REINSURANCE ARRANGEMENTS MINIMIZING THE RISK-ADJUSTED VALUE OF AN INSURER'S LIABILITY

BY

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ABSTRACT

In this paper, we investigate the problem of purchasing a reinsurance policy that minimizes the risk-adjusted value of an insurer's liability, where the valuation is carried out using a cost-of-capital approach. In order to exclude the moral hazard, we assume that both the insurer and reinsurer are obligated to pay more for larger loss in a typical reinsurance treaty. Moreover, the reinsurance premium principle is assumed to satisfy three axioms: law invariance, risk loading and preserving convex order. The proposed class of premium principles is quite general in the sense that it contains all the widely used premium principles except Esscher principle listed in Young (2004). When capital at risk is measured by the value at risk (VaR) or conditional value at risk (CVaR), we find it is optimal for the insurer to cede two separate layers over the prescribed premium principles. By imposing an additional weak constraint on the premium principle, we further get that the reinsurance in the form of a layer is optimal. Finally, to illustrate the applicability of our results, we derive explicitly the optimal one-layer reinsurance for expected value principle and Wang's premium principle, and show that two-layer reinsurance may be optimal for Dutch premium principle.

KEYWORDS

Capital at risk; Cost of capital; Conditional value at risk; Value at risk; Optimal reinsurance; Wang's premium principle; Dutch premium principle; Layer reinsurance.

1. INTRODUCTION

Reinsurance is an effective risk management tool for the insurers, as it can help them to make the return smooth by absorbing the larger losses and reducing the amount of capital that is required to provide coverage. Thus, it sounds meaningful to investigate the problem of purchasing the optimal reinsurance policies from the perspective of an insurer. Technically, the quest for optimal

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reinsurance is typically converted to solving an optimization problem, which brings new challenges to the academics.

Since the seminal work of Borch (1960), the study of optimal reinsurance problems has attracted great attention and has achieved outstanding results. Just to name a few, Borch (1960) finds that the stop-loss reinsurance is optimal under the criterion of minimizing the variance of the insurer's retained loss when the reinsurance premium is calculated by expected value principle. Arrow (1963), who instead considers to maximize the expected utility of the terminal wealth of a risk-averse investor, obtains the similar result in favor of the stop-loss reinsurance. Their results are extended in many papers by assuming other premium principles in an actuarial context. For instance, Kaluszka (2001) generalizes Borch's result by considering mean-variance premium principles, while Young (1999) and Kaluszka and Okolewski (2008) extend Arrow's result by assuming Wang's premium principle and maximal possible claims principle, respectively. More recently, recommended by Swiss Federal Office of Private Insurance (2006) and Risk Margin Working Group (2009), the risk measures such as value at risk (VaR) and conditional value at risk (CVaR) have been widely used by insurance industry for quantifying risk and setting regulatory capital. Due to the popularity of VaR and CVaR risk measures in practice, Cai and Tan (2007) propose two classes of optimal reinsurance models by minimizing the VaR and CVaR of the insurer's total risk exposure, and derive the optimal retention level for the stop-loss reinsurance under the assumption of expected value premium principle. Their results are generalized in a number of important directions either by relaxing the constraint on the ceded loss functions or by considering other elaborate premium principles or both. See e.g. Cai et al. (2008), Tan et al. (2009), Cheung (2010), Chi and Tan (2011a, 2011b), Guerra and Centeno (2012) and Chi (2012).

While VaR and CVaR risk measures are popular in financial and actuarial risk management, they are inconsistent with common risk perception. More precisely, as pointed out by Fu and Khury (2010), fear is not only of severe losses but also of small and moderate losses. Thus, it is necessary to make a more comprehensive assessment of the insurer's liability. More recently, a cost-of-capital approach, which was introduced by Swiss insurance supervisor (see Swiss Federal Office of Private Insurance (2006)), has been recommended by International Actuarial Association (Risk Margin Working Group (2009)) to evaluate the insurer's liability. Under such a method, the risk-adjusted value of liability, which is also known as a market-consistent price of liability, is composed of two parts: best estimate and risk margin. The best estimate is usually represented by the mean of liability, and the insurer is required to hold additional capital to handle the unexpected loss, the difference between the liability and its mean. In practice, the minimum amount of the capital (*capital at risk*) is usually calculated by VaR or CVaR risk measure. Due to the different return required for the shareholders and that from the capital investment, the risk margin is exactly the cost for holding capital at risk. As a result, the risk-adjusted value of liability reflects not only the tail risk but also the average level of liability. To the best of our knowledge, very few academic papers have been devoted to studying the optimal reinsurance problems under the criterion of minimizing the risk-adjusted value of the insurer's liability. It is the objective of this paper to shed some light on this topic.

In this paper, we propose two classes of optimal reinsurance models by minimizing the risk-adjusted value of an insurer's liability where capital at risk is calculated by VaR or CVaR risk measure. In order to exclude the moral hazard, we assume that both the insurer and reinsurer are obligated to pay more for larger losses in a typical reinsurance treaty. We further assume that the reinsurance premium principle satisfies three axioms: law invariance, risk loading and preserving convex order. This assumption is very flexible in the sense that all the premium principles except Esscher principle listed in Young (2004) belong to this particular class. We find it is optimal for an insurer to cede two separate layers over both the VaR and CVaR risk measures and the prescribed premium principles. By imposing an additional constraint on the premium principle, we further get that the reinsurance in the form of a layer is optimal. This additional constraint is very weak in the sense that the result is applicable to vast majority of premium principles in Young (2004). To illustrate the applicability of our results, we derive the optimal one-layer reinsurance explicitly by assuming expected value principle and Wang's premium principle, and show that two-layer reinsurance may be optimal for Dutch premium principle.

We now summarize the key contributions of this paper. First, we propose the optimal reinsurance models by minimizing the risk-adjusted value of an insurer's liability, where the valuation is carried out using a cost-of-capital approach. It should be pointed out that Asimit et al. (2012) take the same criterion to study the optimal risk transfers within an insurance group consisting of two separate legal entities. However, there are significant differences between their study and ours. Specifically, we consider the optimal risk transfers from an insurer to a reinsurer in contrast to intragroup. Moreover, we study the optimal reinsurance problems from the perspective of an insurer instead of a joint party optimality.

Second, we generalize the VaR and CVaR based optimal reinsurance models in Chi and Tan (2011b) and Chi (2012) from two aspects. One is that our objective function includes a term of the mean of liability in addition to the VaR or CVaR of the insurer's risk exposure. As a consequence, it becomes more challenging to derive the optimal reinsurance in this paper. The other is that our proposed class of reinsurance premium principles is quite general and encompasses the premium principles assumed in Chi and Tan (2011b) and Chi (2012). More importantly, we can reproduce their results by technically setting the cost-of-capital rate to be 1.

Third, we study the effect of cost of capital rate on the optimal reinsurance designs. Specifically, under the expected value principle and Wang's premium principle, we find that as capital at risk becomes costly, which is equivalent to saying that the cost-of-capital rate is larger, the insurer would cede more loss in order to reduce the amount of regulatory capital. Our finding is rather intuitive and is consistent with practice.

The rest of this paper is organized as follows. Section 2 introduces two classes of optimal reinsurance models. Section 3 shows that it is optimal for an insurer to cede two separate layers over both the VaR and CVaR risk measures and the prescribed premium principles, and that the analysis can be further simplified to deriving the optimal one-layer reinsurance if an additional constraint is imposed on the premium principle. Section 4 is applied to solve the optimal reinsurance models with the expected value, Wang's or Dutch premium principle. Finally, some concluding remarks are provided in Section 5. All the proofs are given in the Appendix.

2. Model formulation

Let X be the loss initially assumed by an insurer in a given time period. We assume X is a non-negative random variable on the probability space $(\Omega, \mathcal{G}, \mathbb{P})$ with cumulative distribution function (c.d.f.) $F_X(x) \triangleq \mathbb{P}(X \le x)$ and $\mathbb{E}[X] < \infty$. The key issue of reinsurance problems is to optimally split X into $f(X)$ and $R_f(X)$, where $f(X)$ represents the portion of loss that is ceded to a reinsurer while $R_f(X)$ is the residual loss retained by the insurer (cedent). Thus, $f(x)$ and $R_f(x)$ are known as the insurer's ceded and retained loss functions, respectively.

To exclude the moral hazard, we assume that both the insurer and reinsurer are obligated to pay more for larger loss in a typical reinsurance treaty. In other words, both the ceded and retained loss functions are constrained to be increasing. It is worth noting that "increasing" and "decreasing" in this paper mean "non-decreasing" and "non-increasing", respectively. As a result, the set of admissible ceded loss functions is given by

$$\mathfrak{C} \triangleq \{0 \le f(x) \le x : \text{both } R_f(x) \text{ and } f(x) \text{ are increasing functions}\}. \tag{2.1}$$

The admissible ceded loss function has very nice properties. For instance, as shown in Chi and Tan (2011a), $f(x) \in \mathfrak{C}$ is increasing and Lipschitz continuous, i.e.

$$0 \le f(x_2) - f(x_1) \le x_2 - x_1, \ \forall 0 \le x_1 \le x_2. \tag{2.2}$$

Under a typical reinsurance treaty, when ceding part of loss to a reinsurer, the insurer incurs an additional cost in the form of reinsurance premium which is payable to the reinsurer. Let $\pi(.)$ denote the reinsurance premium principle, a mapping from the set of non-negative random variables χ to the set of (extended) non-negative real numbers $\mathbb{R}_+$. In this paper, we assume that $\pi(.)$ satisfies the following axioms:

(i) **Law invariance**: $\pi(Y)$ depends only on the c.d.f. $F_Y(y)$;

(ii) **Risk loading**: $\pi(Y) \geq \mathbb{E}[Y]$ for all $Y \in \chi$;

(iii) **Preserving convex order**: For $Y, Z \in \chi$, $\pi(Y) \leq \pi(Z)$, if $Y \leq_{cx} Z$, i.e.

$$\mathbb{E}[Y] = \mathbb{E}[Z] \quad \text{and} \quad \mathbb{E}[(Y-d)_+] \leq \mathbb{E}[(Z-d)_+],\ \forall d \in \mathbb{R}, \tag{2.3}$$

where $(x)_+ \triangleq \max\{x, 0\}$, provided that the expectations exist.

The first axiom is an implicit assumption in actuarial science; the second axiom is applied to guarantee the safety of the reinsurer according to Strong Law of Large Number; while the axiom of preserving stop-loss order is preferred by Chi and Tan (2011b) as it is consistent with utility framework for a risk-averse reinsurer, it is not satisfied by many famous premium principles such as variance and standard deviation principles. In order to incorporate more premium principles into our analysis, we relax their constraint by assuming that the premium principle satisfies the third axiom in this paper. As a result, the proposed class of premium principles is quite general in the sense that it encompasses all the widely used premium principles listed in Young (2004) except Esscher principle, which fails to preserve convex order as shown in Van Heerwaarden et al. (1989). Specifically, they are net, expected value, exponential, proportional hazards, principle of equivalent utility, Wang's, Swiss, Dutch, variance and standard deviation principles.

By purchasing a reinsurance policy, the liability or risk exposure of the insurer is now given by the sum of the retained loss and the incurred reinsurance premium instead of X. Using $T_f(X)$ to denote the liability of the insurer, we have

$$T_f(X) = R_f(X) + \pi(f(X)).$$

To evaluate the liability of the insurer, we use a cost-of-capital approach that has been widely adopted by the insurance companies in Europe. Specifically, the best estimate of the insurer's liability is usually represented by $\mathbb{E}[T_f(X)]$. However, it is insufficient to cover the risk exposure such that additional capital should be held against the unexpected loss $T_f(X) - \mathbb{E}[T_f(X)]$. If the unexpected loss is quantified by a risk measure ϕ, capital at risk is given by

$$\phi\big(T_f(X) - \mathbb{E}[T_f(X)]\big).$$

In practice, the return from capital investment is much smaller than that required for shareholders. We denote by $\delta \in (0,1)$ the return difference, which is known as the cost-of-capital rate. The risk margin is now given by the product of cost-of-capital rate and capital at risk. According to Risk Margin Working Group (2009), using $L_f^{\phi}(X)$ to denote the risk-adjusted value or market-consistent price of the insurer's liability, we have

$$L_f^{\phi}(X) = \mathbb{E}[T_f(X)] + \delta\phi\big(T_f(X) - \mathbb{E}[T_f(X)]\big). \tag{2.4}$$

For more details on evaluating the insurer's liability with a cost-of-capital approach, see e.g. Swiss Federal Office of Private Insurance (2006), Risk Margin Working Group (2009), Pelsser (2010), Wüthrich et al. (2010), Asimit et al. (2012), and references therein.

In the insurance industry, VaR and CVaR risk measures have been widely used for quantifying risk and setting regulatory capital. They can be defined formally as follows:

Definition 2.1. *The VaR of a random variable Z at a confidence level* $1-\alpha$ *where* $0<\alpha<1$ *is defined as*

$$VaR_\alpha(Z) \triangleq \inf\{z \in \mathbb{R} : \mathbb{P}(Z > z) \le \alpha\}, \tag{2.5}$$

where $\inf\emptyset = \infty$. *Based upon the definition of VaR, CVaR of Z at a confidence level* $1-\alpha$ *is defined as*

$$CVaR_\alpha(Z) \triangleq \frac{1}{\alpha}\int_0^\alpha VaR_s(Z)\,\mathrm{d}s. \tag{2.6}$$

It follows from the definition of $VaR_\alpha(Z)$ that

$$VaR_\alpha(Z) \le z \iff S_Z(z) \le \alpha \tag{2.7}$$

holds for any $z \in \mathbb{R}$, where $S_Z(z) \triangleq 1 - F_Z(z)$. Therefore, for any $X \in \chi$, we have $VaR_\alpha(X) = 0$ for $\alpha \ge S_X(0)$. For this reason, we assume in this paper that the parameter α satisfies $0 < \alpha < S_X(0)$ to avoid the discussion of trivial cases. Another important property associated with $VaR_\alpha(Z)$ is that for any increasing continuous function $H(x)$, we have (see Theorem 1 in Dhaene et al. (2002))

$$VaR_\alpha(H(Z)) = H(VaR_\alpha(Z)). \tag{2.8}$$

CVaR is also known as the "average value at risk" and "expected shortfall", and a key advantage of CVaR over VaR is that CVaR is a coherent risk measure while VaR is not as it fails to satisfy the subadditivity property. More detailed discussions on the properties of VaR and CVaR risk measures can be found in Artzner et al. (1999) and Föllmer and Schied (2004).

By specifying ϕ by VaR and CVaR risk measures at a confidence level $1-\alpha$, the optimal reinsurance models can be formulated as follows:

$$\textbf{VaR-related optimization}: \; L_{f^*}^{VaR_\alpha}(X) = \min_{f \in \mathfrak{C}} L_f^{VaR_\alpha}(X) \tag{2.9}$$

and

$$\textbf{CVaR-related optimization}: \; L_{f^*}^{CVaR_\alpha}(X) = \min_{f \in \mathfrak{C}} L_f^{CVaR_\alpha}(X), \tag{2.10}$$

where f^* is the resulting optimal ceded loss function among $\mathfrak{C}$.

3. Optimal reinsurance: VaR and CVaR risk measures

In this section, we investigate the solutions to the optimal reinsurance models (2.9) and (2.10). To proceed, we introduce several useful notations. Denote a layer $(a, b]$ of a given risk X by

$$I_{(a,b]}(X) \triangleq \min\{(X-a)_+, b-a\}, \quad 0 \le a \le b, \tag{3.1}$$

then it is trivial that $I_{(a,b]}(x) \in \mathfrak{C}$. Moreover, we define a subset of $\mathfrak{C}$ by

$$\mathfrak{C}_v \triangleq \{I_{(0,a]}(x) + I_{(b, VaR_\alpha(X)]}(x) \text{ or } I_{(0,c]}(x) : 0 \le a \le b \le VaR_\alpha(X) \le c\}. \tag{3.2}$$

Theorem 3.1. *For the VaR-related reinsurance model* (2.9)*, it is optimal for an insurer to cede two separate layers in the sense that for any $f \in \mathfrak{C}$, there exists a ceded loss function $\tilde{f} \in \mathfrak{C}_v$ such that*

$$L_{\tilde{f}}^{VaR_\alpha}(X) \le L_f^{VaR_\alpha}(X).$$

As a result, we have

$$\min_{f \in \mathfrak{C}} L_f^{VaR_\alpha}(X) = \min_{f \in \mathfrak{C}_v} L_f^{VaR_\alpha}(X). \tag{3.3}$$

Another set of the ceded loss functions is introduced by

$$\mathfrak{C}_{cv} \triangleq \left\{I_{(0,a]}(x) + I_{(b,c]}(x) : 0 \le a \le b \le VaR_\alpha(X) \le c\right\}. \tag{3.4}$$

Theorem 3.2. *For the CVaR-related reinsurance model* (2.10)*, it is optimal for an insurer to cede two separate layers in the sense that for any $f \in \mathfrak{C}$, there exists a ceded loss function $\hat{f} \in \mathfrak{C}_{cv}$ such that*

$$L_{\hat{f}}^{CVaR_\alpha}(X) \le L_f^{CVaR_\alpha}(X).$$

Thus, we have

$$\min_{f \in \mathfrak{C}} L_f^{CVaR_\alpha}(X) = \min_{f \in \mathfrak{C}_{cv}} L_f^{CVaR_\alpha}(X). \tag{3.5}$$

By Theorems 3.1 and 3.2, the study of infinite-dimensional optimization problems (2.9) and (2.10) is simplified to minimizing the functions of three variables. However, it is generally not an easy task to solve an optimization problem with three variables by either mathematical analysis or numerical methods. In the following corollary, we can further reduce the dimension of the optimization problems by imposing an additional constraint on the premium principle $\pi(.)$.

Corollary 3.3. *Let $\pi_\delta(X) \triangleq \pi(X) - (1-\delta)\mathbb{E}[X]$. If the premium principle $\pi(.)$ further satisfies the axiom of translation invariance, i.e.*

$$\pi(Y+c) = \pi(Y) + c \quad \text{for all} \quad c \geq 0 \;\; \text{and} \;\; Y \in \chi$$

or $\pi_\delta(.)$ preserves first-order stochastic dominance (FOSD), i.e.

$$\pi_\delta(Y) \leq \pi_\delta(Z), \quad \text{if} \quad S_Y(t) \leq S_Z(t), \;\; \forall \;\, t \geq 0,$$

we have

$$\min_{f \in \mathfrak{C}} L_f^{VaR_\alpha}(X) = \min_{f \in \mathfrak{C}_v'} L_f^{VaR_\alpha}(X) \;\; \text{and} \;\; \min_{f \in \mathfrak{C}} L_f^{CVaR_\alpha}(X) = \min_{f \in \mathfrak{C}_{cv}'} L_f^{CVaR_\alpha}(X), \quad (3.6)$$

where

$$\mathfrak{C}_v' \triangleq \{I_{(a, VaR_\alpha(X)]}(x) : 0 \leq a \leq VaR_\alpha(X)\} \subset \mathfrak{C}_v$$

and

$$\mathfrak{C}_{cv}' \triangleq \{I_{(b,c]}(x) : 0 \leq b \leq VaR_\alpha(X) \leq c\} \subset \mathfrak{C}_{cv}.$$

Remark 3.1. *Among the eleven widely used premium principles listed in Young (2004), only expected value, Dutch and Swiss premium principles fail to satisfy the axiom of translation invariance. Further, if $\pi(.)$ follows expected value principle or it is Dutch principle with the loading factor less than δ, it can be shown that $\pi_\delta(.)$ preserves FOSD. Thus, the result of the above corollary is applicable to vast majority of the premium principles in Young (2004). For this reason, we say that the additional constraint on the premium principle in the above corollary is rather weak.*

Further, as δ is technically set to be 1*, our framework recovers the VaR and CVaR based optimal reinsurance models studied in Chi and Tan (2011b) and Chi (2012), i.e.*

$$\min_{f \in \mathfrak{C}} VaR_\alpha(T_f(X)) \quad \text{and} \quad \min_{f \in \mathfrak{C}} CVaR_\alpha(T_f(X)). \quad (3.7)$$

They assume the reinsurance premium is calculated by a principle preserving stop-loss order or a variance related principle. It is easy to show that their assumed premium principles satisfy the conditions of the above corollary for $\delta = 1$, and hence we can reproduce the results of Theorems 3.1 and 3.2 in Chi and Tan (2011b) and Theorems 3.1 and 3.3 in Chi (2012).

In the above corollary, one-layer reinsurance is shown to be optimal over both the VaR and CVaR risk measures and the assumed premium principles. Consequently, the analysis of the reinsurance models (2.9) and (2.10) is simplified to solving one-parameter and two-parameter minimization problems, respectively. It seems difficult to make a further simplification without the specification of the premium principle $\pi(.)$. To illustrate the applicability of the above results, we derive explicitly the optimal one-layer reinsurance under expected value principle and Wang's premium principle and discuss the possibility of the optimality of two-layer reinsurance under Dutch premium principle in the next section.

4. Examples

4.1. Expected value principle

In this subsection, we study the optimal reinsurance models (2.9) and (2.10) with expected value premium principle in the form of

$$\pi(X) = (1 + \rho)\mathbb{E}[X], \tag{4.1}$$

where $\rho > 0$ represents the safety loading factor. While expected value principle is not translation invariant, $\pi_\delta(.)$ defined in Corollary 3.3 is now given by

$$\pi_\delta(X) = (\rho + \delta)\mathbb{E}[X]$$

and obviously it preserves FOSD. As a result, Corollary 3.3 is applicable to expected value premium principle, and hence the study of optimal reinsurance models (2.9) and (2.10) is simplified to solving one-parameter and two-parameter minimization problems in (3.6), respectively. Now, we obtain the main result of this subsection in the following proposition.

Proposition 4.1. *When reinsurance premium is calculated by expected value principle* (4.1)*, the ceded loss function f_v^* that solves the optimal reinsurance model* (2.9) *is given by*

$$f_v^*(x) = \begin{cases} I_{(d^*, VaR_\alpha(X)]}(x), & d^* < VaR_\alpha(X); \\ 0, & otherwise, \end{cases} \tag{4.2}$$

where $d^* \triangleq VaR_{\frac{\delta}{\delta+\rho}}(X)$. Moreover, the ceded loss function f_{cv}^* that solves the optimal reinsurance model (2.10) *is given by*

$$f_{cv}^*(x) = \begin{cases} (x - d^*)_+, & \alpha < \frac{\delta}{\delta+\rho}; \\ 0, & otherwise. \end{cases} \tag{4.3}$$

Remark 4.1. *We know from the above proposition that under the expected value principle, the optimal retention level d^* is an increasing function of ρ/δ. In other words, if reinsurance becomes more costly or the cost of capital becomes lower, the insurer would cede less loss. The finding is rather intuitive and is consistent with practice. Moreover, the above proposition reproduces the results of Theorems 3.2 and 4.1 in Chi and Tan (2011a) when δ is technically set to be* 1.

4.2. Wang's premium principle

In this subsection, we assume that the reinsurance premium is calculated by Wang's principle, which was introduced by Wang et al. (1997) in the form of

$$\pi(X) = \int_0^\infty g(S_X(t))\mathrm{d}t, \tag{4.4}$$

where the *distortion function* $g : [0,1] \to [0,1]$ is increasing and concave with

$$g(x) \geq x, \quad g(0) = 0 \quad \text{and} \quad g(1) = 1.$$

Wang's premium principle has been used to study the optimal reinsurance problems by many papers such as Young (1999), Kaluszka (2005), Cheung (2010) and Chi and Tan (2011b).

Proposition 4.2. *Denote the right-derivative of $g(x)$ by $g'_+(x)$. For Wang's premium principle* (4.4)*, $\pi_\delta(.)$ preserves FOSD if and only if $g'_+(1-) \geq 1-\delta$.*

Example 4.1. *The proportional hazards premium principle introduced by Wang (1995) is a special case of Wang's premium principle with $g(x) = x^\varepsilon$ for $0 < \varepsilon \leq 1$. The above proposition shows that $\pi_\delta(.)$ for the proportional hazards premium principle preserves FOSD if and only if $\varepsilon \geq 1-\delta$.*

While the above proposition shows that not all the $\pi_\delta(.)$ preserves FOSD, the result of Corollary 3.3 is still applicable to Wang's premium principle as it is translation invariant according to Young (2004). Thus, we can obtain the solutions to optimal reinsurance models (2.9) and (2.10) with Wang's premium principle by solving one-parameter and two-parameter minimization problems in (3.6), respectively.

To proceed, we introduce several useful notations. Let

$$w(x) \triangleq g(x) - (1-\delta)x - \delta, \; x \in [0,1], \tag{4.5}$$

then $w(x)$ is a concave function with $w(0) = -\delta < 0$ and $w(1) = 0$, and hence $w'_+(x)$ is decreasing and right-continuous. If $w'_+(1-) \geq 0$, i.e. $g'_+(1-) \geq 1-\delta$, we have $w(x) \leq 0$ for any $x \in [0,1]$. Otherwise, if $g'_+(1-) < 1-\delta$, the equation $w(x) = 0$, $x \in (0,1)$ has a unique solution, which is denoted by ξ.

Further, let

$$\beta \triangleq \begin{cases} VaR_{\xi\vee\alpha}(X), & g'_{+}(1-) < 1-\delta; \\ 0, & \text{otherwise}, \end{cases} \tag{4.6}$$

where $x \vee y \triangleq \max\{x, y\}$, and denote

$$\gamma \triangleq \inf\left\{ VaR_{\alpha}(X) \leq c < ess\,sup\,X : \frac{g(S_X(c))}{S_X(c)} \geq 1-\delta+\delta/\alpha \right\}, \tag{4.7}$$

where the *essential supremum* of random variable X is defined by

$$ess\,sup\,X \triangleq \sup\{x \in \mathbb{R} : F_X(x) < 1\}.$$

Proposition 4.3. *β is decreasing in δ, while γ is an increasing function of δ.*

Based upon the previous preparation, we obtain the main result of this subsection in the following proposition.

Proposition 4.4. *When reinsurance premium is calculated according to Wang's principle* (4.4)*, the ceded loss function $f_v^*(x)$ that solves the optimal reinsurance model* (2.9) *is given by*

$$f_v^*(x) = I_{(\beta,\, VaR_\alpha(X)]}(x). \tag{4.8}$$

Moreover, the ceded loss function $f_{cv}^*(x)$ that solves the optimal reinsurance model (2.10) *is given by*

$$f_{cv}^*(x) = I_{(\beta,\gamma]}(x). \tag{4.9}$$

Remark 4.2. *By Propositions 4.3 and 4.4, we know that both the optimal ceded loss functions $f_v^*(x)$ and $f_{cv}^*(x)$ are increasing in δ. Not surprisingly, the insurer would cede more loss in order to reduce the amount of regulatory capital when the capital becomes more costly. Especially, when δ is technically set to be* 1*, Proposition 4.4 provides the solutions to VaR and CVaR based optimal reinsurance models with Wang's premium principle in* (3.7)*. In this case, the optimal retention level becomes zero, i.e. $\beta = 0$. In other words, the insurer would reinsure against all the small and moderate losses, but suffer tail risk as it is too costly to be reinsured under Wang's principle. However, the cost-of-capital rate in practice is set to be relatively small, i.e. $\delta = 6\% \ll 1$, then the retention level of optimal layer reinsurance may be positive, which is equivalent to saying that the insurer may be optimal to retain part of small loss in addition to the severe loss. For instance, when reinsurance premium is calculated by proportional hazards* *premium principle, the optimal retention level β is positive for $\varepsilon < 1 - \delta$ and the loss X with continuous and strictly increasing distribution function. As pointed out by Wang (1995), ε represents the risk-averse attitude of the reinsurer, and the smaller ε, the more costly the reinsurance. Thus, the insurer would retain more loss for the smaller ε.*

4.3. Dutch premium principle

While Corollary 3.3 implies that the reinsurance in the form of a layer is optimal over the vast majority of widely used premium principles in Young (2004), we will provide an exception by studying the optimal reinsurance model (2.9) with Dutch premium principle in this subsection.

The Dutch premium principle, which was introduced by Van Heerwaarden and Kaas (1992), is given by

$$\pi(X) = \mathbb{E}[X] + \theta\mathbb{E}[(X - \lambda\mathbb{E}[X])_+], \; 0 < \theta \leq 1 \;\text{ and }\; \lambda \geq 1, \tag{4.10}$$

where θ is the loading factor and $\lambda\mathbb{E}[X]$ represents a threshold amount of loss. If $\lambda = 1$, Dutch principle is translation invariant. Moreover, $\pi_\delta(.)$ in Corollary 3.3 is given by

$$\pi_\delta(X) = \delta(\mathbb{E}[X] + \frac{\theta}{\delta}\mathbb{E}[(X - \lambda\mathbb{E}[X])_+]).$$

As shown in Van Heerwaarden and Kaas (1992), Dutch premium principle preserves FOSD. So does $\pi_\delta(.)$ if $\delta \geq \theta$. Thus, when $\lambda = 1$ or $\delta \geq \theta$, Corollary 3.3 implies that the reinsurance in the form of a layer is optimal.

Proposition 4.5. *When reinsurance premium is calculated by Dutch principle* (4.10), *if $\delta \geq \theta$ or $\lambda \geq VaR_\alpha(X) / \int_0^{VaR_\alpha(X)} S_X(t)\,\mathrm{d}t$, the ceded loss function that solves the optimal reinsurance model* (2.9) *is given by*

$$f_v^*(x) = I_{(0,\, VaR_\alpha(X)]}(x).$$

Moreover, if $\lambda = 1$, we have

$$f_v^*(x) = \begin{cases} I_{(0,\, VaR_\alpha(X)]}(x), & \delta/\theta \geq S_X\left(\int_0^{VaR_\alpha(X)} S_X(t)\,\mathrm{d}t\right); \\ I_{(b^*,\, VaR_\alpha(X)]}(x), & \textit{otherwise}, \end{cases} \tag{4.11}$$

where

$$b^* \triangleq \sup\left\{0 \leq b \leq VaR_\alpha(X) : b + \int_b^{VaR_\alpha(X)} S_X(t)\,\mathrm{d}t \leq VaR_{\delta/\theta}(X)\right\}. \tag{4.12}$$

Now, the residual task of this subsection is to study the reinsurance model (2.9) under Dutch premium principle with

$$\delta < \theta \quad \text{and} \quad 1 < \lambda < VaR_\alpha(X) \Big/ \int_0^{VaR_\alpha(X)} S_X(t)\mathrm{d}t. \tag{4.13}$$

For the sake of simplicity, we make the following assumption:

Assumption 4.1. *The survival distribution function $S_X(t)$ is strictly decreasing and continuous on* $[0, \infty)$.

Under the above assumption and the condition (4.13), the equation

$$\lambda \int_0^a S_X(t)\mathrm{d}t = a, \quad 0 < a < VaR_\alpha(X) \tag{4.14}$$

has a unique solution, which is denoted by a_0. We define a set composed of two segments by

$$\begin{aligned} \mathcal{D} &\triangleq \{(a,b) : G(a,b) = 0, a_0 \le a \le b \le VaR_\alpha(X)\} \\ &\bigcup \{(a, VaR_\alpha(X)) : a_0 < a < VaR_\alpha(X)\}, \end{aligned} \tag{4.15}$$

where

$$G(a,b) \triangleq \lambda\left(\int_0^a S_X(t)\mathrm{d}t + \int_b^{VaR_\alpha(X)} S_X(t)\mathrm{d}t\right) - a, \ 0 \le a \le b \le VaR_\alpha(X). \tag{4.16}$$

Proposition 4.6. *Under Assumption 4.1, when reinsurance premium is calculated by Dutch principle with the constraint* (4.13)*, we have*

$$\min_{f \in \mathfrak{C}} L_f^{VaR_\alpha}(X) = \min_{f \in \mathfrak{C}_v} L_f^{VaR_\alpha}(X) = \min_{(a,b) \in \mathcal{D}} H(a,b) + (1-\delta)\,\mathbb{E}[X],$$

where

$$\begin{aligned} H(a,b) \triangleq \frac{\delta}{\lambda}(G(a,b) + a) &+ \theta \int_b^{VaR_\alpha(X)} S_X(t)\mathrm{d}t \\ &+ \theta \int_{G(a,b)+a}^a S_X(t)\mathrm{d}t + \delta(b-a) \end{aligned}$$

for $a_0 \le a \le b \le VaR_\alpha(X)$.

Under our model assumptions, the above proposition implies that the analysis of optimal reinsurance model (2.9) can be simplified to minimizing a function of two variables over two segments, which is easy to be solved numerically. In contrast to the results of Corollary 3.3 and Proposition 4.5, we show in the following example that two-layer reinsurance could be optimal.

Example 4.2. *The loss X is assumed to follow the Pareto distribution with probability density function*

$$p(x) = \frac{2}{(x+1)^3}, \quad x > 0,$$

then we have $S_X(t) = 1/(t+1)^2$. *Further, we let*

$$\alpha = 5\%, \quad \delta = 0.1, \quad \theta = 0.9, \quad \textit{and} \quad \lambda = 1.5,$$

then the conditions of Proposition 4.6 are satisfied and

$$VaR_\alpha(X) = 3.472, \;\; a_0 = \lambda - 1 = 0.5.$$

By simple numerical calculation, we obtain that the minimum of $H(a,b)$ *over* $\mathcal{D}$ *is attainable at* $(a,b) = (0.901, 1.854)$. *Consequently, in this example, the ceded loss function that solves the optimal reinsurance model* (2.9) *is given by*

$$f_v^*(x) = \min\{x, 0.901\} + I_{(1.854,\,3.472]}(x).$$

5. Concluding remarks

In this paper, we have introduced two classes of optimal reinsurance models by minimizing the risk-adjusted value of an insurer's liability, where the valuation is carried out using a cost-of-capital approach. In order to exclude the moral hazard, we have assumed that both the insurer and reinsurer are obligated to pay more for larger loss in a typical reinsurance treaty. When reinsurance premium principle satisfies the axioms of law invariance, risk loading and preserving convex order, we have shown that it is optimal for the insurer to cede two separate layers. Further, we have found that the reinsurance in the form of a layer is optimal when an additional constraint is imposed on premium principle, whereas the constraint is rather weak in the sense that it excludes very few widely used premium principles in Young (2004). Under expected value principle and Wang's premium principle, the optimal one-layer reinsurance has been derived explicitly and the effect of cost of capital on the optimal reinsurance designs has been analyzed. Finally, we have discussed the optimality of two-layer reinsurance under the reinsurance model (2.9) with Dutch premium principle.

Appendix A. The proofs

We first introduce a useful definition:

Definition A.1. *A function $f_1(x)$ is said to up-cross a function $f_2(x)$, if there exists an $x_0 \in \mathbb{R}$ such that*

$$\begin{cases} f_1(x) \le f_2(x), & x < x_0; \\ f_1(x) \ge f_2(x), & x > x_0. \end{cases}$$

We rewrite Lemma 3 in Ohlin (1969) as the following lemma, which is a very useful criterion of convex order.

Lemma A.1. *Let Y be a random variable and $\{f_i(y); i = 1,2\}$ two increasing functions with $\mathbb{E}[f_1(Y)] = \mathbb{E}[f_2(Y)]$. If $f_1(y)$ up-crosses $f_2(y)$, we have $f_2(Y) \le_{cx} f_1(Y)$.*

We are now ready to prove Theorem 3.1.

Proof of Theorem 3.1.

It is well-known that VaR risk measure is translation invariant, then it follows from (2.2), (2.4) and (2.8) that

$$\begin{aligned} L_f^{VaR_\alpha}(X) &= (1-\delta)\mathbb{E}[R_f(X)] + \pi(f(X)) + \delta VaR_\alpha(R_f(X)) \\ &= (1-\delta)\mathbb{E}[X] - (1-\delta)\mathbb{E}[f(X)] + \pi(f(X)) + \delta R_f(VaR_\alpha(X)) \\ &= (1-\delta)\mathbb{E}[X] + \delta VaR_\alpha(X) + Q_v(f) \end{aligned} \tag{A.1}$$

for any $f \in \mathfrak{C}$, where

$$Q_v(f) \triangleq \pi(f(X)) - (1-\delta)\,\mathbb{E}[f(X)] - \delta f(VaR_\alpha(X)). \tag{A.2}$$

Thus, the optimal reinsurance model (2.9) is equivalent to

$$\min_{f \in \mathfrak{C}} Q_v(f). \tag{A.3}$$

For any $f \in \mathfrak{C}$, denote

$$h(x) \triangleq I_{(0,\, f(VaR_\alpha(X))]}(x) = \min\{x, f(VaR_\alpha(X))\}, \quad x \ge 0.$$

If $\mathbb{E}[h(X)] < \mathbb{E}[f(X)]$, there exists an a satisfying $a > f(VaR_\alpha(X))$ such that

$$\mathbb{E}[I_{(0,\,a]}(X)] = \mathbb{E}[f(X)].$$

In this case, due to the increasing and Lipschitz-continuous properties of the ceded loss functions as stated in (2.2), it is easy to get that $I_{(0,a]}(VaR_\alpha(X)) = \min\{a, VaR_\alpha(X)\} \geq f(VaR_\alpha(X))$ and $f(x)$ up-crosses $I_{(0,a]}(x)$, then Lemma A.1 implies $I_{(0,a]}(X) \leq_{cx} f(X)$. Consequently, note that $\pi(.)$ preserves convex order, then we have $Q_v(f) \geq Q_v(I_{(0,a]})$.

Otherwise, if $\mathbb{E}[h(X)] \geq \mathbb{E}[f(X)]$, it follows from (2.2) that there exists a $b \in [0, f(VaR_\alpha(X))]$ such that $\mathbb{E}[f(X)] = \mathbb{E}[f_b(X)]$, where

$$f_b(x) \triangleq I_{(0,b]}(x) + I_{(VaR_\alpha(X)-f(VaR_\alpha(X))+b,\,VaR_\alpha(X)]}(x), \quad x \geq 0.$$

In this case, $f_b(VaR_\alpha(X)) = f(VaR_\alpha(X))$ and $f(x)$ up-crosses $f_b(x)$ such that $f_b(X) \leq_{cx} f(X)$ according to Lemma A.1, then (A.2) implies $Q_v(f) \geq Q_v(f_b)$.

It is trivial that $I_{(0,a]}(x), f_b(x) \in \mathfrak{C}_v$, where $\mathfrak{C}_v$ is given in (3.2). Thus, collecting all the above arguments, together with (A.1), yields

$$\min_{f\in\mathfrak{C}} L_f^{VaR_\alpha}(X) \;\geq\; \min_{f\in\mathfrak{C}_v} L_f^{VaR_\alpha}(X).$$

Further, the above inequality is exactly an identity as $\mathfrak{C}_v \subset \mathfrak{C}$. The proof is thus complete.

Proof of Theorem 3.2.

As pointed out by Föllmer and Schied (2004), CVaR risk measure is translation invariant, then it follows from (2.2), (2.4) and (2.8) that

$$\begin{aligned} L_f^{CVaR_\alpha}(X) &= \mathbb{E}[R_f(X)] + \pi(f(X)) + \delta(CVaR_\alpha(R_f(X)) - \mathbb{E}[R_f(X)]) \\ &= (1-\delta)\,\mathbb{E}[X] - (1-\delta)\,\mathbb{E}[f(X)] + \pi(f(X)) + \frac{\delta}{\alpha}\int_0^\alpha VaR_s(R_f(X))\mathrm{d}s \\ &= (1-\delta)\,\mathbb{E}[X] + \delta CVaR_\alpha(X) + Q_{cv}(f) \end{aligned} \tag{A.4}$$

for any $f \in \mathfrak{C}$, where

$$Q_{cv}(f) \triangleq \pi(f(X)) - (1-\delta)\,\mathbb{E}[f(X)] - \delta\, CVaR_\alpha(f(X)). \tag{A.5}$$

Thus, the optimal reinsurance problem (2.10) is equivalent to

$$\min_{f\in\mathfrak{C}} Q_{cv}(f).$$

The following proof is a slight modification to that of Theorem 3.2 in Chi and Tan (2011b). Specifically, for any $f \in \mathfrak{C}$, recall that $f(x)$ is increasing and Lipschitz continuous, then there exists a c satisfying $c \geq VaR_\alpha(X)$ such that

$$CVaR_\alpha(f(X)) = CVaR_\alpha(f_1(X)),$$

where

$$f_1(x) \triangleq \begin{cases} f(x), & 0 \le x \le VaR_\alpha(X); \\ f(VaR_\alpha(X)) + I_{(VaR_\alpha(X),\,c]}(x), & x > VaR_\alpha(X). \end{cases}$$

Moreover, it is easy to verify that $f(x)$ up-crosses $f_1(x)$, and

$$\begin{aligned} \mathbb{E}[f(X)] &= \mathbb{E}[VaR_U(f(X))] = \int_0^\alpha VaR_u(f(X))\,\mathrm{d}u + \int_\alpha^1 VaR_u(f(X))\,\mathrm{d}u \\ &= \alpha CVaR_\alpha(f(X)) + \int_\alpha^1 f(VaR_u(X))\,\mathrm{d}u \\ &= \alpha CVaR_\alpha(f_1(X)) + \int_\alpha^1 f_1(VaR_u(X))\,\mathrm{d}u = \mathbb{E}[f_1(X)], \end{aligned}$$

where U is uniformly distributed on $[0,1]$, and the first equality is derived by the fact that X and $VaR_U(X)$ have the same distribution. Thus, it follows from Lemma A.1 that $f_1(X) \le_{cx} f(X)$.

Further, building upon $f_1(x)$, we can construct a ceded loss function in the form of

$$f_2(x) \triangleq I_{(0,\,a]}(x) + I_{(VaR_\alpha(X) - f(VaR_\alpha(X)) + a,\,c]}(x), \quad x \ge 0$$

for some $0 \le a \le f(VaR_\alpha(X))$ such that $\mathbb{E}[f_2(X)] = \mathbb{E}[f_1(X)]$. Since $f_2(x) = f_1(x)$ for any $x \ge VaR_\alpha(X)$, we have $CVaR_\alpha(f_2(X)) = CVaR_\alpha(f_1(X))$. Moreover, it is easy to verify that $f_1(x)$ up-crosses $f_2(x)$, then Lemma A.1 implies $f_2(X) \le_{cx} f_1(X)$. Consequently, it follows from (A.5) that

$$Q_{cv}(f) \ge Q_{cv}(f_1) \ge Q_{cv}(f_2),$$

which results in

$$\min_{f \in \mathfrak{C}} L_f^{CVaR_\alpha}(X) \ge \min_{f \in \mathfrak{C}_{cv}} L_f^{CVaR_\alpha}(X),$$

where $\mathfrak{C}_{cv}$ is defined in (3.4). The above inequality is exactly an identity as $\mathfrak{C}_{cv} \subset \mathfrak{C}$. The proof is therefore complete.

Before giving the proof of Corollary 3.3, we introduce a useful lemma.

Lemma A.2. $I_{(0,c]}(X) - \mathbb{E}[I_{(0,c]}(X)]$ *is increasing in the sense of convex order.*

Proof. Since $\mathbb{E}[(X-d)_+] = \int_d^\infty S_X(t)\,\mathrm{d}t$ for any $d \in \mathbb{R}$, it is easy to get that

$$\mathbb{E}[(I_{(0,c]}(X)-\mathbb{E}[I_{(0,c]}(X)]-d)_+]=\begin{cases}0, & d\geq\int_0^c F_X(y)\,\mathrm{d}y;\\ \int_{d+\int_0^c S_X(y)\mathrm{d}y}^{c} S_X(x)\,\mathrm{d}x, & \text{otherwise.}\end{cases}$$

Moreover, the derivative of $\int_{d+\int_0^c S_X(y)\mathrm{d}y}^{c} S_X(x)\mathrm{d}x$ with respect to (w.r.t.) c is given by

$$S_X(c)\left(1-S_X\left(d+\int_0^c S_X(y)\mathrm{d}y\right)\right)\geq 0, \quad a.s.$$

then $\mathbb{E}[(I_{(0,c]}(X)-\mathbb{E}[I_{(0,c]}(X)]-d)_+]$ is increasing in c for any $d\in\mathbb{R}$. Consequently, it follows from (2.3) that $I_{(0,c]}(X)-\mathbb{E}[I_{(0,c]}(X)]$ is increasing in the sense of convex order. The proof is therefore complete. □

Now, we have enough preparation to carry out the proof of Corollary 3.3.

Proof of Corollary 3.3.

First, we assume $\pi(.)$ is translation invariant. For any $c\geq VaR_\alpha(X)$, (A.2) implies

$$\begin{aligned}Q_v(I_{(0,c]}) &= \pi(I_{(0,c]}(X)-\mathbb{E}[I_{(0,c]}(X)]+\mathbb{E}[X])+\delta\mathbb{E}[I_{(0,c]}(X)]\\ &\quad -\delta VaR_\alpha(X)-\mathbb{E}[X].\end{aligned}$$

It is trivial that $\mathbb{E}[I_{(0,c]}(X)]$ is increasing in c. So is $\pi(I_{(0,c]}(X)-\mathbb{E}[I_{(0,c]}(X)]+\mathbb{E}[X])$ according to Lemma A.2 and the convex order preserving property of $\pi(.)$. Consequently, we have $Q_v(I_{(0,c]})\geq Q_v(I_{(0,VaR_\alpha(X)]})$.

If the ceded loss function in $\mathfrak{C}_v$ is $f(x)=I_{(0,a]}(x)+I_{(b,VaR_\alpha(X)]}(x)$ for some $0\leq a\leq b\leq VaR_\alpha(X)$, we can get the result by a slight modification to the proof of Theorem 3.1 in Chi (2012). Specifically, $Q_v(f)$ in (A.2) can now be rewritten by

$$\begin{aligned}Q_v(f) &= \pi(f(X)-f(VaR_\alpha(X))+VaR_\alpha(X))-(1-\delta)\,\mathbb{E}[f(X)-f(VaR_\alpha(X))]\\ &\quad -VaR_\alpha(X).\end{aligned}$$

It is easy to find an m satisfying

$$VaR_\alpha(X)-b=f(VaR_\alpha(X))-a\leq m\leq f(VaR_\alpha(X))\leq VaR_\alpha(X)$$

such that

$$\mathbb{E}[I_{(VaR_\alpha(X)-m,\,VaR_\alpha(X)]}(X)-m]=\mathbb{E}[f(X)-f(VaR_\alpha(X))].$$

Furthermore, it follows from Definition A.1 that $f(x) - f(VaR_\alpha(X))$ up-crosses the function $I_{(VaR_\alpha(X)-m,\, VaR_\alpha(X)]}(x) - m$, then Lemma A.1 implies

$$I_{(VaR_\alpha(X)-m,\, VaR_\alpha(X)]}(X) - m \leq_{cx} f(X) - f(VaR_\alpha(X)).$$

Recall that $\pi(.)$ preserves convex order, then we have $Q_v(f) \geq Q_v(I_{(VaR_\alpha(X)-m,\, VaR_\alpha(X)]})$.

Similarly, for the ceded loss function $f(x) = I_{(0,a]}(x) + I_{(b,c]}(x)$ with $c \geq VaR_\alpha(X)$, we have

$$I_{(VaR_\alpha(X)-m,\, c]}(X) - m \leq_{cx} f(X) - f(VaR_\alpha(X)).$$

Moreover, we have

$$I_{(VaR_\alpha(X)-m,\, c]}(x) - m = I_{(VaR_\alpha(X),\, c]}(x) = f(x) - f(VaR_\alpha(X)), \quad \forall x \geq VaR_\alpha(X).$$

Consequently, since $Q_{cv}(f)$ in (A.5) can be rewritten by

$$\begin{aligned} Q_{cv}(f) \;=\; & \pi(f(X) - f(VaR_\alpha(X)) + VaR_\alpha(X)) - (1-\delta)\,\mathbb{E}[f(X) - f(VaR_\alpha(X))] \\ & - \frac{\delta}{\alpha}\int_0^\alpha f(VaR_s(X)) - f(VaR_\alpha(X))\,\mathrm{d}s - VaR_\alpha(X), \end{aligned}$$

then we have $Q_{cv}(f) \geq Q_{cv}(I_{(VaR_\alpha(X)-m,\, c]})$.

Collecting all the above arguments, together with (A.1), (A.4) and Theorems 3.1 and 3.2, leads to (3.6) for translation invariant $\pi(.)$.

Next, we prove (3.6) for the case that $\pi_\delta(.)$ preserves FOSD. Specifically, in this case, we have

$$Q_v(I_{(0,c]}) = \pi_\delta(I_{(0,c]}(X)) - \delta VaR_\alpha(X), \;\; c \geq VaR_\alpha(X),$$

which is increasing in c. Thus, (A.1) implies

$$L^{VaR_\alpha}_{I_{(0,c]}}(X) \;\geq\; L^{VaR_\alpha}_{I_{(0,VaR_\alpha(X)]}}(X), \quad \forall c \geq VaR_\alpha(X).$$

Furthermore, if the ceded loss function in $\mathfrak{C}_v$ is $f(x) = I_{(0,a]}(x) + I_{(b,\, VaR_\alpha(X)]}(x)$ for some $0 \leq a \leq b \leq VaR_\alpha(X)$, let $k_v(x) \triangleq I_{(b-a,\, VaR_\alpha(X)]}(x)$, $x \geq 0$, then we have $k_v(VaR_\alpha(X)) = f(VaR_\alpha(X))$ and $k_v(x) \leq f(x)$ for any $x \geq 0$. Consequently, we have

$$Q_v(f) \;=\; \pi_\delta(f(X)) - \delta f(VaR_\alpha(X)) \;\geq\; \pi_\delta(k_v(X)) - \delta k_v(VaR_\alpha(X)) = Q_v(k_v).$$

Since $\mathfrak{C}'_v \subset \mathfrak{C}_v$, we get the first equation in (3.6) by using (A.1) and Theorem 3.1.

Similarly, if the ceded loss function in $\mathfrak{C}_{cv}$ is $f(x) = I_{(0,a]}(x) + I_{(b,c]}(x)$ for some $0 \le a \le b \le VaR_\alpha(X) \le c$, define $k(x) \triangleq I_{(b-a,c]}(x)$, then we have $k(x) \le f(x)$, $\forall x \ge 0$ and the inequality is exactly an identity for $x \ge VaR_\alpha(X)$. Moreover, (A.5) implies

$$Q_{cv}(f) = \pi_\delta(f(X)) - \delta CVaR_\alpha(f(X)) \ge \pi_\delta(k(X)) - \delta CVaR_\alpha(k(X)) = Q_{cv}(k).$$

As a consequence, note that $\mathfrak{C}'_{cv} \subset \mathfrak{C}_{cv}$, then the second equation in (3.6) can be obtained by using (A.4) and Theorem 3.2. The proof is thus complete.

Proof of Proposition 4.1.

When reinsurance premium is calculated by expected value principle (4.1), it follows from (A.2) that

$$Q_v(I_{(d, VaR_\alpha(X)]}) = (\rho+\delta)\int_d^{VaR_\alpha(X)} S_X(t)\mathrm{d}t + \delta d - \delta VaR_\alpha(X)$$

for any $0 \le d \le VaR_\alpha(X)$, then we have

$$\frac{\partial Q_v(I_{(d, VaR_\alpha(X)]})}{\partial d} = (\delta+\rho)\left(\frac{\delta}{\delta+\rho} - S_X(d)\right), \ a.s.$$

The above equation, together with (2.7), implies that the minimum of $Q_v(I_{(d, VaR_\alpha(X)]})$ over $[0, VaR_\alpha(X)]$ is attainable at $d = \min\{VaR_{\frac{\delta}{\delta+\rho}}(X), VaR_\alpha(X)\} = \min\{d^*, VaR_\alpha(X)\}$. As a consequence, recall that $\pi_\delta(.)$ for expected value principle preserves FOSD, then using (3.6) and (A.1), we get that the ceded loss function f_v^* defined in (4.2) is optimal under the reinsurance model (2.9).

Moreover, for any $0 \le b \le VaR_\alpha(X) \le c$, we have

$$\begin{aligned} CVaR_\alpha(I_{(b,c]}(X)) &= \frac{1}{\alpha}\int_0^\alpha (VaR_s(X) - b)_+ - (VaR_s(X) - c)_+ \,\mathrm{d}s \\ &= \frac{1}{\alpha}\int_0^\alpha (VaR_s(X) - b)\mathrm{d}s - \frac{1}{\alpha}\int_0^1 (VaR_s(X) - c)_+ \,\mathrm{d}s \\ &= CVaR_\alpha(X) - b - \frac{1}{\alpha}\mathbb{E}\big[(X-c)_+\big], \end{aligned} \tag{A.6}$$

where the last equality is implied by the fact that X and $VaR_U(X)$ have the same distribution, then (A.5) implies

$$\begin{aligned} Q_{cv}(I_{(b,c]}) &= (\delta+\rho)\,\mathbb{E}[I_{(b,c]}(X)] - \delta CVaR_\alpha(I_{(b,c]}(X)) \\ &= (\delta+\rho)\,\mathbb{E}[(X-b)_+] + \delta b + \left(\frac{\delta}{\alpha} - (\delta+\rho)\right)\mathbb{E}[(X-c)_+] - \delta CVaR_\alpha(X). \end{aligned}$$

Similarly, given a $c \geq VaR_\alpha(X)$, the minimum of $Q_{cv}(I_{(b,c]})$ over $[0, VaR_\alpha(X)]$ is attainable at $b = \min\{d^*, VaR_\alpha(X)\}$. On the other hand, fixed a $b \in [0, VaR_\alpha(X)]$, it follows from the above equation that the minimum value of $Q_{cv}(I_{(b,c]})$ over $[VaR_\alpha(X), \infty]$ is attainable at $c = \infty$ for $\alpha < \frac{\delta}{\delta+\rho}$ and $c = VaR_\alpha(X)$ for $\alpha \geq \frac{\delta}{\delta+\rho}$. Collecting the above arguments, together with (3.6) and (A.4), yields that f^*_{cv} given in (4.3) is optimal under the reinsurance model (2.10) with expected value principle. The proof is thus complete.

Proof of Proposition 4.2.

By Theorem 22 in Kaluszka (2005), Wang's premium principle (4.4) can be rewritten by

$$\pi(Y) = \int_0^1 VaR_s(Y)\, g(\mathrm{d}s)$$

for all $Y \in \chi$. For any non-negative random variables Y and Z with $Y \leq_{st} Z$, i.e. $S_Y(t) \leq S_Z(t)$, $\forall t \geq 0$, we have $VaR_s(Y) \leq VaR_s(Z)$ for any $s \in (0,1)$ and

$$\pi_\delta(Z) - \pi_\delta(Y) = \int_0^1 (VaR_s(Z) - VaR_s(Y))\, w(\mathrm{d}s),$$

where $w(s)$ is given in (4.5). If $g'_+(1-) \geq 1 - \delta$, $w(s)$ is increasing and continuous on [0,1]. Thus, the above equation implies $\pi_\delta(Z) \geq \pi_\delta(Y)$, and hence $\pi_\delta(.)$ preserves FOSD.

Otherwise, if $g'_+(1-) < 1 - \delta$, we can show that $\pi_\delta(.)$ fails to preserve FOSD by constructing two non-negative random variables Y and Z which satisfy $Y \leq_{st} Z$ but $\pi_\delta(Z) < \pi_\delta(Y)$. Specifically, denote $s_0 \triangleq \inf\{0 \leq s \leq 1 : g'_+(s) < 1 - \delta\}$, then we have $0 < s_0 < 1$. Let Y have the same distribution with U and

$$F_Z(t) = \begin{cases} t, & 1 - s_0 \leq t \leq 1; \\ 0, & 0 \leq t < 1 - s_0, \end{cases}$$

then we have $Y \leq_{st} Z$ but

$$\pi_\delta(Z) - \pi_\delta(Y) = \int_{s_0}^1 (s - s_0)\, w(\mathrm{d}s) < 0,$$

where the inequality is implied by $w'_+(s) = g'_+(s) - (1 - \delta) < 0$ for any $s \in (s_0, 1)$. The proof is thus complete.

Proof of Proposition 4.3.

Since $w(x)$, ξ, β are parameterized by δ, we rewrite them as $w_\delta(x)$, $\xi(\delta)$, $\beta(\delta)$ to emphasize this dependence. We first show $\beta(\delta_1) \geq \beta(\delta_2)$ for any $0 < \delta_1 < \delta_2 < 1$.

Specifically, if $g'_+(1-) \geq 1-\delta_2$, it follows from (4.6) that $\beta(\delta_2)=0$, then the result is trivial; otherwise, we have $g'_+(1-) < 1-\delta_2 < 1-\delta_1$, then the equation $w_{\delta_i}(x)=0$, $x \in (0,1)$ has a unique solution $\xi(\delta_i)$ for any $i=1,2$. In this case, as $w_\delta(x)$ in (4.5) is strictly decreasing in δ for any $x \in (0,1)$, we have $w_{\delta_2}(\xi(\delta_1)) < w_{\delta_1}(\xi(\delta_1)) = 0$ which results in $\xi(\delta_1) < \xi(\delta_2)$, then (4.6) implies the result.

Finally, it is easy to see from (4.7) that γ is increasing in δ. The proof is thus complete.

Proof of Proposition 4.4.

When reinsurance premium is calculated by Wang's principle (4.4), it follows from (A.2) that

$$Q_v(I_{(d,VaR_\alpha(X))]}) = \int_d^{VaR_\alpha(X)} g(S_X(t))\mathrm{d}t - (1-\delta)\int_d^{VaR_\alpha(X)} S_X(t)\mathrm{d}t + \delta d - \delta VaR_\alpha(X)$$

for any $d \in [0, VaR_\alpha(X)]$, then we have

$$\frac{\partial Q_v(I_{(d,VaR_\alpha(X))]})}{\partial d} = -w(S_X(d)), \ \ a.s.$$

where $w(x)$ is given in (4.5). If $g'_+(1-) \geq 1-\delta$, we have $w(x) \leq 0$ for any $x \in [0,1]$ such that $\frac{\partial Q_v(I_{(d,VaR_\alpha(X))]})}{\partial d} \geq 0$. In this case, the minimum value of $Q_v(I_{(d,VaR_\alpha(X))]})$ over $[0, VaR_\alpha(X)]$ is attainable at $d=0$. Otherwise, if $g'_+(1-) < 1-\delta$, note that $w(s)=0$, $0<s<1$ has a unique solution ξ, then the above equation, together with (2.7), implies that $Q_v(I_{(d,VaR_\alpha(X))]})$ can attain the minimum value at $d = \min\{VaR_\alpha(X), VaR_\xi(X)\} = VaR_{\alpha \vee \xi}(X)$. Consequently, recall that Wang's premium principle is translation invariant, then it follows from Corollary 3.3 and (A.1) that $f_v^*(x)$ in (4.8) is a solution to the optimal reinsurance model (2.9).

Further, for any $0 \leq d \leq VaR_\alpha(X) \leq c$, it follows from (A.5) that

$$\begin{aligned} Q_{cv}(I_{(d,c]}) &= \int_d^c g(S_X(t))\mathrm{d}t - (1-\delta)\,\mathbb{E}[I_{(d,c]}(X)] - \delta CVaR_\alpha(I_{(d,c]}(X)) \\ &= \int_d^c g(S_X(t))\mathrm{d}t - (1-\delta)\,\mathbb{E}[(X-d)_+] + \delta d + (1-\delta+\frac{\delta}{\alpha})\,\mathbb{E}[(X-c)_+] \\ &\quad - \delta CVaR_\alpha(X), \end{aligned}$$

where the second equality is implied by (A.6), then the partial derivatives of $Q_{cv}(I_{(d,c]})$ are given by

$$\begin{cases} \dfrac{\partial Q_{cv}(I_{(d,c]})}{\partial d} = -w(S_X(d)); \\ \dfrac{\partial Q_{cv}(I_{(d,c]})}{\partial c} = S_X(c)\left(\dfrac{g(S_X(c))}{S_X(c)} - (1-\delta+\delta/\alpha)\right); \end{cases} \quad a.s.$$

Similarly, we can prove that given a $c \geq VaR_\alpha(X)$, the minimum value of $Q_{cv}(I_{(d,c]})$ over $[0, VaR_\alpha(X)]$ is attainable at $d = \beta$ where β is defined in (4.6). Furthermore, as pointed out by Chi and Tan (2011b), $\frac{g(S_X(c))}{S_X(c)}$ is an increasing right-continuous function over $[0, ess\,sup\, X)$, then the above equation implies that the minimum value of $Q_{cv}(I_{(\beta,c]})$ over $[VaR_\alpha(X), \infty]$ is attainable at $c = \gamma$ where γ is given by (4.7). As a result, we know from Corollary 3.3 and (A.4) that $f^*_{cv}(x)$ in (4.9) is a solution to the optimal reinsurance model (2.10) with Wang's premium principle. The proof is therefore complete.

Proof of Proposition 4.5.

When reinsurance premium is calculated by Dutch principle (4.10) with $\delta \geq \theta$ or $\lambda = 1$, Corollary 3.3, together with (A.1), implies that the optimal reinsurance model (2.9) is equivalent to

$$\min_{0 \leq d \leq VaR_\alpha(X)} Q_v(I_{(d, VaR_\alpha(X)]}), \tag{A.7}$$

where $Q_v(f)$ is given in (A.2).

We first solve the above minimization problem for the case: $\delta \geq \theta$. If $\lambda \geq \frac{VaR_\alpha(X) - d}{\int_d^{VaR_\alpha(X)} S_X(t)\mathrm{d}t}$, we have

$$Q_v(I_{(d, VaR_\alpha(X)]}) = -\delta \int_d^{VaR_\alpha(X)} F_X(t)\,\mathrm{d}t$$

and obviously it is increasing in d; otherwise, $Q_v(I_{(d, VaR_\alpha(X)]})$ could be rewritten as

$$Q_v(I_{(d, VaR_\alpha(X)]}) = \theta \int_{d + \lambda \int_d^{VaR_\alpha(X)} S_X(t)\mathrm{d}t}^{VaR_\alpha(X)} S_X(t)\,\mathrm{d}t - \delta \int_d^{VaR_\alpha(X)} F_X(t)\,\mathrm{d}t$$

and we can show it is also increasing in d. Specifically, taking the derivatives of $Q_v(I_{(d, VaR_\alpha(X)]})$ w.r.t. d yields

$$\frac{\partial Q_v(I_{(d, VaR_\alpha(X)]})}{\partial d} = \delta F_X(d) - \theta S_X\left(d + \lambda \int_d^{VaR_\alpha(X)} S_X(t)\,\mathrm{d}t\right)(1 - \lambda S_X(d)), \; a.s.$$

If $1 - \lambda S_X(d) \leq 0$, it is trivial that $\frac{\partial Q_v(I_{(d, VaR_\alpha(X)]})}{\partial d} \geq 0$; otherwise, the above equation implies $\frac{\partial Q_v(I_{(d, VaR_\alpha(X)]})}{\partial d} \geq \delta(\lambda - 1)\, S_X(d) \geq 0$. Consequently, we have

$$Q_v(I_{(d, VaR_\alpha(X)]}) \geq Q_v(I_{(0, VaR_\alpha(X)]}), \quad \forall 0 \leq d \leq VaR_\alpha(X).$$

Next, we proceed to solve the optimization problem (A.7) for the case: $\lambda = 1$ and $\delta < \theta$. Similarly, we have

$$\frac{\partial Q_v(I_{(d,VaR_\alpha(X)]})}{\partial d} = \theta F_X(d)\left(\frac{\delta}{\theta} - S_X\left(d + \int_d^{VaR_\alpha(X)} S_X(t)\,\mathrm{d}t\right)\right),\ a.s. \tag{A.8}$$

If $S_X(\int_0^{VaR_\alpha(X)} S_X(t)\,\mathrm{d}t) \le \delta/\theta$, note that $d + \int_d^{VaR_\alpha(X)} S_X(t)\,\mathrm{d}t$ is an increasing function, then we have $\frac{\partial Q_v(I_{(d,VaR_\alpha(X)]})}{\partial d} \ge 0$ such that

$$Q_v(I_{(d,VaR_\alpha(X)]}) \ge Q_v(I_{(0,VaR_\alpha(X)]}),\ \forall 0 \le d \le VaR_\alpha(X).$$

Otherwise, it is easy to see from (A.8) that the minimum of $Q_v(I_{(d,VaR_\alpha(X)]})$ over $[0, VaR_\alpha(X)]$ is attainable at $d = b^*$ where b^* is given in (4.12).

Now, the residual task is to explore the solutions to the optimal reinsurance model (2.9) under Dutch premium principle with

$$\lambda \ge \frac{VaR_\alpha(X)}{\int_0^{VaR_\alpha(X)} S_X(t)\,\mathrm{d}t} \quad \text{and} \quad \delta < \theta.$$

In this case, neither Dutch principle is translation invariant nor $\pi_\delta(.)$ preserves FOSD, then Corollary 3.3 is inapplicable. Thus, we can only use the result of Theorem 3.1 and derive the optimal parameters of two-layer reinsurance.

For $c \ge VaR_\alpha(X)$, we have

$$Q_v(I_{(0,c]}) = \delta\int_0^c S_X(t)\,\mathrm{d}t + \theta\int_{c\wedge\lambda\int_0^c S_X(t)\mathrm{d}t}^c S_X(t)\,\mathrm{d}t - \delta VaR_\alpha(X),$$

where $x \wedge y \triangleq \min\{x,y\}$. If $c \le \lambda\int_0^c S_X(t)\,\mathrm{d}t$, it is trivial that $Q_v(I_{(0,c]})$ is increasing in c; otherwise, if $c > \lambda\int_0^c S_X(t)\,\mathrm{d}t$, we have

$$\frac{\mathrm{d}Q_v(I_{(0,c]})}{\mathrm{d}c} = \delta S_X(c) + \theta S_X(c)\left(1 - \lambda S_X\left(\lambda\int_0^c S_X(t)\,\mathrm{d}t\right)\right) \ge 0, \quad a.s.$$

where the inequality is implied by

$$S_X\left(\lambda\int_0^c S_X(t)\,\mathrm{d}t\right) = \mathbb{P}\left(X \wedge c > \lambda\int_0^c S_X(t)\,\mathrm{d}t\right) \le \frac{\mathbb{E}[X \wedge c]}{\lambda\int_0^c S_X(t)\,\mathrm{d}t} = \frac{1}{\lambda}.$$

As a result, we get $Q_v(I_{(0,c]}) \ge Q_v(I_{(0,VaR_\alpha(X)]})$, $\forall c \ge VaR_\alpha(X)$.

Following, if $f(x) = I_{(0,a]}(x) + I_{(b,VaR_\alpha(X)]}(x)$ for some $0 \le a \le b \le VaR_\alpha(X)$, note that $\int_0^a S_X(t)\,\mathrm{d}t/a$ is a decreasing function, then we have

$$\lambda\mathbb{E}[f(X)] \ge \lambda\int_0^a S_X(t)\,\mathrm{d}t \ge a.$$

Denote

$$V(a,b) \triangleq \lambda\int_0^a S_X(t)\mathrm{d}t + \lambda\int_b^{VaR_\alpha(X)} S_X(t)\mathrm{d}t - (VaR_\alpha(X) - b + a) \qquad \text{(A.9)}$$

for $0 \leq a \leq b \leq VaR_\alpha(X)$, then we have $V(b,b) = \lambda\int_0^{VaR_\alpha(X)} S_X(t)\mathrm{d}t - VaR_\alpha(X) \geq 0$ and $\frac{\partial V(a,b)}{\partial a} = \lambda S_X(a) - 1$, *a.s.* Thus, we can prove $\lambda S_X(a) - 1 \geq 0$ when $V(a,b) < 0$. Specifically, if not, we get $V(a,b) \geq V(b,b) \geq 0$ as $\frac{\partial V(t,b)}{\partial a} \leq 0$ for any $a \leq t \leq b$.

Given a $b \in [0, VaR_\alpha(X)]$, if $V(a,b) \geq 0$, we have

$$Q_v(f) = -\delta\int_0^a F_X(t)\mathrm{d}t - \delta\int_b^{VaR_\alpha(X)} F_X(t)\mathrm{d}t \qquad \text{(A.10)}$$

and trivially it is decreasing in a; otherwise, if $V(a,b) < 0$, we have

$$Q_v(f) = \theta\int_{V(a,b)+VaR_\alpha(X)}^{VaR_\alpha(X)} S_X(t)\mathrm{d}t - \delta\int_0^a F_X(t)\mathrm{d}t - \delta\int_b^{VaR_\alpha(X)} F_X(t)\mathrm{d}t.$$

Taking the derivatives of $Q_v(f)$ w.r.t. a yields

$$\frac{\partial Q_v(f)}{\partial a} = -\delta F_X(a) - \theta S_X\big(V(a,b) + VaR_\alpha(X)\big)\big(\lambda S_X(a) - 1\big) \leq 0, \quad a.s.$$

Consequently, we get $Q_v(f) \geq Q_v(I_{(0,VaR_\alpha(X)]})$, and the final result can be derived by Theorem 3.1 and (A.1). The proof is thus complete.

Proof of Proposition 4.6.

As stated in Young (2004), Dutch premium principle (4.10) satisfies the axioms of law invariance, risk loading and preserving convex order, then using Theorem 3.1 and (A.1), we get that the optimal reinsurance model (2.9) is equivalent to

$$\min_{f\in\mathfrak{C}_v} Q_v(f),$$

where $\mathfrak{C}_v$ and $Q_v(f)$ are given in (3.2) and (A.2), respectively.

First, using the same proof as that of Proposition 4.5, we have

$$Q_v(I_{(0,c]}) \geq Q_v(I_{(0,VaR_\alpha(X)]}), \quad \forall c \geq VaR_\alpha(X).$$

Thus, we only need to derive the optimal parameters of two-layer reinsurance in the form of

$$f(x) = I_{(0,a]}(x) + I_{(b,VaR_\alpha(X)]}(x), \quad 0 \leq a \leq b \leq VaR_\alpha(X).$$

For this case, we rewrite $Q_v(f)$ by $W(a,b)$ as it is a function of a and b.

To proceed, we define

$$\mathcal{F} \triangleq \{(a,b): 0 \le a \le b \le VaR_\alpha(X)\}; \qquad \mathcal{F}_2 \triangleq \{(a,b) \in \mathcal{F}: V(a,b) \ge 0\};$$

$$\mathcal{F}_1 \triangleq \{(a,b) \in \mathcal{F}: G(a,b) \le 0\} \setminus \{(0, VaR_\alpha(X))\}; \quad \mathcal{F}_3 \triangleq \mathcal{F} \setminus (\mathcal{F}_1 \cup \mathcal{F}_2),$$

where $G(a,b)$ and $V(a,b)$ are given in (4.16) and (A.9) respectively and we have the following properties:

- $V(a,a) = \lambda \int_0^{VaR_\alpha(X)} S_X(t)\mathrm{d}t - VaR_\alpha(X) < 0$ for any $0 \le a \le VaR_\alpha(X)$ as it is assumed in this proposition that

$$1 < \lambda < VaR_\alpha(X) \Big/ \int_0^{VaR_\alpha(X)} S_X(t)\mathrm{d}t.$$

  Moreover, under Assumption 4.1, it is easy to get

$$V(0, VaR_\alpha(X)) = V(a_0, VaR_\alpha(X)) = 0$$

  where a_0 is the unique solution to the equation (4.14), and $a_0 > VaR_{1/\lambda}(X)$.
- For any $(a,b) \in \mathcal{F}_2$, we have

$$0 \le a \le a_0 \ \text{ and } \ b > VaR_{1/\lambda}(X).$$

  Specifically, if there exists an $(\bar{a},\bar{b}) \in \mathcal{F}_2$ with $\bar{b} \le VaR_{1/\lambda}(X)$, note that $\frac{\partial V(a,b)}{\partial b} = 1 - \lambda S_X(b)$, then it leads to a contradiction $0 > V(\bar{a},\bar{a}) \ge V(\bar{a},\bar{b}) \ge 0$. Further, for any $a_0 < a \le b \le VaR_\alpha(X)$, recall that $a_0 > VaR_{1/\lambda}(X)$, then we have $\frac{\partial V(a,b)}{\partial a} < 0 < \frac{\partial V(a,b)}{\partial b}$ such that $V(a,b) < V(a_0, VaR_\alpha(X)) = 0$. Thus, we have $(a,b) \notin \mathcal{F}_2$.
- We have $a \ge a_0$ for any $(a,b) \in \mathcal{F}_1$ as $0 \ge G(a,b) \ge \lambda \int_0^a S_X(t)\mathrm{d}t - a$.

Based upon the above properties, the partitioning of the set $\mathcal{F}$ is depicted in Figure A.1.

Following, we try to figure out the minimum points of $W(a,b)$ located in three subsets of $\mathcal{F}$, separately. First, we find no minimum points of $W(a,b)$ are in $\mathcal{F}_2 \setminus \{(a_0, VaR_\alpha(X))\}$. Specifically, for any $(a,b) \in \mathcal{F}_2$, $W(a,b) = Q_v(f)$ is given in (A.10). Under Assumption 4.1, it is easy to see that $W(a,b)$ is strictly decreasing in a and strictly increasing in b. Moreover, recall that $b > VaR_{1/\lambda}(X)$ for any $(a,b) \in \mathcal{F}_2$, then given an a satisfying $0 \le a \le a_0$, the equation $V(a,b) = 0$, $a < b \le VaR_\alpha(X)$ has a unique solution $b(a)$ and $b'(a) = \frac{1 - \lambda S_X(a)}{1 - \lambda S_X(b(a))}$. Thus, we have

$$\frac{\mathrm{d}W(a,b(a))}{\mathrm{d}a} = \delta(\lambda - 1)(S_X(b(a)) - S_X(a)) / (1 - \lambda S_X(b(a))) < 0.$$

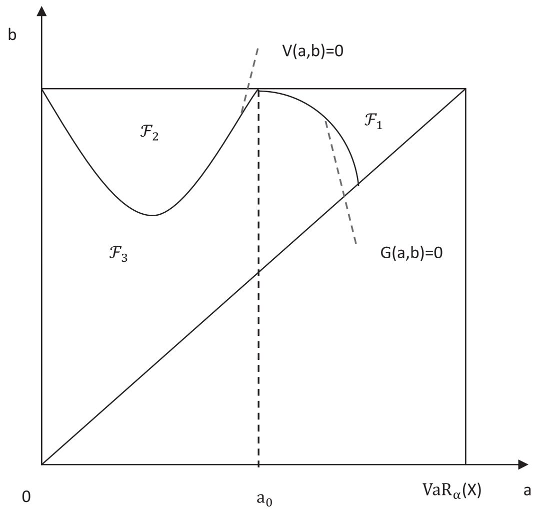

FIGURE A.1: A decomposition of the set $\mathcal{F}$.

Consequently, we get $W(a,b) > W(a_0, VaR_\alpha(X))$ for any $(a,b) \in \mathcal{F}_2 \setminus \{(a_0, VaR_\alpha(X))\}$.

Next, for $(a,b) \in \mathcal{F}_3$, $W(a,b)$ could be rewritten as

$$W(a,b) = \theta \int_{G(a,b)+b}^{VaR_\alpha(X)} S_X(t)\,\mathrm{d}t - \delta \int_0^a F_X(t)\,\mathrm{d}t - \delta \int_b^{VaR_\alpha(X)} F_X(t)\,\mathrm{d}t.$$

The partial derivatives of $W(a,b)$ are given by

$$\begin{cases} \dfrac{\partial W(a,b)}{\partial a} = -\delta F_X(a) - \theta S_X(G(a,b)+b)(\lambda S_X(a) - 1); \\ \dfrac{\partial W(a,b)}{\partial b} = \delta F_X(b) + \theta S_X(G(a,b)+b)(\lambda S_X(b) - 1). \end{cases}$$

If (a,b) is a stationary point of $W(a,b)$ on $\mathcal{F}_3$, we have $\frac{\partial W(a,b)}{\partial a} = \frac{\partial W(a,b)}{\partial b} = 0$, which results in $S_X(a) = S_X(b)$. Thus, we have $a = b$ under Assumption 4.1, which is contradicted to the assumption of stationary points that (a,b) is an interior point of $\mathcal{F}_3$. Moreover, as $\lambda > 1$, it follows from the above equation that $\left.\frac{\partial W(a,b)}{\partial a}\right|_{a=0} < 0$. Thus, $W(a,b)$ has no minimum points located in $\mathcal{F}_3 \setminus \mathcal{B}$, where $\mathcal{B} \triangleq \{(a,a) : 0 \le a \le VaR_\alpha(X)\}$. Moreover, since

$$W(a,a) = Q_v(I_{(0, VaR_\alpha(X)]}), \quad \forall 0 \le a \le VaR_\alpha(X), \tag{A.11}$$

then we have $\min_{(a,b) \in \mathcal{B}} W(a,b) \ge \min_{(a,b) \in \mathcal{F}_1} W(a,b)$.

Finally, collecting all the above arguments yields that the minimum of $W(a,b)$ over $\mathcal{F}$ must be attainable on the compact set $\mathcal{F}_1$. For any $(a,b) \in \mathcal{F}_1$, we have

$$W(a,b) = -\delta\int_0^a F_X(t)\mathrm{d}t - \delta\int_b^{VaR_\alpha(X)} F_X(t)\mathrm{d}t + \theta\left(\int_{G(a,b)+a}^a S_X(t)\mathrm{d}t + \int_b^{VaR_\alpha(X)} S_X(t)\mathrm{d}t\right),$$

then the partial derivatives of $W(a,b)$ are given by

$$\begin{cases} \dfrac{\partial W(a,b)}{\partial a} = -\delta F_X(a) + \theta S_X(a)\left(1 - \lambda S_X(G(a,b)+a)\right); \\ \dfrac{\partial W(a,b)}{\partial b} = \delta F_X(b) - \theta S_X(b)\left(1 - \lambda S_X(G(a,b)+a)\right). \end{cases}$$

Thus, if (a,b) is a stationary point of $W(d,c)$ over $\mathcal{F}_1$, we have $\frac{\partial W(a,b)}{\partial a} = \frac{\partial W(a,b)}{\partial b} = 0$, which results in $a = b$. It is contradicted to the definition of stationary points, and hence $W(a,b)$ over $\mathcal{F}_1$ has no stationary points. By Fermat's theorem, we know that the minimum of $W(a,b)$ over $\mathcal{F}_1$ can only be attainable on the boundary. As a consequence, it follows from (A.11) that

$$\min_{(a,b)\in\mathcal{F}} W(a,b) = \min_{(a,b)\in\mathcal{F}_1} W(a,b) = \min_{(a,b)\in\mathcal{D}} W(a,b),$$

where $\mathcal{D}$ is given in (4.15). The proof is thus complete.

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