{|L(t)|^2 - |\theta(t)|^2}\,|V(t)|\sqrt{(1 - N(t-))\,\lambda(t)}\,dt \\ &\quad + Z(t)\,dW(t) + V(t)\,d\tilde{N}(t),\ Y(T) = (1 - N(T))f(T,S(T)). \end{aligned} \tag{6.1}$$

We remark that the BSDE (6.1) could reduce to a simpler (and straightforward to solve) linear equation if the sign of V is constant. For conditions under which $V(t) \geq 0$ or $V(t) \leq 0$ we refer to Delong (2011).

In most of the applications our BSDE (6.1) would not reduce to a linear equation (as the process V changes its sign) and a numerical scheme for (6.1) must be used. We propose to adapt the discretization scheme from Bouchard and Elie (2008) who construct a numerical scheme for finding a solution to a BSDE driven by a Brownian motion and a Poisson random measure.

Notice that our BSDE (6.1) is equivalent to the following BSDE

$$\begin{aligned} dY(t) &= Y(t-)r(t)dt - (1-N(t-))h(t,S(t))dt - g(t,S(t))(1-N(t-))\lambda(t)dt \\ &\quad + Z(t)\theta(t)dt - \sqrt{|L(t)|^2 - |\theta(t)|^2}\,|Q(t) + g(t,S(t))|\sqrt{(1-N(t-))\lambda(t)}\,dt \\ &\quad + Z(t)dW(t) + Q(t)d\tilde{N}(t),\ Y(T) = (1-N(T))f(T,S(T)), \end{aligned} \tag{6.2}$$

where we introduce the process Q defined as $Q(t) = V(t) - g(t,S(t))$, $0 \leq t \leq T$. The unique solution to the BSDE (6.2) could be approximated as follows. Choose a time grid h. Set $Y(T) = (1-N(T))f(T,S(T))$. The backward procedure from $t = T-h$ till $t = 0$ is of the form

$$\begin{aligned} Z(t) &= \frac{1}{h}\mathbb{E}\big[Y(t+h)\triangle W(t)\,\big|\,\mathcal{F}_t\big], \\ Q(t) &= \frac{1}{h\lambda(t)}\mathbb{E}\big[Y(t+h)\triangle\tilde{N}(t)\,\big|\,\mathcal{F}_t\big], \\ Y(t) &= \frac{1}{1+r(t)h}\mathbb{E}\big[Y(t+h) - \big(-(1-N(t))h(t,S(t)) - g(t,S(t))(1-N(t))\lambda(t) \\ &\quad + Z(t)\theta(t) - \sqrt{|L(t)|^2 - |\theta(t)|^2}\,|Q(t) + g(t,S(t))|\sqrt{(1-N(t))\lambda(t)}\big)h\,\big|\,\mathcal{F}_t\big], \end{aligned} \tag{6.3}$$

where $\triangle W$ and $\triangle\tilde{N}$ denote the increments of the Brownian motion and the compensated point process over the corresponding time grid $[t, t+h]$. The process Z, Q, Y have to be obtained by Monte-Carlo simulations. We remark that the proposed scheme (6.3) could be adapted to the more general dynamics from Sections 3-5.

Let us consider a numerical example. We deal with a unit-linked policy with duration of $T = 1$ year. The asset S follows a geometric Brownian motion with the dynamics

$$\frac{dS(t)}{S(t)} = 0.05dt + 0.1dW(t),\ S(0) = 100.$$

The risk-free rate equals $r = 0.02$. We assume that if the policyholder lapses the policy then the current asset value is paid, $h(t,S(t)) = S(t)$, and if the policyholder does not lapse the policy then the asset value is paid with the

guarantee that the minimal risk-free return has to be earned, $f(T, S(T)) = \max\{S(T), S(0)e^{0.02T}\}$. The intensity follows the linear process

$$\lambda(t) = 0.15 + 0.1S(t), \quad 0 \leq t \leq T.$$

The parameters and the dynamics for λ are chosen arbitrarily for the case of this numerical example. To guarantee that the price derived from solving the BSDE (6.1) is arbitrage-free we only consider the parameters L from the interval $L \in [0.3; 0.489]$. We apply the numerical scheme (6.3) to solve the BSDE. To calculate the inner expected values and avoid nested simulations in (6.3) we use Monte-Carlo regression methods, see Bender and Denk (2007), and we approximate the expectations by regression splines (natural cubic splines) with $S(t)$ as the explanatory variable. In our study we use 10000 simulated sample paths. The numerical scheme works efficiently.

Table 1 compares the prices $Y(0)$ at the inception of the contract for different parameters L from the non-arbitrage range. The price is increasing in L as proved in Lemma 3.1. Notice that the insurer can buy the stock S and face an unhedgeable risk only if the policyholder does not lapse the policy and the terminal guarantee (the put option) is in force. The differences between the initial prices in Table 1 are small due to the fact that the embedded put option is rather cheap at the inception of the contract and small changes in L (which represents a compensation for taking the unhedgeable risk) do not substantially increase the price for the whole contract. We investigate in more detail

TABLE 1

THE PRICE FOR THE CONTRACT AT THE INITIAL TIME $t = 0$

Parameter L	The price
0.3	102.66
0.35	102.93
0.4	103.06
0.45	103.18

TABLE 2

PERCENTILES OF THE TERMINAL SURPLUS (THE NET ASSET WEALTH) FOR $L = 0.45$

Percentile 0.001	– 70.37
Percentile 0.01	– 38.74
Percentile 0.05	– 25.071
Percentile 0.95	32.18
Percentile 0.99	51.56
Percentile 0.999	124.15

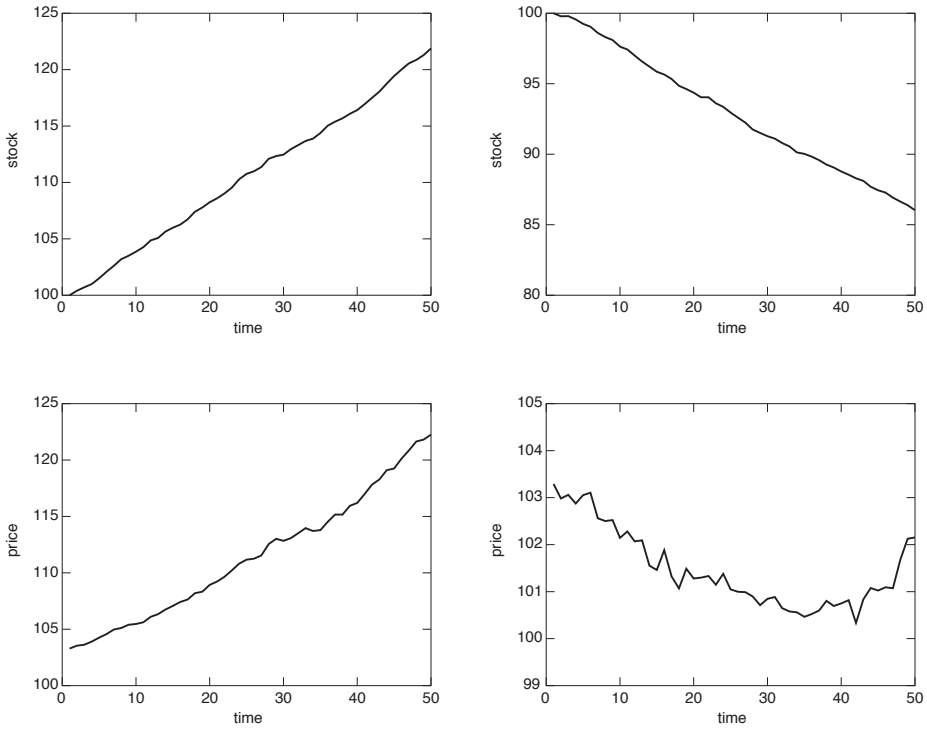

FIGURE 1: The price processes for $L = 0.45$ under the two stock scenarios.

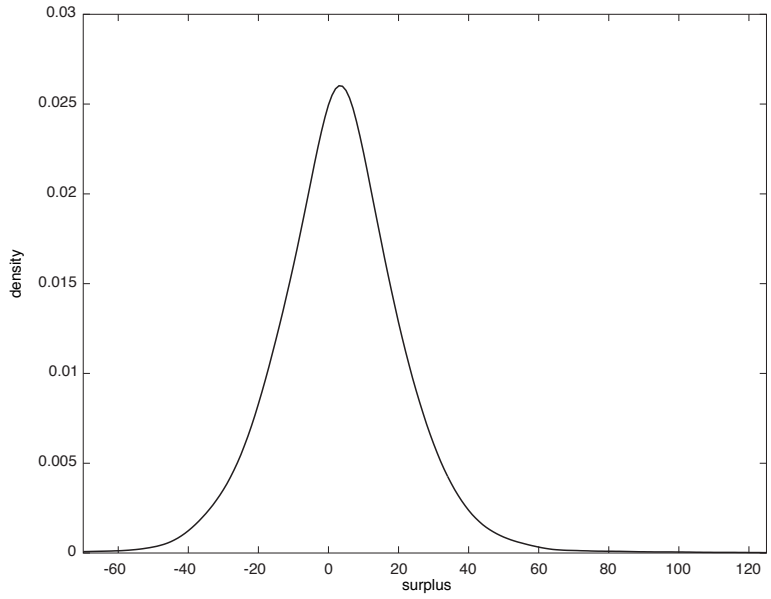

FIGURE 2: Empirical density of the terminal surplus (the net asset wealth) for $L = 0.45$.

the case of $L = 0.45$. In Figure 1 we present how the price process Y evolves over time under the two stock scenarios. Both dynamics agree with our intuition. If the stock increases then the surrender value and the unguaranteed terminal benefit should both increase and they should dominate the decreasing value of the put option. Hence, the price should increase along with the stock. If the stock decreases then the surrender value and the unguaranteed terminal benefit should both decrease but at the same time the value of the put option should increase. Hence, the price should slightly decrease, much less than the stock, and next increase to fulfill the terminal guarantee. In Figure 1 we can observe that our approximation of the price process is very sensitive to downward changes in the stock over the time. This suggests that smoothing over time-dependent coefficients of the fitted natural cubic splines might be useful. Finally, in Figure 2 we give a smoothed version of the empirical density of the surplus (the net asset wealth) at the terminal time $T = 1$ arising from applying the optimal investment strategy (4.5). From Figure 2 and Table 2 we can deduce that the distribution of the terminal net asset wealth is right-skewed which is very favorable from the point of possible applications. The expected net asset wealth arising from collecting the premium 103.18 equals 4.02 which gives a reasonable profit margin of 3.9% for the insurer. We can conclude that our pricing and hedging strategy could be very useful for insurers and could improve their asset-liability positions.

7. Conclusion

In this paper we studied pricing and hedging for an insurance payment process. We succeeded in establishing a general modeling framework for no-good-deal, local mean-variance and ambiguity risk pricing and hedging. We characterized the price and the hedging strategy as a solution to a backward stochastic differential equation. We believe that our results reinforce a derivation of robust hedging strategies in sophisticated models and improve asset-liability management in insurance companies.

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A. Appendix

Proof of Theorem 3.1: Choose $(\beta, \gamma) \in \mathcal{Q}$ and the corresponding measure $\mathbb{Q}^{\beta,\gamma} \in \mathcal{Q}$, which we simply denote by $\mathbb{Q}$. There exists unique square integrable solutions $(R^{\beta,\gamma}, Z^{\beta,\gamma}, U^{\beta,\gamma}, V^{\beta,\gamma})$, (R^*, Z^*, U^*, V^*) to the BSDEs (3.6) and (3.7), see Becherer (2006), Delong (2010). Define $\bar{R}(t) = R^{\beta,\gamma}(t) - R^*(t)$ and $\bar{Z}$, $\bar{U}$, $\bar{V}$ in the analogous way. We obtain

$$
\begin{aligned}
d\bar{R}(t) &= \bar{R}(t-)r(t)dt + \bar{Z}(t)\theta(t)dt + \bar{Z}(t)dW(t) + \bar{U}(t)\,dB(t) + \bar{V}(t)d\tilde{N}(t) \\
&\quad + \Big(U^{\beta,\gamma}(t)\beta(t) + V^{\beta,\gamma}(t)\gamma(t)\eta(t) + \sqrt{|L(t)|^2 - |\theta(t)|^2}\sqrt{|U^*(t)|^2 + |V^*(t)|^2\eta(t)}\Big)dt \\
&= \bar{R}(t-)r(t)dt + \bar{Z}(t)(dW(t) + \theta(t)dt) + \bar{U}(t)\,(dB(t) + \beta(t)dt) + \bar{V}(t)(d\tilde{N}(t) + \gamma(t)\eta(t)dt) \\
&\quad + \Big(U^*(t)\beta(t)dt + V^*(t)\gamma(t)\eta(t)dt + \sqrt{|L(t)|^2 - |\theta(t)|^2}\sqrt{|U^*(t)|^2 + |V^*(t)|^2\eta(t)}\Big)dt, \\
\bar{R}(T) &= 0.
\end{aligned}
$$

By changing the measure to $\mathbb{Q}$ and discounting we arrive at

$$
\begin{aligned}
&d\left(e^{-\int_0^t r(s)ds}\bar{R}(t)\right) \\
&\quad = e^{-\int_0^t r(s)ds}\bar{Z}(t)dW^{\mathbb{Q}}(t) + e^{-\int_0^t r(s)ds}\bar{U}(t)dB^{\mathbb{Q}}(t) + e^{-\int_0^t r(s)ds}\bar{V}(t)d\tilde{N}^{\mathbb{Q}}(t) \\
&\qquad + e^{-\int_0^t r(s)ds}\Big(U^*(t)\beta(t) + V^*(t)\gamma(t)\eta(t) \\
&\qquad\quad + \sqrt{|L(t)|^2 - |\theta(t)|^2}\sqrt{|U^*(t)|^2 + |V^*(t)|^2\eta(t)}\Big)dt.
\end{aligned}
\tag{A.1}
$$

We can show that the stochastic integrals in (A.1) are $\mathbb{Q}$-martingales, see Delong (2011). Integrating (A.1) and taking the expected value we can derive

$$
\begin{aligned}
\bar{R}(t) &= -\mathbb{E}^{\mathbb{Q}}\Big[\int_t^T e^{-\int_t^s r(u)du}(U^*(s)\beta(s) + V^*(s)\gamma(s)\eta(s) \\
&\quad + \sqrt{|L(s)|^2 - |\theta(s)|^2}\sqrt{|U^*(s)|^2 + |V^*(s)|^2\eta(s)})\,ds\,|\,\mathcal{F}_t\Big],\ 0 \le t \le T.
\end{aligned}
\tag{A.2}
$$

One can show that the pair

$$
\begin{aligned}
x^* &= --\frac{u}{\sqrt{u^2 + v^2\eta}}\,\delta\mathbf{1}\{u \neq 0\}, \\
y^* &= -\frac{v}{\sqrt{u^2 + v^2\eta}}\,\delta\mathbf{1}\{v\eta \neq 0\},
\end{aligned}
\tag{A.3}
$$

is a solution to the optimization problem

$$\begin{aligned} &ux + vy\eta \to \min \\ &x^2 + y^2\eta \le \delta^2, \end{aligned} \tag{A.4}$$

and the minimum in (A.4) equals $\delta\sqrt{u^2 + v^2\eta}$. We conclude now that $\bar{R}(t) \le 0$, $0 \le t \le T$. The first part of the theorem is proved.

Define the processes

$$\beta^*(t) = -\frac{U^*(t)}{\sqrt{|U^*(t)|^2 + |V^*(t)|^2\eta(t)}} \sqrt{|L(t)|^2 - |\theta(t)|^2}\ \mathbf{1}\{|U^*(t)| \neq 0\}$$

$$\gamma^*(t) = -\frac{V^*(t)}{\sqrt{|U^*(t)|^2 + |V^*(t)|^2\eta(t)}} \sqrt{|L(t)|^2 - |\theta(t)|^2}\ \mathbf{1}\{|V^*(t)\,\eta(t)| \neq 0\},$$

We can check that under our assumptions we have $\gamma^*(t) < 1$ and $(\beta^*, \gamma^*) \in \mathcal{Q}$. Notice that a solution $(R^{\beta^*,\gamma^*}, Z^{\beta^*,\gamma^*}, U^{\beta^*,\gamma^*}, V^{\beta^*,\gamma^*})$ to (3.6), defined with the processes (A.5), must coincide with a solution (R^*, Z^*, U^*, V^*) to (3.7) by the uniqueness of solutions. We conclude that $\sup_{(\beta,\gamma)\in\mathcal{Q}} R^{\beta,\gamma}(t) = R^{\beta^*,\gamma^*}(t) = R^*(t)$. Finally, we find a representation of $R^{\beta,\gamma}$ for $(\beta, \gamma) \in \mathcal{Q}$. Changing the measure in (3.6) leads to

$$\begin{aligned} dR^{\beta,\gamma}(t) \;=\; & R^{\beta,\gamma}(t-)\,r(t)\,dt - H(t)\,dt - G(t)\,dN(t) \\ & + Z^{\beta,\gamma}(t)\,dW^{\mathbb{Q}}(t) + U^{\beta,\gamma}(t)\,dB^{\mathbb{Q}}(t) + V^{\beta,\gamma}(t)\,d\tilde{N}^{\mathbb{Q}}(t),\ R^{\beta,\gamma}(T) = F, \end{aligned} \tag{A.6}$$

and by taking the expected value we find that

$$\begin{aligned} R^{\beta,\gamma}(t) \;=\; & \mathbb{E}^{\mathbb{Q}}\Big[e^{-\int_t^T r(s)ds} F \\ & + \int_t^T e^{-\int_t^s r(u)du} H(s)\,ds + \int_t^T e^{-\int_t^s r(u)du} G(s)\,dN(s)\,\Big|\,\mathcal{F}_t\Big],\ 0 \le t \le T. \end{aligned}$$

The proof is completed as $R(t) = \sup_{(\beta,\gamma)\in\mathcal{Q}} R^{\beta,\gamma}(t)$ by (3.5). □

Proof of Lemma 3.1: Let $\bar{Y}(t) = Y(t) - Y'(t)$. The processes $\bar{Z}$, $\bar{U}$, $\bar{V}$, $\bar{F}$, $\bar{G}$, $\bar{H}$ are defined analogously. We can derive

$$\begin{aligned} d\bar{Y}(t) \;=\; & \bar{Y}(t-)\,r(t)\,dt - \bar{H}(t)\,dt - \bar{G}(t)\,dN(t) + \bar{Z}(t)\,\theta(t)\,dt \\ & - \sqrt{|L(t)|^2 - |\theta(t)|^2}\ \sqrt{|U(t)|^2 + |V(t)|^2\eta(t)}\ dt \\ & + \sqrt{|L'(t)|^2 - |\theta(t)|^2}\ \sqrt{|U'(t)|^2 + |V'(t)|^2\eta(t)}\ dt \end{aligned}$$

$$
\begin{aligned}
&+ \bar{Z}(t)\,dW(t) + \bar{U}(t)\,dB(t) + \bar{V}(t)\,d\tilde{N}(t)\\
&= \bar{Y}(t-)\,r(t)\,dt - \bar{H}(t)\,dt - \bar{G}(t)\,dN(t) + \bar{Z}(t)(dW(t) + \theta(t)\,dt)\\
&\quad + \bar{U}(t)\Big(dB(t) + \sqrt{|L'(t)|^2 - |\theta(t)|^2}\cdot\\
&\qquad \cdot \frac{\sqrt{|U'(t)|^2 + |V'(t)|^2\eta(t)} - \sqrt{|U(t)|^2 + |V'(t)|^2\eta(t)}}{\bar{U}(t)}\,\mathbf{1}\{U(t)\neq 0\}\,dt\Big)\\
&\quad + \bar{V}(t)\Big(d\tilde{N}(t) + \sqrt{|L'(t)|^2 - |\theta(t)|^2}\cdot\\
&\qquad \cdot \frac{\sqrt{|U(t)|^2 + |V'(t)|^2\eta(t)} - \sqrt{|U(t)|^2 + |V(t)|^2\eta(t)}}{\bar{V}(t)\,\eta(t)}\,\eta(t)\mathbf{1}\{\bar{V}(t)\eta(t)\neq 0\}\,dt\Big)\\
&\quad - \Big(\sqrt{|L(t)|^2 - |\theta(t)|^2} - \sqrt{|L'(t)|^2 - |\theta(t)|^2}\Big)\sqrt{|U(t)|^2 + |V(t)|^2\eta(t)}\;dt\\
&= \bar{Y}(t-)\,r(t)\,dt - \bar{H}(t)\,dt - \bar{G}(t)\,dN(t) + \bar{Z}(t)\,dW^{\bar{\mathbb{Q}}}(t) + \bar{U}(t)\,dB^{\bar{\mathbb{Q}}}(t) + \bar{V}(t)\,d\tilde{N}^{\bar{\mathbb{Q}}}(t)\\
&\quad - \Big(\sqrt{|L(t)|^2 - |\theta(t)|^2} - \sqrt{|L'(t)|^2 - |\theta(t)|^2}\Big)\sqrt{|U(t)|^2 + |V(t)|^2\eta(t)}\;dt,
\end{aligned}
$$

where we change the measure to $\bar{\mathbb{Q}} \in \mathcal{P}$. It is not difficult to check that the processes defining $\bar{\mathbb{Q}}$ fulfill $(\theta, \beta, \gamma) \in \mathcal{P}$. By taking the expected value we obtain

$$
\begin{aligned}
\bar{Y}(t) = \mathbb{E}^{\bar{\mathbb{Q}}}\Big[&\bar{F}e^{-\int_t^T r(s)ds} + \int_t^T \bar{H}(s)e^{-\int_t^s r(u)du}ds + \int_t^T \bar{G}(s)e^{-\int_t^s r(u)du}dN(s)\\
&+ \int_t^T e^{-\int_t^s r(u)du}\Big(\sqrt{|L(s)|^2 - |\theta(s)|^2} - \sqrt{|L'(s)|^2 - |\theta(s)|^2}\Big)\sqrt{|U(s)|^2 + |V(s)|^2\eta(s)}\,ds\Big] \geq 0,
\end{aligned}
$$

for all $0 \leq t \leq T$. □

Proof of Theorem 4.1: The result follows from the properties of the function

$$
w(\pi) = L\sqrt{(\pi\sigma - z)^2 + u^2 + v^2\eta} - \pi(\mu - r) - ry + \varphi,
$$

investigated under the assumption that $L > \theta = \frac{\mu - r}{\sigma} \geq 0$. It is straightforward to find a unique minimizer π^* of w and φ such that $w(\pi^*) = 0$. The strategy (4.5) is admissible, $\pi \in \Pi$. Its predictability and square integrability follow from predictability and square integrability of the solution to (4.6) and the assumptions (A1), (A6). □

Proof of Lemma 4.1: By substituting the optimal investment strategy and the generator f into (4.2) we obtain the following dynamics of the surplus process

$$dC(t) = C(t-)\,r(t)\,dt + \frac{|L(t)|^2}{\sqrt{|L(t)|^2 - |\theta(t)|^2}}\sqrt{|U(t)|^2 + |V(t)|^2\,\eta(t)}\,dt$$

$$+ \sqrt{\frac{|\theta(t)|^2}{|L(t)|^2 - |\theta(t)|^2}}\sqrt{|U(t)|^2 + |V(t)|^2\,\eta(t)}\;dW(t) - U(t)\,dB(t) - V(t)\,d\tilde{N}(t).$$

For all $0 \le s \le t \le T$, by discounting and taking the expected value (which is finite by (A6) and square integrability of U, V) we arrive at

$$\mathbb{E}\left[e^{-\int_0^t r(u)du}C(t) - e^{-\int_0^s r(u)du}C(s)\,\middle|\,\mathcal{F}_s\right]$$

$$= \mathbb{E}\left[\int_s^t e^{-\int_0^u r(v)dv}\frac{|L(u)|^2}{\sqrt{|L(u)|^2 - |\theta(u)|^2}}\sqrt{|U(u)|^2 + |V(u)|^2\eta(u)}\,du\,\middle|\,\mathcal{F}_s\right] \ge 0.$$

The submartingale property is proved. □

We can prove the results of Theorems 5.1 and 5.2 similarly to Theorem 3.1, Theorem 4.1 and Lemma 3.1.

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