ARE FLEXIBLE PREMIUM VARIABLE ANNUITIES UNDER-PRICED?

BY

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ABSTRACT

A variable annuity (VA) is a deferred annuity that allows an annuitant to invest his/her contributions into a range of mutual funds. A separate account termed as sub-account is set up for the investment. Unlike a mutual fund, a VA offers a guaranteed minimum death benefit or GMDB and often offers a guaranteed minimum living benefit or GMLB during the accumulation phase of the VA contract. Almost all the research to date has focused on single premium variable annuities (SPVAs), i.e. it is assumed that an annuitant makes a single lump-sum contribution at the time of issue. In this paper, we study flexible premium variable annuities (FPVAs) that allow contributions during the accumulation phase. We derive a valuation formula for guarantees embedded in FPVAs and show that the delta hedging strategy for an FPVA is substantially different from that for an SPVA. The numerical examples illustrate that the cost in the form of mortality and expense (M&E) fee for an FPVA in many situations is significantly higher than the cost for a similar SPVA. This finding suggests that the current pricing practice by most VA providers that charges the same M&E fee for both should be re-examined.

KEYWORDS

Variable Annuity; Flexible Premium; GMDB; GMLB; Arithmetic Asian Option; Mortality and Expense Fee.

1. INTRODUCTION

A variable annuity (VA) is a deferred annuity that allows an annuitant to invest his/her contributions into a range of mutual funds. A separate account termed as sub-account is set up for the investment. Unlike a mutual fund, a VA offers a guaranteed minimum death benefit or GMDB and often offers a guaranteed minimum living benefit or GMLB during the accumulation phase of the VA contract. GMDB guarantees the beneficiaries of a VA the greater of (a) the sub-account value or (b) the total accumulated (with interest if any) premiums paid in the past, upon the death of the annuitant. GMLB provides accumulation

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or income protection for a fixed number of years contingent on survival. Typically, an insurance charge in the form of mortality and expense (M&E) fee proportional to the sub-account value is applied daily to cover the guarantees[1]. Variable annuities have accounted for a significant portion of life insurers' premium incomes in the North America. According to the 2011 IRI Fact Book published by the Insured Retirement Institute (formerly the National Association of Variable Annuities), variable annuity industry total sales in 2010 were $138.3 billion while fixed annuity sales were $71.7 billion. Hence, proper valuation and effective hedging of variable annuities are central to annuity providers' risk management.

Most VAs offer several premium contribution options in the name of Flexible Premium Variable Annuity. An annuitant can either make a single lump-sum contribution at the time of issue or make a series of contributions during the accumulation phase. The M&E fee remains the same, regardless which payment option is chosen. For example, New York Life offers three contribution options for tax-qualified VA policies and two payment options for non-qualified policies with a M&E fee of 1.40%.

TABLE 1

MINIMUM INITIAL PREMIUM OF NEW YORK LIFE FLEXIBLE PREMIUM VARIABLE ANNUITY

Non-Qualified Policies
• $2500 plus pre-authorized monthly deductions of $50 per month; or
• $5000 single premium.
Individual Retirement Account (IRA)
• $1200 plus pre-authorized monthly deductions of $100 per month; or
• Pre-authorized monthly deductions of $165 per month; or
• $2000 single premium.

See New York Life Fact Sheet (2011) for more details.

There has been considerable amount of research on modeling, pricing and hedging VAs in recent years. See Coleman et al. (2006), Milevsky and Salisbury (2006), Bauer et al. (2008), Chen et al. (2008), Dai et al. (2008), Lin et al. (2009), Hürlimann (2010), and references therein. However, almost all the research to date has focused on single premium variable annuities (SPVAs), i.e. it is assumed that an annuitant makes a single lump-sum contribution at the time of issue. Flexible premium variable annuities (FPVAs) that allow contributions during the accumulation phase are fundamentally different in terms

1 In this paper, we ignore the administrative and distribution charges as well as management fees associated with the underlying mutual funds for simplicity.

of their payoff structure. The payoff of an SPVA is similar to the payoff of a European (put) option. On the other hand, since the payoff of an FPVA depends on the premium contributions over the accumulation phase its payoff is path-dependent and resembles the payoff of an Asian option as shown in the next section. An immediate consequence is that the delta hedging strategy for an FPVA will be significantly different from the delta hedging strategy for an SPVA with the same guarantee. A similar insurance product to FPVAs is unit-linked insurance contracts with periodic/regular premium that are sold in many European countries. The guarantees embedded in these contracts are also Asian put options. See Nielsen and Sandmann (2002), Schrager and Pelsser (2004), and references therein.

The aim of this paper is (a) to examine whether or not charging the same M&E fee for both FPVA and SPVA with the same guarantee is justified; and (b) to identify the delta hedging strategy for an FPVA. We assume that the return of the underlying sub-account of an FPVA follows the Black-Scholes model and the FPVA has the enhanced return of premium guarantee. Furthermore, the premium contributions are made continuously at a constant rate. Using a modified PDE approach of Vecer (2002), we are able to obtain a valuation formula for the fair M&E fee and to identify the associated delta hedging strategy. Although the valuation formula does not provide a closed form for the fair M&E fee in general, it can be computed efficiently due to the simplicity of the PDEs involved. Using reasonable model parameters, we find that the fair M&E fees for FPVAs in many situations are significantly higher than those for similar SPVAs. This important finding suggests that the current pricing practice by most VA providers that charges the same M&E fee for both should be re-examined.

While the valuation problem of FPVAs in this paper is investigated by using a continuous-time approach, this problem may also be investigated in a discrete-time framework. In the latter situation, we may adapt the valuation approach in Nielsen and Sandmann (2002) or in Schrager and Pelsser (2004), since the unit-linked insurance contracts considered in their papers are similar to the FPVA. Our PDE approach and the approximation method in Nielsen and Sandmann (2002) and Schrager and Pelsser (2004) have their own pros and cons. The PDE approach allows us to compute the fair M&E fee in an extremely fast and accurate manner. This is because that the value of the embedded Asian-type put option satisfies a one-dimensional PDE that can be easily numerically solved. Further, the PDE approach will give us an explicit delta hedging strategy. On the other hand, because of the analytical nature of the PDE approach its applicability is limited in terms of what models we can use. The approaches in Nielsen and Sandmann (2002) and Schrager and Pelsser (2004) are in general more flexible. They allow for the use of a general Gaussian process for the underlying mutual fund and for the incorporation of stochastic interest rates. However, the valuation of Asian-type guarantees can be challenging when a discrete-time approach is used. In their papers, the method of Rogers and Shi (1995) is adapted to obtain a tight lower bound for guarantee values at the time of issue. The upper bounds for guarantee values in Nielsen and Sandmann (2002) and Schrager and Pelsser (2004) are given by the lower bound plus a correction and by applying the method of Thompson (1999), respectively. The numerical illustrations in both papers and those in Schrager and Pelsser (2004) in particular show that the bounds can provide satisfactory approximations to the guarantee values, but theoretically the performance of the bounds can not be evaluated. Furthermore, the calculation of the fair M&E fee might be difficult if we adapt their approach, especially when the number of time periods is very large.

This paper is organized as follows. In the next section, we describe the stochastic model for the sub-account of a VA and the payoff function of the enhanced return of premium guarantee. In Section 3, we derive a valuation formula for the guarantee and associated delta hedging strategy in the absence of mortality risk. We show that the delta hedging strategy for an FPVA is substantially different from that for an SPVA. In Section 4, we derive formulas that are used to calculate fair M&E fees and identify the delta hedging strategy. Numerical illustrations are given in Section 5. Finally, some concluding remarks are made in Section 6.

2. Mathematical Formulation

In this section, we describe the stochastic model for the sub-account of a variable annuity and the enhanced return of premium guarantee. We begin by assuming that the value of a unit of the underlying mutual fund at time t, S_t, follows a geometric Brownian motion under the risk-neutral probability measure $\mathbb{Q}$:

$$dS_t = rS_t dt + \sigma S_t \, dW_t, \tag{1}$$

where r is the risk-free interest rate compounded continuously, σ is the volatility, $\{W_t; t \geq 0\}$ is a standard Brownian motion, and t is measured in year. In other words, the Black-Scholes model is used to model the mutual fund. Without the loss of generality, we assume the initial value $S_0 = 1$ and there are no dividends.

An annuitant is assumed to make an initial contribution of A_0 dollars and subsequent annual contributions of $k \geq 0$ dollars payable continuously over the accumulation phase of duration T. If $k = 0$, the VA is an SPVA. Otherwise, it is an FPVA. Let c be the annual M&E fee payable continuously, and A_t be the value of the sub-account at time t. Then we have the following stochastic differential equation (SDE) for the sub-account:

$$dA_t = (r - c)A_t dt + \sigma A_t \, dW_t + k dt. \tag{2}$$

Here, we assume no withdrawal nor surrender during the accumulation phase. It is easy to see that SDE (2) can be solved by Itô's lemma and rewritten as

$$A_t = A_0 e^{(r - c - \frac{\sigma^2}{2})t + \sigma W_t} + k \int_0^t e^{(r - c - \frac{\sigma^2}{2})(t - s) + \sigma(W_t - W_s)} ds, \ t \geq 0. \tag{3}$$

We consider one of the most common embedded guarantees in a variable annuity during the accumulation phase: the enhanced return of premium guarantee. It guarantees that the beneficiaries or the annuitant of the VA will receive the greater of the current sub-account value or the accumulated value of the total premiums paid at a rate of $g \geq 0$ (rising floor rate) at the event of death or at the end of the accumulation phase. This is a combination of a GMDB and a GMAB (guaranteed minimum accumulation benefit). We assume $g < r$ to avoid a negative spread. If $g = 0$, the guarantee is the standard return of premium guarantee. Let now $G(t)$ be the level of minimum guarantee at time t. Then $G(t)$ follows the differential equation

$$\mathrm{d}G(t) = gG(t)\mathrm{d}t + k\mathrm{d}t, \quad 0 \leq t \leq T \quad \text{with} \quad G(0) = A_0, \tag{4}$$

or

$$G(t) = \begin{cases} A_0 e^{gt} + k\,\dfrac{e^{gt}-1}{g}, & g > 0; \\ A_0 + kt, & g = 0. \end{cases}$$

Obviously, if the guarantee is exercised at time t, then its payoff is given by

$$P(t) = (G(t) - A_t)_+ \triangleq \max\{G(t) - A_t, 0\}. \tag{5}$$

At first glance, $P(t)$ is the payoff of a put option with the underlying asset being A_t. However, the usual option pricing technique does not apply as expression (3) shows that $\{A_t;\, t \geq 0\}$ is not a geometric Brownian motion. Instead, it is the sum of a geometric Brownian motion and an arithmetic average of the geometric Brownian motion over time. Thus, this payoff can be viewed as the payoff of an arithmetic Asian put option with a correlated floating strike.

Many methods have been developed to price arithmetic Asian options. See Thompson (1999) for tight analytic bounds of the price of Asian options, Geman and Yor (1993) and Yor (2001) for the Laplace transformation methods, Rogers and Shi (1995) for PDE approaches, and Broadie et al. (1999) and references therein for Monte-Carlo methods. More recently, Vecer (2002) developed a very clever PDE approach for Asian options. This method has been proven to be extremely fast and stable in numerical calculation of the price of arithmetic Asian options. More importantly, it can be easily modified to price the put option with payoff (5).

3. Pricing and Hedging the Put Option

In this section, we derive a valuation formula for the put option that has the payoff (5) in the absence of mortality risk by slightly modifying the PDE approach developed in Vecer (2002). This approach allows for efficient computation of the fair M&E fee and for easy identification of the delta hedging strategy as we will see in this and next sections.

Proposition 1. *Denote*

$$q_t(u) = \frac{k}{r-c}[e^{(r-c)(t-u)} - 1], \quad 0 \le u \le t.$$

Then, the time-0 price of the put option with payoff (5) is given by

$$\Pi(0,t) = e^{-rt} U_t\left(0, \left(A_0 + \frac{k}{r-c}\right)e^{(r-c)t} - \frac{k}{r-c}\right), \tag{6}$$

where $U_t(u,z)$ satisfies PDE

$$\frac{\partial U_t(u,z)}{\partial u} + \frac{1}{2}\sigma^2[z - q_t(u)]^2 \frac{\partial^2 U_t(u,z)}{\partial z^2} = 0, \ \ 0 \le u \le t \tag{7}$$

with terminal condition $U_t(u,z) = (G(t) - z)_+$.

Especially, when $k = 0$, we have

$$\Pi(0,t) = e^{-rt} U_t(0, A_0 e^{(r-c)t}) = A_0 e^{-(r-g)t} N(-\xi_1 \sqrt{t}\,) - A_0\, e^{-ct} N(-\xi_2 \sqrt{t}\,), \tag{8}$$

where $N(x)$ is the cumulative distribution function of the standard normal distribution, and

$$\xi_1 = \frac{r - c - g - \sigma^2/2}{\sigma} \quad \textit{and} \quad \xi_2 = \xi_1 + \sigma. \tag{9}$$

Proof. Our proof is a modification of that in Vecer (2002). Let $\{Y_u; 0 \le u \le t\}$ be a solution of SDE

$$\mathrm{d}Y_u = q_t(u)\mathrm{d}S_u^{-1} - r[Y_u - q_t(u)S_u^{-1}]\mathrm{d}u, \tag{10}$$

where the initial value Y_0 and the deterministic differentiable function $\{q_t(u); 0 \le u \le t\}$ are to be determined. Rewrite SDE (10) as

$$\begin{aligned}\mathrm{d}(e^{ru}Y_u) &= e^{ru}q_t(u)\mathrm{d}S_u^{-1} + re^{ru}q_t(u)S_u^{-1}\mathrm{d}u \\ &= \mathrm{d}(e^{ru}q_t(u)S_u^{-1}) - e^{ru}S_u^{-1}\mathrm{d}q_t(u).\end{aligned}$$

Thus, Y_u can be solved and expressed as

$$Y_u = e^{-ru}(Y_0 - q_t(0)) + q_t(u)S_u^{-1} - \int_0^u e^{-r(u-v)} S_v^{-1} q_t'(v)\mathrm{d}v. \tag{11}$$

In order to reproduce the payoff of the put option, let $q_t(u)$ be a function such that

$$q_t'(u) = -ke^{(r-c)(t-u)} \quad \text{with} \quad q_t(t) = 0.$$

Hence,

$$q_t(u) = \frac{k}{r-c}[e^{(r-c)(t-u)} - 1], \; 0 \le u \le t.$$

Further, choose the initial value of Y_u to be

$$Y_0 = q_t(0) + A_0 e^{(r-c)t} = \left(A_0 + \frac{k}{r-c}\right) e^{(r-c)t} - \frac{k}{r-c}.$$

From (11), we have

$$Y_u = e^{r(t-u)-ct} A_0 + ke^{r(t-u)} \int_0^u e^{-c(t-v)} S_v^{-1} \mathrm{d}v + q_t(u) S_u^{-1}. \tag{12}$$

In particular, when $u = t$, we have

$$Y_t = A_0 e^{-ct} + ke^{-ct} \int_0^t e^{cv} S_v^{-1} \mathrm{d}v. \tag{13}$$

Therefore, the price of the put option $\Pi(0, t)$ can now be written as

$$\Pi(0,t) = e^{-rt}\mathbb{E}[(G(t) - A_t)_+] = e^{-rt}\mathbb{E}[(G(t) - Y_t \times S_t)_+].$$

Let

$$Z_u = Y_u \times S_u, \; 0 \le u \le t. \tag{14}$$

From (10), straightforward algebra leads to

$$\mathrm{d}Z_u = Y_u \mathrm{d}S_u + S_u \mathrm{d}Y_u + < \mathrm{d}S_u, \mathrm{d}Y_u >= \sigma(Z_u - q_t(u)) \mathrm{d}W_u. \tag{15}$$

Denote

$$U_t(u, Z_u) = \mathbb{E}[(G(t) - Z_t)_+ \,|\, \mathcal{F}_u] \tag{16}$$

for $0 \le u \le t$, where $\{\mathcal{F}_u; u \ge 0\}$ is the filtration generated by $\{W_u; u \ge 0\}$, then we have $\Pi(0, t) = e^{-rt} U_t(0, Z_0)$ and it follows from (15) that $U_t(u, z)$ satisfies the PDE in (7).

In particular, when $k = 0$, PDE (7) is reduced to

$$\frac{\partial U_t(u,z)}{\partial u} + \frac{1}{2}\sigma^2 z^2 \frac{\partial^2 U_t(u,z)}{\partial z^2} = 0,$$

which has the closed-form solution, then we obtain the explicit expression of $\Pi(0,t)$ in (8). The proof is therefore complete. □

In the next proposition, we identify the delta hedging strategy for the put option.

Proposition 2. *For* $0 \le u < t$, *the time-*u price $\Pi(u,t)$ *of the put option is given by* $\Pi(u,t) = e^{-r(t-u)} U_t(u, Z_u)$, *where* $U_t(u,z)$ *is the solution of* (7) *and* $\{Z_u;\ 0 \le u \le t\}$ *is given in* (14). *The put option can be perfectly replicated using the underlying* S_u *and the money market account, and the delta at time* u *is given by*

$$h_t(u) = e^{-r(t-u)} S_u^{-1} [Z_u - q_t(u)] \frac{\partial U_t(u, Z_u)}{\partial z}. \tag{17}$$

Proof. It follows from (7) that

$$\mathrm{d}U_t(u, Z_u) = \frac{\partial U_t(u, Z_u)}{\partial z} \,\mathrm{d}Z_u = \sigma [Z_u - q_t(u)] \frac{\partial U_t(u, Z_u)}{\partial z} \,\mathrm{d}W_u.$$

Since $\Pi(u,t) = e^{-r(t-u)} \mathbb{E}[P(t)|\mathcal{F}_u] = e^{-r(t-u)} U_t(u, Z_u)$, then applying Itô's lemma to $\Pi(u,t)$ we have

$$\begin{aligned} \mathrm{d}\Pi(u, t) &= r\Pi(u, t)\mathrm{d}u + \sigma e^{-r(t-u)} [Z_u - q_t(u)] \frac{\partial U_t(u, Z_u)}{\partial z} \,\mathrm{d}W_u \\ &= r(\Pi(u, t) - h_t(u) \times S_u) \,\mathrm{d}u + h_t(u) \mathrm{d}S_u, \end{aligned}$$

where $h_t(u)$ is defined in (17). Therefore, $h_t(u)$ is the number of units of S_u or the delta at time u of the replicating portfolio. □

We remark that when k is non-zero, i.e., the VA is an FPVA, the delta is path-dependent. This is because Z_u is a function of Y_u from (14) and formula (12) shows that Y_u is path-dependent. However, if $k = 0$, the delta is path-independent as the middle and last terms of (12) disappear.

4. Evaluation of Minimum Guarantees

In this section, we evaluate the combined GMDB and GMAB guarantee described in Section 2. We first determine the fair M&E fee for the guarantee and then identify the delta hedging strategy.

It is assumed in this section that the mortality risk is diversifiable. This is a common assumption used in insurance literature and is also a reasonable assumption for VA valuation as the majority of VA issuers have a few hundred thousands of VA policies in their portfolio and mortality risk is often a secondary concern. With this assumption, the liability-at-issue associated with the guarantee can be written as

$$\mathcal{L}_0 = \int_0^T e^{-rt} P(t)\,{}_tp_x \mu_{x+t}\,\mathrm{d}t + e^{-rT} P(T)\,{}_Tp_x - c\int_0^T e^{-rt} A_t\,{}_tp_x\,\mathrm{d}t, \tag{18}$$

where ${}_tp_x$ and μ_{x+t} are the actuarial symbols for the survival probability and the force of mortality, respectively.

The expected present value of the liability as a function of the M&E fee is then

$$\begin{aligned} L_0(c) &= \mathbb{E}[\mathcal{L}_0] \\ &= \int_0^T \Pi(0,t)\,{}_tp_x \mu_{x+t}\,\mathrm{d}t + \Pi(0,T)\,{}_Tp_x \\ &\quad - cA_0\,\bar{a}_{x:\bar{T}|c} - \frac{kc}{r-c}\,(\bar{a}_{x:\bar{T}|c} - \bar{a}_{x:\bar{T}|r}), \end{aligned} \tag{19}$$

where $\bar{a}_{x:\bar{T}|r} = \int_0^T e^{-rt}{}_tp_x\,\mathrm{d}t$ is the actuarial symbol for the net single premium of the continuous T-year temporary life annuity for a life aged x. The fair M&E fee c^* is such that the expected present value of the liability at time 0 is zero, i.e.

$$L_0(c^*) = 0. \tag{20}$$

The fair M&E fee c^* should exist in practice but it is not so simple in theory. When the M&E fee c increases, both the income from the M&E charges and the value of the guarantee increase. The latter is due to the decrease in sub-account value caused by the M&E fee deduction. The next proposition confirms the existence of the fair M&E fee c^* and describes the relationship between the liability and the key parameters of the VA guarantee.

Proposition 3. *$L_0(c)$, as a function of the M&E fee c, the rising floor rate g and the mutual fund volatility σ, is strictly decreasing in c and strictly increasing in both g and σ. As a result of monotonicity in c, the fair M&E fee c^* in* (20) *uniquely exists. Furthermore, the fair M&E fee c^* as a function of g and σ is increasing in both. In other words, one must raise the M&E fee if the rising floor rate is set higher and/or the financial market becomes more volatile.*

The proof of Proposition 3 and the proof of Proposition 4 below are lengthy and mathematically complicated. We delegate them to Appendix A. The above proposition reconfirms our intuition about the relationship between the policy liability and the three variables: the M&E fee, the guarantee rate and the fund volatility. It also provides a useful guidance on solving (20) numerically for the fair M&E fee.

Once the M&E fee is determined, the delta hedging strategy can be identified easily using the result in Proposition 2, as in the following proposition.

Proposition 4. *Under the assumption that the mortality risk is diversifiable, the policy liability $\mathcal{L}_0$ defined in* (18) *can be hedged using the underlying S_u and the money market account, and the delta at time-u is given by*

$$\Delta(u) = {}_T p_x h_T(u) + \int_u^T {}_t p_x \mu_{x+t} h_t(u)\mathrm{d}t - c \times {}_u p_x \bar{a}_{x+u:\overline{T-u}|c} A_u / S_u, \tag{21}$$

where $h_t(u)$ is given in (17).

We want to point out that the above delta hedging strategy for an FPVA is fundamentally different from the delta hedging strategy for an SPVA. As shown in Proposition 2 and the remark afterwards, $h_t(u)$ depends on the values of the sub-account over the entire period $[0, u]$ and so is $\Delta(u)$ as a result of it. Thus, the delta hedging strategy for an FPVA is path-dependent. On the other hand, for an SPVA the guarantee is essentially a series of European put options with different expiration times. The corresponding delta at time u for the guarantee therefore depends only on the value of S_u. Thus, the hedging strategy for an SPVA is path-independent.

5. Numerical Illustrations

In this section, we calculate fair M&E fees for a range of parameter values. As mentioned in the beginning of this paper, it is a common practice among

TABLE 2

CIA 1997-2004 Insurance Mortality Table

Attained Age (Last Birthday)	CIA 97-04		Attained Age (Last Birthday)	CIA 97-04	
	Male	Female		Male	Female
50	0.00238	0.00181	65	0.01304	0.01152
51	0.00263	0.00204	66	0.01462	0.01265
52	0.00291	0.00229	67	0.01632	0.01377
53	0.00323	0.00259	68	0.01816	0.01489
54	0.00359	0.00292	69	0.02013	0.01602
55	0.00400	0.00329	70	0.02225	0.01715
56	0.00445	0.00371	71	0.02450	0.01827
57	0.00498	0.00419	72	0.02691	0.01940
58	0.00558	0.00472	73	0.02946	0.02053
59	0.00626	0.00530	74	0.03216	0.02166
60	0.00705	0.00595	75	0.03501	0.02278
61	0.00801	0.00706	76	0.03802	0.02391
62	0.00908	0.00817	77	0.04119	0.02504
63	0.01028	0.00929	78	0.04453	0.02617
64	0.01160	0.01040	79	0.04802	0.02729

VA providers to charge the same M&E fee for both SPVA and FPVA. However, our numerical results in this section show that the fair M&E fee for an FPVA can be significantly higher than that for an SPVA with the same benefit.

In our numerical illustrations, we assume that a VA policyholder is 50 years old at the time of issue and the policyholder's mortality follows the CIA 1997-2004 Age Last Birthday Aggregate Ultimate Life Table given in the Table 2.

Fair M&E fees are calculated according to formula (20). We assume the uniform distribution of death within each year of age (the UDD Assumption) in calculation. We denote by R the ratio of the initial contribution to the amount of annual contribution, i.e. $R = A_0/k$. Thus, $R = 0$ represents the case of no initial contribution and $R = \infty$ corresponds to an SPVA. The period of the accumulation phase ranges from 10 years to 25 years and the volatility of the underlying mutual fund ranges from 10% (low volatility) to 25% (high volatility). In the following tables, we present fair M&E fees for interest rates of 3% and 6%.

As shown in the both tables, the fair M&E fee is a decreasing function of variable R. Recall that R is the ratio of the initial contribution to the annual contribution and $R = \infty$ corresponds to an SPVA. The decrease in M&E fee implies that an FPVA is always more costly than a similar SPVA. Moreover, the numerical results show that sometimes the difference in cost is significant and an FPVA can cost up to 45% more than a similar SPVA.

TABLE 3

FAIR M&E FEES FOR A 50 YEARS OLD MALE

σ	R	$r = 0.03$, $g = 0$				$r = 0.06$, $g = 0.03$				$r = 0.06$, $g = 0$			
		$T = 10$	15	20	25	$T = 10$	15	20	25	$T = 10$	15	20	25
10%	0	47.22	24.09	14.19	9.01	45.36	22.84	12.55	8.56	6.37	1.88	0.67	0.31
	1	41.70	21.68	12.64	8.72	40.03	20.43	11.47	7.23	4.67	1.53	0.56	0.27
	5	34.60	17.17	10.86	6.49	33.68	16.89	9.37	6.40	3.03	1.05	0.41	0.20
	10	31.68	15.64	9.32	5.76	31.28	15.07	9.07	6.00	2.99	0.89	0.34	0.19
	∞	30.08	14.55	8.28	5.33	30.08	14.55	8.28	5.33	2.87	0.70	0.31	0.16
15%	0	128.99	75.12	49.11	34.55	124.51	70.96	45.68	32.15	33.86	13.89	6.89	3.42
	1	116.73	69.20	45.32	32.49	113.29	65.98	42.88	30.05	29.47	11.98	6.16	3.33
	5	98.29	57.78	38.86	27.67	96.84	55.56	36.79	26.20	22.24	9.65	4.78	2.25
	10	92.67	53.10	35.14	25.37	90.94	51.85	33.68	23.94	20.26	8.13	3.66	2.23
	∞	87.31	48.55	30.94	21.78	87.31	48.55	30.94	21.78	19.00	7.27	3.43	2.00
20%	0	231.41	141.69	96.45	70.89	225.08	135.57	91.07	65.58	81.36	39.14	21.69	13.07
	1	210.96	130.78	90.70	66.95	204.43	125.63	85.28	61.64	71.18	34.84	20.07	12.56
	5	180.33	111.11	76.09	57.47	176.70	107.89	72.59	53.16	57.73	28.19	15.68	9.41
	10	169.08	100.59	67.53	51.23	167.91	99.31	65.81	47.05	53.3	24.95	13.38	8.73
	∞	160.47	94.23	62.77	45.70	160.47	94.23	62.77	45.70	49.23	22.16	11.77	7.27
25%	0	346.40	216.49	149.10	112.08	337.27	207.46	141.79	102.93	143	74.11	43.28	28.47
	1	316.13	201.36	139.41	105.33	308.20	193.29	133.02	97.21	126.64	66.72	39.38	26.11
	5	271.48	169.11	114.96	88.72	266.92	164.92	111.49	81.59	103.80	54.29	31.63	20.39
	10	250.86	148.78	109.53	81.15	249.42	147.50	106.88	78.52	94.91	48.28	25.95	18.71
	∞	241.86	145.69	99.00	73.23	241.86	145.69	99.00	73.23	89.39	43.47	24.70	15.84

TABLE 4

FAIR M&E FEES FOR A 50 YEARS OLD FEMALE

σ	R	$r = 0.03$, $g = 0$				$r = 0.06$, $g = 0.03$				$r = 0.06$, $g = 0$			
		$T = 10$	15	20	25	$T = 10$	15	20	25	$T = 10$	15	20	25
10%	0	47.50	24.59	14.28	9.32	45.70	23.15	12.90	7.94	5.92	1.81	0.63	0.28
	1	41.98	22.18	12.90	8.09	40.37	20.74	11.81	7.54	4.86	1.48	0.55	0.25
	5	33.89	17.67	10.73	6.80	33.06	17.20	9.99	5.77	3.56	1.02	0.38	0.19
	10	31.96	16.14	8.85	6.07	31.62	15.38	9.03	5.58	2.86	0.83	0.33	0.17
	∞	29.97	14.40	8.12	5.13	29.97	14.40	8.12	5.13	2.81	0.73	0.24	0.12
15%	0	128.29	74.65	48.61	33.93	124.85	71.27	46.02	31.52	34.14	13.80	6.77	3.68
	1	116.03	68.73	45.79	31.87	112.67	65.36	42.25	29.43	28.76	12.11	5.86	3.32
	5	97.59	57.31	38.36	27.04	96.22	55.87	36.17	25.57	22.52	9.15	3.96	1.98
	10	91.96	52.63	34.64	23.81	91.28	52.16	33.05	23.31	20.54	7.63	3.49	1.87
	∞	86.98	48.22	30.58	21.24	86.98	48.22	30.58	21.24	18.86	7.14	3.30	1.83
20%	0	230.71	141.22	95.95	70.27	224.46	134.94	90.45	64.96	81.64	39.61	21.41	13.38
	1	210.25	130.32	90.20	65.39	204.77	125.00	84.65	61.02	71.47	35.30	19.41	11.94
	5	179.63	110.64	75.59	54.97	177.05	107.26	71.96	52.54	57.03	27.69	15.62	8.79
	10	168.38	100.12	67.03	44.98	167.28	98.68	65.18	46.42	52.60	24.45	12.81	8.10
	∞	159.97	93.69	62.13	44.74	159.97	93.69	62.13	44.74	49.01	21.85	11.54	6.96
25%	0	345.70	216.02	148.60	110.52	336.64	206.83	141.16	102.30	143.28	73.59	43.28	27.84
	1	316.41	200.89	138.91	100.02	307.58	192.67	131.43	95.64	126.93	67.19	39.38	25.48
	5	270.78	168.64	114.46	82.47	266.30	164.30	110.87	80.03	103.10	53.79	31.63	19.77
	10	250.16	148.31	109.06	79.69	248.80	146.87	105.94	77.07	95.19	46.84	25.37	18.08
	∞	241.21	144.90	98.02	71.72	241.21	144.90	98.02	71.72	88.99	43.03	24.23	15.26

6. CONCLUDING REMARKS

In this paper, we investigated the valuation problem of flexible premium variable annuities. Assuming a geometric Brownian motion for the underlying mutual fund of a variable annuity, we used a PDE approach to solve the problem. We found that the cost of the return of premium guarantee for flexible premium variable annuities is significantly higher than the cost of the same guarantee for single premium variable annuities. This suggests that the current pricing practice in the North America needs to be re-examined. Moreover, the delta hedging strategy we have identified implies that when designing a hedging program for an FPVA portfolio, one needs to use not only the current market information but also the past market information.

We recognize the shortcoming that we assume constant interest rate and constant volatility of the underlying mutual fund in our paper. As a VA contract is often long term, it would be more realistic to consider both the interest rate and volatility stochastic. We intend to extend our work in this paper to incorporate stochastic interest rates and stochastic volatility in our future research.

Appendix

A. Proofs of Propositions in Section 4

We use theory of stochastic orders to prove Proposition 3. A nonnegative random variable X is said to be smaller than another nonnegative random variable Y in the convex order (denoted by $X \leq_{cx} Y$), if $\mathbb{E}[X] = \mathbb{E}[Y]$ and the following inequality

$$\mathbb{E}[(d-X)_+] \leq \mathbb{E}[(d-Y)_+], \ \forall d \geq 0 \tag{22}$$

holds. See Shaked and Shanthikumar (2007) for more details of stochastic orders.

To prove Proposition 3, we first prove two lemmas.

Lemma 1. *Denote*

$$H_t \triangleq \exp\left\{(\tilde{r} - \frac{\sigma^2}{2})t + \sigma W_t\right\}, \ \forall t \geq 0, \ \tilde{r} \in \mathbb{R} \ \textit{and} \ \sigma > 0. \tag{23}$$

Then H_t is increasing in σ in the convex order.

Proof. It is easy to see that $\mathbb{E}[H_t] = exp\{\tilde{r}t\}$ does not depend on volatility σ. Further, it is well-known that the price of a European put option under the Black-Scholes model has a positive vega, which implies that $\mathbb{E}[(d-H_t)_+]$ is increasing in σ. These two properties together imply that H_t is increasing in σ in the convex order. □

With Lemma 1, we can prove the following lemma.

Lemma 2. *For any finite positive measure π defined on $(0,t]$, $\int_0^t H_s \pi(\mathrm{d}s)$ is increasing in σ in the convex order.*

Proof. For any partition $0 = t_0 < t_1 < \cdots < t_n = t$ over the interval $[0,t]$, let $a_i = \pi((t_{i-1}, t_i])$ for $1 \leq i \leq n$. We first show that partial sum $\sum_{i=1}^n a_i H_{t_i}$ is increasing in σ in the convex order. We denote $X_i \triangleq H_{t_i} / H_{t_{i-1}}$ for any $1 \leq i \leq n$, then $\{X_i; 1 \leq i \leq n\}$ are mutually independent and each X_i is equal to $H_{t_i - t_{i-1}}$ in distribution. Thus, it follows from Lemma 1 that $\{X_i; 1 \leq i \leq n\}$ are increasing in σ in the convex order. Further, Theorem 7.A.24 in Shaked and Shanthikumar (2007) shows that random vector $\mathbf{X} \triangleq (X_1, \cdots, X_n)$ increases in σ in the sense of componentwise convex order, which is equivalent to saying that $\mathbb{E}[\phi(\mathbf{X})]$ increases in σ for any componentwise convex function $\phi : \mathbb{R}^n \to \mathbb{R}$, provided that the expectation exists. It is easy to see that $\psi(\mathbf{x}) \triangleq d - \sum_{i=1}^n a_i \prod_{j=1}^i x_j$ is componentwise linear and thus componentwise convex for any $d \geq 0$. Let $\phi(\mathbf{x}) = \max\{\psi(\mathbf{x}), 0\}$. Then, $\phi(\mathbf{x})$ is componentwise convex. Consequently, we have that $\mathbb{E}[(d-\sum_{i=1}^{n} a_i H_{t_i})_+] = \mathbb{E}[\phi(\mathbf{X})]$ is increasing in σ. It is also easy to see that $\mathbb{E}[\sum_{i=1}^{n} a_i H_{t_i}]$ does not depend on σ. These two properties together imply $\sum_{i=1}^{n} a_i H_{t_i}$ is increasing in σ in the convex order.

Next, since Brownian motion has almost surely continuous paths, the partial sum converges almost surely to the integral $\int_0^t H_s \pi(\mathrm{d}s)$ as $\max_{1\le i\le n}|t_i - t_{i-1}| \to 0$. It is obvious that the convex order property preserves after taking the limit. The proof is thus complete. □

Proof of Proposition 3. Since $\{W_t - W_{t-s};\ 0\le s\le t\}$ and $\{W_s;\ 0\le s\le t\}$ are equal in distribution, A_t expressed in (3) has the same distribution as

$$A_0 e^{(r-c-\frac{\sigma^2}{2})t+\sigma W_t} + k\int_0^t e^{(r-c-\frac{\sigma^2}{2})s+\sigma W_s}\,\mathrm{d}s.$$

We remark that a similar identity in discrete time is given in Proposition 1 of Schrager and Pelsser (2004). Thus, $\Pi(0,t)$ can be written as

$$\Pi(0,t) = e^{-rt}\mathbb{E}\left[\left(G(t) - A_0 e^{(r-c-\frac{\sigma^2}{2})t+\sigma W_t} - k\int_0^t e^{(r-c-\frac{\sigma^2}{2})s+\sigma W_s}\,\mathrm{d}s\right)_+\right].$$

Recall that $G(t)$ is strictly increasing in g, then it follows from the above equation that $\Pi(0,t)$ is also strictly increasing in g. Moreover, $\Pi(0,t)$ is also strictly increasing in σ due to Lemma 2, where the positive measure π has density k over $[0,t)$ and a mass of A_0 at t. As a result, $L_0(c)$ is strictly increasing in both g and σ because of (19).

To show that $L_0(c)$ is decreasing in c and c^* exists, denote

$$K(t) \triangleq e^{-rt}\mathbb{E}\left[\left(A_0 e^{(r-c-\frac{\sigma^2}{2})t+\sigma W_t} + k\int_0^t e^{(r-c-\frac{\sigma^2}{2})s+\sigma W_s}\,\mathrm{d}s\right)\vee G(t)\right],\ t\ge 0,$$

where $x\vee y = \max\{x, y\}$, then we have

$$\Pi(0,t) = K(t) - (A_0 + \frac{k}{r-c})\, e^{-ct} + \frac{k}{r-c}\, e^{-rt}.$$

And $L_0(c)$ in (19) can be rewritten by

$$L_0(c) = \int_0^T K(t)\, {}_tp_x\, \mu_{x+t}\,\mathrm{d}t + K(T)\, {}_Tp_x - A_0 - k\bar{a}_{x:\bar{T}|r}.$$

From the definition of $K(t)$, we know $K(t)$ is strictly decreasing in c and so is $L_0(c)$.

Finally, since $\lim_{c\to\infty} K(t) = e^{-rt}G(t)$, then the above equation implies

$$\begin{aligned}\lim_{c\to\infty} L_0(c) &= -A_0(r-g)\,\bar{a}_{x:\bar{T}|(r-g)} - \frac{k}{g}(r-g)(\bar{a}_{x:\bar{T}|r-g} - \bar{a}_{x:\bar{T}|r})\\ &\le -A_0(r-g)\,\bar{a}_{x:\bar{T}|(r-g)} < 0.\end{aligned}$$

On the other hand, it is obvious from (19) that $L_0(0) > 0$. Consequently, as $L_0(c)$ is continuous, the equation (20) has a unique solution c^*. Recalling that $L_0(c)$ is strictly increasing in g and σ, c^*, as a function of g and σ, has the same property.

Proof of Proposition 4. According to (3), it is easy to show that for any $0 \le u \le t$, we have

$$\mathbb{E}[e^{-r(t-u)}A_t \,|\, \mathcal{F}_u] = (A_u + \frac{k}{r-c})\, e^{-c(t-u)} - \frac{k}{r-c}\, e^{-r(t-u)}.$$

Thus, from (18) we have

$$\begin{aligned}
&\mathbb{E}[\mathcal{L}_0 \,|\, \mathcal{F}_u] \\
&= {}_T p_x \,\mathbb{E}[e^{-rT}P(T)\,|\,\mathcal{F}_u] + \int_u^T {}_t p_x\, \mu_{x+t}\, \mathbb{E}[e^{-rt}P(t)\,|\,\mathcal{F}_u]\mathrm{d}t - c\int_u^T {}_t p_x\, \mathbb{E}[e^{-rt}A_t\,|\,\mathcal{F}_u]\mathrm{d}t \\
&\quad + \int_0^u {}_t p_x\, \mu_{x+t}\, e^{-rt}P(t)\mathrm{d}t - c\int_0^u {}_t p_x\, e^{-rt}A_t\,\mathrm{d}t \\
&= {}_T p_x \,\mathbb{E}[e^{-rT}P(T)\,|\,\mathcal{F}_u] + \int_u^T {}_t p_x\, \mu_{x+t}\, \mathbb{E}[e^{-rt}P(t)\,|\,\mathcal{F}_u]\mathrm{d}t - ce^{-ru}A_u\int_u^T {}_t p_x\, e^{-c(t-u)}\mathrm{d}t \\
&\quad + \int_0^u {}_t p_x\, \mu_{x+t}\, e^{-rt}P(t)\mathrm{d}t - c\int_0^u {}_t p_x\, e^{-rt}A_t\,\mathrm{d}t - \frac{kc}{r-c}\, e^{-ru}\int_u^T {}_t p_x(e^{-c(t-u)} - e^{-r(t-u)})\mathrm{d}t
\end{aligned}$$

for any $0 \le u \le T$. Since

$$\begin{aligned}
\mathrm{d}(e^{-ru}A_u) &= -re^{-ru}A_u\mathrm{d}u + e^{-ru}\mathrm{d}A_u \\
&= (k - cA_u)\, e^{-ru}\mathrm{d}u + \sigma e^{-ru}A_u\,\mathrm{d}W_u \\
&= (k - cA_u)\, e^{-ru}\mathrm{d}u + A_u/S_u\mathrm{d}(e^{-ru}S_u)
\end{aligned}$$

and $\{\mathbb{E}[\mathcal{L}_0|\mathcal{F}_u];\ 0 \le u \le T\}$ is a martingale process, Ito's Lemma yields

$$\begin{aligned}
&\mathrm{d}\mathbb{E}[\mathcal{L}_0|\mathcal{F}_u] \\
&= {}_T p_x\,\mathrm{d}\mathbb{E}[e^{-rT}P(T)\,|\,\mathcal{F}_u] + \int_u^T {}_t p_x\,\mu_{x+t}\mathrm{d}(\mathbb{E}[e^{-rt}P(t)\,|\,\mathcal{F}_u])\mathrm{d}t \\
&\qquad\qquad - c\int_u^T e^{-c(t-u)}{}_t p_x\,\mathrm{d}t \times A_u/S_u\mathrm{d}(e^{-ru}S_u) \\
&= \left({}_T p_x\, h_T(u) + \int_u^T {}_t p_x\,\mu_{x+t}h_t(u)\mathrm{d}t - c\times {}_u p_x\,\bar{a}_{x+u:\overline{T-u}|c}\,A_u/S_u\right)\mathrm{d}(e^{-ru}S_u) \\
&= \Delta(u)\mathrm{d}(e^{-ru}S_u),
\end{aligned}$$

where the second equality is due to Proposition 2. Therefore, the proof is complete.

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