AVERAGE VALUE-AT-RISK MINIMIZING REINSURANCE UNDER WANG'S PREMIUM PRINCIPLE WITH CONSTRAINTS

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ABSTRACT

In the present work, we study the optimal reinsurance decision problem in which the Average Value-at-Risk of the retained loss is minimized under Wang's premium principle and is also subject to either (1) a budget constraint on reinsurance premium, or (2) a reinsurer's probabilistic benchmark constraint of his potential loss. We show that the optimal reinsurance is a single-insurance layer under Constraint (1), and a cap insurance or a double-insurance layer under Constraint (2); moreover, under Constraint (2), we further establish that under most common circumstances (see Remark after Theorem 3), a cap insurance will suffice to be optimal. Finally, some numerical illustrations will be provided.

KEYWORDS

Optimal reinsurance; Average Value-at-Risk; Value-at-Risk; Wang's premium principle; Single and double insurance layers.

1. INTRODUCTION

Insurance and reinsurance are effective risk management tools that are primarily used to protect against contingent losses of market participants; their use cannot reduce the underlying (non-hedgeable) risk but only shift a portion of it from the risk-bearer to the insurance seller. In the last fifty years, both theoretical and empirical studies have been dedicated to determine the most favorable form of insurance to both parties. In his seminal work, Arrow [2] used the expected utility (Neumann and Morgenstern [31]) to quantify the risk averse insured's satisfaction of his own uncertain terminal wealth, and under the actuarial pricing principle, he established the optimality of stop-loss insurances which maximizes the insureds' expected utilities. In earlier time, Borch [7] also obtained similar result when utility is replaced by the variance of the terminal loss. In other words, a rational risk averse insured prefers full protection on potentially large losses to that on small amount of losses even though

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the probability of their occurrence is not negligible. Similar results had been obtained from the perspective of Pareto optimality, for example see the works of Borch [8], Buhlmann and Jewell [9], and Raviv [32]. Further extensions of their models and settings, subject to different objective functions/criteria or premium pricing principles, or with a relaxation of constraints on feasible insurances, or with an additional budget constraints, can also be found in the works of Balbás et al. [4], Blazenko [6], Gerber [22], Gollier [24], Guerra and Centeno [23], Kaluszka [27, 28, 29], Moore and Young [30], Sung et al. [34] and Young [40], and the references therein. Regarding the determination of insurance premium, Wang et al. [37] and Wang [36] proposed a list of natural axioms which suggests that a "sounding" premium price should be a Choquet integral of the indemnity which has a close connection with the dual theory of risk first proposed by Yaari [38]. As an application, Young [40] considered the problem of maximization of the expected utility of the terminal wealth of an insurer under Wang's premium principle.

Since the last decade, the theory of risk measure has become a popular topic in both research and practice in financial economics. The paper by Artzner et al. [3] on coherent measures of risk pioneered the axiomatic approach to the theory of risk measures. Since then, many authors have made various contributions in this direction. As a generalization of coherent measures of risk, the notion of convex risk measures was studied by Föllmer and Schied [19], Frittelli and Rosazza Gianin [21] and Heath and Ku [25]; for further analysis, see Delbaen [14] and [15], Föllmer and Schied [20]. As one of the most popular measures of risk, Value-at-Risk ($V@R$) has achieved the highest status of being written into industry regulations (*Basel II* and *Solvency II*). However, it suffers from being unstable and difficult to numerically compute without normality assumption of the underlying losses. Besides, $V@R$ only measures the contingency of the occurrence of the underlying potential loss but not the "average" magnitude of the loss. Another limitation of the $V@R$ is its lack of subadditivity, and hence not coherent. These limitations of $V@R$ have already been pointed out by Embrechts [18] and Acerbi and Tasche [1]. To remedy these shortcomings, an alternative risk measure that does quantify the losses that might be encountered in the tail is Average Value-at-Risk ($AV@R$); indeed, $AV@R$ is a law-invariant coherent risk measure[1]. More details of $AV@R$ can be found in, for example, Acerbi and Tasche [1], Delbaen [15], and Rockafellar and Uryasev [33]. Recently, optimal reinsurance decision problem

[1] Recall that a risk measure $\rho : L^\infty \to \mathbb{R}$ is called a coherent risk measure if the following axioms are satisfied for any $Y_1, Y_2 \in L^\infty$:

- *Monotonicity* If $Y_1 \le Y_2$, then $\rho(Y_1) \le \rho(Y_2)$.
- *Translation Invariance* For any $m \in \mathbb{R}$, $\rho(Y_1 + m) = \rho(Y_1) + m$.
- *Subadditivity* $\rho(Y_1 + Y_2) \le \rho(Y_1) + \rho(Y_2)$.
- *Positive Homogeneity* For any $\lambda \ge 0$, $\rho(\lambda Y_1) = \lambda \rho(Y_1)$.

Besides, a risk measure ρ is also said to be law-invariant if $\rho(Y_1) = \rho(Y_2)$ whenever Y_1 and Y_2 have the same distribution under the real-world probability measure $\mathbb{P}$.

has been revisited under different risk measures; for instance, Cai and Tan [10], and Tan et al. [35] sought for the optimal stop-loss contracts and optimal quota-share contracts under various premium pricing principles. Cai et al. [11] also considered the extension of the previous works under either Value at Risk or Conditional Tail Expectation of retained loss in which all reinsurances with non-decreasing convex indemnities are regarded as feasible; Cheung [12] extended their results under Wang's premium principle. Cheung et al. [13] recently resolved the long lasting optimal reinsurance decision problem under most general convex risk measures subject to the actuarial pricing principle.

In this paper, we study the optimal reinsurance decision problem such that the Average Value-at-Risk of the retained loss is minimized under Wang's premium principle and is also subject to either (1) a budget constraint on reinsurance premium, or (2) a reinsurer's probabilistic benchmark constraint of his potential loss. With no doubt, the constraint (2) is reasonable from a practical point of view; indeed, to ensure a reliable risk management of a reinsurance company, the reinsurer takes the incentive to limit his potential loss below a predetermined level, at least, in concern with Solvency II (i.e. $V@R$-based risk management, for example, see Basak and Shapiro [5]). In Section 2, we provide some preliminary results and lay down the problem formulation. Section 3 discusses the optimal reinsurance decision problem under constraint (1), and we show that a single insurance layers are optimal. Section 4 investigates the same decision problem subject to constraint (2), we show that a single insurance layer can still be optimal in most practical cases; however, if the reinsurer tightens his risk exposure by reducing his tolerance level, instead of a single insurance layer, the optimal reinsurance schedules become double insurance layers. Numerical examples will be given to supplement our theoretical results. Section 5 is the conclusion, and Section 6 contains proofs of some technical results.

2. Preliminaries and Problem Formulation

Let X be a non-negative random variable representing the potential loss of the insurance company which aims to effectively reduce its risk exposure by purchasing a reinsurance. We assume that the distribution function (resp., survival function) of X, denoted by F_X (resp., S_X), is absolutely continuous on the positive real-line. Let $I:[0,\infty)\to[0,\infty)$ be a reinsurance policy. We say that I is *feasible* if it is (1) non-decreasing and continuous, (2) $0\le I(x)\le x$ for all $x\ge 0$, and (3) also satisfies the relation:

$$I(x_1)-I(x_2)\le x_1-x_2,\quad \text{for any}\quad 0\le x_2\le x_1,$$

that is to say, I is 1-Lipschitz and hence is differentiable a.e.. The third property will be referred to as the "slowly-growing" property. Both the ceded and retained loss functions are non-decreasing such that the higher the incurred loss, the greater the loss to both the insurer and reinsurer, and hence moral hazard can be avoided; otherwise, the insurer might take the advantage of twisting the actual loss amount. Such properties were shown to be necessary for any optimal contracts in the expected utility framework under Wang's premium principle in Young [40], and they are

> (...) desirable because if the indemnity benefit were to decrease with losses, then insureds would have an incentive to underreport their losses. If the indemnity benefit were to increase more rapidly than losses increase, then insureds would have an incentive to create incremental losses. (These two moral hazards exist when an insurer can costlessly verify losses that are reported, but an insured can hide a loss by not reporting it).

Note that the same class of feasible reinsurance contracts was also considered in Denuit and Vermandele [16], and we do not require reinsurance contracts to be convex, as opposed to Cai et al. [11] and Cheung [12].

Let $\mathcal{I}$ be the set of all feasible reinsurances. For any $\hat{\alpha} \in (0,1)$, we define $AV@R_{\hat{\alpha}}(X)$ as the $\alpha-$level Average Value-at-Risk of X, i.e.

$$AV@R_{\hat{\alpha}}(X) \triangleq \frac{1}{1-\hat{\alpha}} \int_{\hat{\alpha}}^{1} V@R_p(X)\,dp,$$

where $V@R_p(X) \triangleq F_X^{-1}(p)$ is the Value-at-Risk ($V@R$) of X at the probability level p. Denote $\alpha \triangleq 1 - \hat{\alpha}$ and $a \triangleq S_X^{-1}(\alpha)$, a change-of-variable gives:

$$AV@R_{\hat{\alpha}}(X) = \frac{1}{\alpha}\int_0^{\alpha} S_X^{-1}(p)\,dp = \frac{1}{\alpha}\int_a^{\infty} x\,dF_X(x). \tag{1}$$

For more properties of V@R and AV@R, see Dhaene et al. [17].

The objective of this paper is to seek for an optimal reinsurance contract within the class $\mathcal{I}$ that minimizes the $AV@R$ of the insurer's retained loss $X - I(X) + P_I$, where P_I is the reinsurance premium of I. We assume that P_I is calibrated by *Wang's Premium Principle*:

$$P_I = (1+\theta)\int_0^{\infty} g \circ S_{I(X)}(t)\,dt$$

for some *distortion* $g : [0,1] \to [0,1]$ and a risk loading $\theta \geq 0$. Here, g is non-decreasing, differentiable and concave on $[0,1]$, with $g(0) = 0$ and $g(1) = 1$. In Section 3, we consider the problem when there is a budget constraint on premium; while in Section 4, the same problem is considered when there is only a probabilistic benchmark constraint on reinsurer's risk. As a remark, under a special case of Wang's Premium Principle where $g(x) = x$, known as the *actuarial pricing principle*, and without imposing a budget constraint on the underlying reinsurance premium (a "free premium" setting), a similar problem has been considered in a recent work of Cheung et al. [13] in which the optimality of stop-loss contracts has been established.

3. Optimal Reinsurance with Budget Constraint on Premium

In this section, we study the optimal reinsurance decision problem of minimizing the AV@R of retained loss under a budget constraint on Wang's premium charged:

$$\begin{aligned} \min_{I\in\mathcal{I}} \quad & AV@R_{\hat{\alpha}}(X - I(X)) \\ \text{such that} \quad & (1+\theta)\int_0^\infty g \circ S_{I(X)}(t)\,dt \le P. \end{aligned} \tag{2}$$

In this formulation, $\hat{\alpha} \in (0,1)$ and $P > 0$ are some fixed constants. It follows from (1) that Problem 2 can be rewritten as

$$\begin{aligned} \min_{I\in\mathcal{I}} \quad & \frac{1}{\alpha}\int_a^\infty x - I(x)\,dF_X(x) \\ \text{such that} \quad & (1+\theta)\int_0^\infty I(x) g' \circ S_X(x)\,dF_X(x) \le P. \end{aligned} \tag{3}$$

For the expression on the left hand side of the budget constraint, by using the simple change-of-variable formula and integration-by-parts, we can obtain an alternative expression which can make our later analysis easier:

$$\int_0^\infty g \circ S_{I(X)}(t)\,dt = \int_0^\infty I(x) g' \circ S_X(x)\,dF_X(x) = \int_0^\infty I'(x)\, g \circ S_X(x)\,dx.$$

In order to establish the explicit form of an optimal solution of Problem 3 (or equivalently, Problem 2), two key steps will be carried out. Firstly, given an arbitrary feasible reinsurance I_0, we show that I_0 can always be modified to another feasible reinsurance that lead to a smaller $AV@R$ of the retained loss but requires less premium. Secondly, a certain number of control parameters for the modification will then be determined.

Proposition 1. *For any feasible reinsurance $I_0 \in \mathcal{I}$, there exists another $I^* \in \mathcal{I}$ in the form*

$$I^*(x) = (x - d_1)^+ - (x - d_2)^+,$$

for some $d_1 \in [0, a]$ and $d_2 \in [a, \infty]$ which may depend on I_0, such that

$$AV@R_{\hat{\alpha}}(X - I^*(X)) \le AV@R_{\hat{\alpha}}(X - I_0(X)),$$

and $P_{I^* \le P}$, where *P_{I^*} is calibrated by Wang's premium principle.*

Proof. Denote

$$\mathcal{I}_a(I_0) \triangleq \{I \in \mathcal{I} : I(a) = I_0(a)\}.$$

We look for another contract from $\mathcal{I}_a(I_0)$ which does not need an additional premium yet has a smaller $AV@R$ of the retained loss. We can achieve this by modifying carefully I_0 on the intervals $[0,a)$ and $(a,\infty]$ separately. Indeed, it is clear that the objective function is not affected by values of a contract on $[0,a)$, we can simply modify I_0 on $[0,a)$ to $I(x)=(x-a+I_0(a))^+$ which is the smallest, and hence the cheapest possible contract on $[0,a)$ within the class $\mathcal{I}_a(I_0)$. To modify I_0 on the interval $(a,\infty]$, we first consider the following minimization problem:

$$\begin{aligned} \min_{I\in\mathcal{I}_a(I_0)} \quad & \frac{1}{\alpha}\int_a^\infty x - I(x)dF_X(x) \\ \text{such that} \quad & \int_a^\infty I(x)g'\circ S_X(x)dF_X(x) \le P_1, \end{aligned} \tag{4}$$

where $P_1 = \int_a^\infty I_0(x)g'\circ S_X(x)dF_X(x)$. The optimal solution is a contract in the form

$$I^*(x) = (x-a+I_0(a))^+ - (x-d_2)^+, \ \ x \ge a,$$

for some $d_2 \in [a,\infty]$. A proof of this optimality, based on Lagrangian duality approach, can be found in Section 6 A.1. Combining the two modifications, we obtain the contract

$$I^*(x) = (x-a+I_0(a))^+ - (x-d_2)^+, \ \ x \ge 0,$$

for some $d_2 \in [a,\infty]$ that lead to a smaller $AV@R$ of the retained loss. Finally, we verify that the premium required by I^* is less than that of I_0:

$$\begin{aligned} P_{I^*} &= (1+\theta)\int_0^\infty I^*(x)g'\circ S_X(x)dF_X(x) \\ &\le (1+\theta)\int_0^a I_0(x)g'\circ S_X(x)dF_X(x) + (1+\theta)P_1 = P_{I_0}. \end{aligned}$$

The result follows. □

Since the modification process as stated in Proposition 1 can be adopted for any feasible reinsurance contract, the optimal solution of Problem 3 must be in the form:

$$I^*(x) = (x-d_1)^+ - (x-d_2)^+, \text{ for some } d_1\in[0,a] \text{ and } d_2 \in [a,\infty]. \tag{5}$$

It should be noted that Expression (5) may represent a cap insurance, full insurance, stop-loss insurance or proper insurance layer, depending on different values of the parameters d_1 and d_2. As a consequence of Proposition 1, the search for optimal reinsurances reduces to the determination of the optimal parameters d_1 and d_2. To this end, by substituting (5) into Problem 3, the latter becomes:

$$\min_{\substack{0\le d_1\le a\\ a\le d_2\le\infty}} \quad \frac{1}{\alpha}\int_a^\infty x dF_X(x) - (a-d_1) - \frac{1}{\alpha}\int_a^{d_2} S_X(x)\,dx$$
$$\text{such that} \quad (1+\theta)\int_{d_1}^{d_2} g\circ S_X(x)\,dx \le P.$$

This minimization problem can be solved by analyzing its dual problem:

$$\max_{\lambda\ge 0} D_2(\lambda),$$

where $D_2(\lambda)$ is the Lagrangian dual function defined by:

$$\begin{aligned} D_2(\lambda) &= \min_{\substack{0\le d_1\le a\\ a\le d_2\le\infty}} \left\{\frac{1}{\alpha}\int_a^\infty x dF_X(x) - (a-d_1)\right. \\ &\qquad \left. -\frac{1}{\alpha}\int_a^{d_2} S_X(x)\,dx + \lambda\left(\int_{d_1}^{d_2} g\circ S_X(x)\,dx - \frac{P}{1+\theta}\right)\right\} \\ &= \min_{\substack{0\le d_1\le a\\ a\le d_2\le\infty}} L_\lambda(d_1,d_2) - a + \frac{1}{\alpha}\int_a^\infty x dF_X(x) - \frac{\lambda P}{1+\theta}, \end{aligned}$$

where

$$\begin{aligned} L_\lambda(d_1,d_2) \triangleq\ & d_1 + \int_{d_1}^{a} \lambda g\circ S_X(x)\,dx \\ & + \int_a^{d_2} \lambda g\circ S_X(x) - \frac{1}{\alpha} S_X(x)dx. \end{aligned} \tag{6}$$

For every $\lambda\ge 0$, denote by $(d_1^\lambda, d_2^\lambda)$ the minimizer of L_λ. The existence of $(d_1^\lambda, d_2^\lambda)$ is shown in Section 6 A.2. Before we move on, we first make a remark on the particular case where the distortion function g is linear, i.e. $g(x)=x$. In this case, the premium charged is calibrated under the actuarial pricing principle, and the corresponding optimal reinsurance will be in the form:

$$I^*(x) = \begin{cases} 0, & \text{when } \lambda > \frac{1}{\alpha}; \\ \left(x - S_X^{-1}(\frac{1}{\lambda})\right)^+, & \text{when } \frac{1}{S_X(0)} < \lambda < \frac{1}{\alpha}; \\ x, & \text{when } 0 < \lambda < \frac{1}{S_X(0)}. \end{cases}$$

Therefore, the optimal reinsurance is reduced to no insurance or a stop-loss insurance, and this result agrees with Cheung et al. [13].

We now turn back to the case of general distortion g. Without loss of generality, we assume that $g'(0) = +\infty$[2]. By applying the results in Section 6 A.2, the dual problem $\max_{\lambda \geq 0} D_2(\lambda)$ can be solved as follows.

Theorem 1. *The optimal reinsurance of Problem 2 is either an insurance layer or a cap insurance.*

Proof. Using the assumption that $g'(0) = +\infty$ and the fact that S_X is continuous on $\mathbb{R}$, we have

$$\frac{1}{\alpha g'(0)} = 0 < 1 = \frac{1}{g \circ S_X(0)}.$$

Recalling (18) and (19) for the expressions for d_1^λ and d_2^λ as given in Section 6 A.2:

$$(d_1^\lambda, d_2^\lambda) = \begin{cases} (a, a), & \text{when } \lambda > \frac{1}{g(\alpha)}; \\ (S_X^{-1} \circ g^{-1}(\frac{1}{\lambda}), x_1^\lambda), & \text{when } 1 \leq \lambda \leq \frac{1}{g(\alpha)}; \\ (0, x_1^\lambda), & \text{when } 0 \leq \lambda < 1. \end{cases} \tag{7}$$

Here x_1^λ is defined in Section 6 A.1, and $x_1^\lambda = \infty$ if and only if $\lambda = 0$. It follows that

$$D_2(\lambda) = \lambda\left(\int_{d_1^\lambda}^{d_2^\lambda} g \circ S_X(x)dx - \frac{P}{1+\theta}\right) + d_1^\lambda - a + \frac{1}{\alpha}\int_a^\infty x dF_X(x) - \frac{1}{\alpha}\int_a^{d_2^\lambda} S_X(x)dx,$$

and its derivative is given by

$$D_2'(\lambda) = \int_{d_1^\lambda}^{d_2^\lambda} g \circ S_X(x)dx - \frac{P}{1+\theta}.$$

Since d_1^λ and d_2^λ are respectively non-decreasing and non-increasing in λ (see Section 6 A.1), it follows that $D_2'(\lambda)$ is non-increasing in λ, and so $D_2(\lambda)$ is concave. If P is not less than the premium charged for the full insurance $I(x) = x$, the budget constraint in Problem (2) would not be effective; without loss of generality, we therefore assume that $P < (1+\theta)\int_0^\infty g \circ S_X(x)dx$, and hence

$$\lim_{\lambda \downarrow 0} D_2'(\lambda) = \int_0^\infty g \circ S_X(x)dx - \frac{P}{1+\theta} > 0.$$

[2] This technical assumption is imposed only for the sake of convenience of the presentation of this paper. The form of the optimal contracts remains the same for the general case without such assumption.

On the other hand, $D_2'(\lambda) = -\frac{P}{1+\theta} < 0$ when $\lambda > \frac{1}{g(\alpha)}$. Therefore, by continuity, there must exists a $\lambda^* \in (0, \frac{1}{g(\alpha)})$ such that $D_2'(\lambda^*) = 0$; together with the concavity of $D_2(\lambda)$, it can achieve its maximum at λ^*. Hence, using Proposition 1, we conclude that the optimal reinsurance to Problem 2 is given by $I^*(x) = (x - d_1^{\lambda^*})^+ - (x - d_2^{\lambda^*})^+$, where $(d_1^{\lambda^*}, d_2^{\lambda^*})$ can be evaluated by (7) at λ^*, and hence I^* is either an insurance layer or a cap insurance.

In the rest of this section, we provide two numerical examples which demonstrate that both cap insurance and insurance layer could serve as optimal. Suppose that the loss X faced by the insured follows an exponential distribution and the distortion function is a power function; more precisely, we take

$$S_X(x) = \begin{cases} e^{-mx}, & \text{when } x \geq 0; \\ 1, & \text{when } x < 0; \end{cases}$$

and $g(x) = x^k$ for some $m > 0$ and $k \in (0,1)$. Under these assumptions, the objective function is

$$\begin{aligned} AV@R_{\hat{\alpha}}(X - I(X)) &= \frac{1}{\alpha}\int_a^\infty x dF_X(x) - (a - d_1) - \frac{1}{\alpha}\int_a^{d_2} S_X(x)dx \\ &= \frac{1}{\alpha}\int_a^\infty mxe^{-mx}dx - a + d_1 - \frac{1}{\alpha}\int_a^{d_2} e^{-mx}dx \\ &= d_1 + \frac{1}{m\alpha}e^{-md_2}, \end{aligned}$$

and the budget constraint is

$$P = (1+\theta)\int_{d_1}^{d_2} e^{-kmx}dx = (1+\theta)\frac{1}{mk}(e^{-mkd_1} - e^{-mkd_2}).$$

Express d_2 in term of d_1 via

$$e^{-md_2} = \left(e^{-mkd_1} - \frac{mkP}{1+\theta}\right)^{\frac{1}{k}},$$

then

$$AV@R_{\hat{\alpha}}(X - I(X)) = d_1 + \frac{1}{m\alpha}\left(e^{-mkd_1} - \frac{mkP}{1+\theta}\right)^{\frac{1}{k}}, \tag{8}$$

which is a real-valued function of $d_1 \in [0, a]$, and its first order derivative with respect to d_1 is simply

$$(AV@R_{\hat{\alpha}}(X - I(X)))' = 1 - \frac{1}{\alpha}\left(e^{-mkd_1} - \frac{mkP}{1+\theta}\right)^{\frac{1}{k}-1} e^{-mkd_1}. \tag{9}$$

Example 1. Let $\theta = 0$, $\alpha = 0.8879$, $k = 0.75$, $m = 0.02$ and $P = 20$. Equation (9) could be computed to be as follows:

$$(AV@R_{\hat{\alpha}}(X - I(X)))' = 1 - \frac{1}{0.8879}\left(e^{-\frac{3}{200}d_1} - \frac{3}{10}\right)^{\frac{1}{3}} e^{-\frac{3}{200}d_1},$$

which is increasing in d_1, and hence $d_1 = 0$ is the unique root of the equation

$$(AV@R_{\hat{\alpha}}(X - I(X)))' = 0.$$

Further, we can also show that the objective function

$$AV@R_{\hat{\alpha}}(X - I(X)) = d_1 + \frac{50}{0.8879}\left(e^{-\frac{3}{200}d_1} - 0.3\right)^{\frac{1}{3}}$$

achieves its minimum at $d_1^* = 0$. By solving Equation (8), $d_2^* = 23.778$ and the cap reinsurance $I^*(x) = x - (x - 23.778)^+$ is the optimal solution of Problem 2.

Example 2. Let $\theta = 0$, $\alpha = 0.7097$, $k = 0.75$, $m = 0.02$ and $P = 20$. By applying similar argument as in Example 1, the first order derivative of the objective function is given by

$$(AV@R_{\hat{\alpha}}(X - I(X)))' = 1 - \frac{1}{0.7097}\left(e^{-\frac{3}{200}d_1} - 0.3\right)^{\frac{1}{3}} e^{-\frac{3}{200}d_1}.$$

which is increasing in d_1 and has its unique root at $d_1 = 10$. It then follows that $d_2 = 38.57$, and hence $I^*(x) = (x - 10)^+ - (x - 38.57)^+$, which is an insurance layer instead of a cap insurance, is the optimal solution of Problem 2.

4. Optimal Reinsurance with Reinsurer's Risk Constraint

Solvency II, being a regulatory framework for insurance and reinsurance industry, would soon be implemented by most European countries and would also prescribe minimum capital levels (calibrated by V@R) for investment, underwriting and operational risks. The essential features of Solvency II will be very much similar to that of Basel II (see Chapter 11 in Hull [26] for more details). In the future, from the perspective of the reinsurer, any issuance of reinsurance contracts has to strictly comply with Solvency II. Under such a regulatory constraint, the set of feasible reinsurances has significantly reduced to such a form that the optimal reinsurance previously obtained in Section 3 might not be feasible anymore; this naturally leads us to study the optimal reinsurance decision problem subject to this reinsurer's risk (probabilistic) constraint on the potential terminal loss.

Let L be the threshold level of acceptable loss by the reinsurer, and β be its tolerance probability. We now formulate our new optimal reinsurance decision problem as:

$$\begin{aligned} \min_{I \in \mathcal{I}} \quad & AV@R_{\hat{\alpha}}(X - I(X) + P_I) \\ \text{such that} \quad & \mathbb{P}\big(I(X) - P_I \geq L\big) \leq \beta. \end{aligned} \tag{10}$$

Here, the probability constraint is equivalent to

$$V@R_{1-\beta}(I(X)) \leq L + P_I,$$

which can be simplified further as a functional inequality:

$$I(b) \leq L + P_I \;\text{ where }\; b \triangleq S_X^{-1}(\beta).$$

Applying the law-invariant property of $AV@R$, Problem 10 admits the following simpler formulation:

$$\begin{aligned} \min_{I \in \mathcal{I}} \quad & \left\{ \frac{1}{\alpha} \int_a^\infty x - I(x)\, dF_X(x) + P_I \right\} \\ \text{such that} \quad & I(b) \leq L + P_I. \end{aligned} \tag{11}$$

In what follows, we shall first consider the case in which the premium charged is subject to a budget constraint with linear distortion function (i.e. under the actuarial pricing principle). Secondly, we shall turn to another problem under free premium setting with general distortion.

4.1. Fixed Premium Problem Under Actuarial Pricing Principle

Suppose that $g(x) = x$ for $0 \leq x \leq 1$, and there is a budget constraint on the premium charged, i.e. $(1 + \theta)\mathbb{E}[I(X)] = P$, for some fixed $\theta \geq 0$ and fixed $P \in (0, (1 + \theta)\mathbb{E}[X])$. Given $\hat{\alpha}_1$, $\hat{\alpha}_2$ and $C > 0$, the optimal reinsurance decision problem becomes:

$$\begin{aligned} \min_{I \in \mathcal{I}} \quad & AV@R_{\hat{\alpha}_1}(X - I(X)) \\ \text{such that} \quad & (1+\theta)\mathbb{E}[I(X)] = P,\; V@R_{\hat{\alpha}_2}(I(X)) \leq C. \end{aligned} \tag{12}$$

Let $\alpha_1 \triangleq 1 - \hat{\alpha}_1$, $\alpha_2 \triangleq 1 - \hat{\alpha}_2$, $a_1 \triangleq S_X^{-1}(\alpha_1)$, $a_2 \triangleq S_X^{-1}(\alpha_2)$ and

$$\mathcal{I}_1 \triangleq \big\{ I \in \mathcal{I} : (1+\theta)\mathbb{E}[I(X)] = P \quad \text{and} \quad I(a_2) \leq C \big\}.$$

Similar to the discussion in the previous section, Problem 12 can be simplified to:

$$\min_{I \in \mathcal{I}_1} \frac{1}{\alpha_1} \int_{a_1}^{\infty} x - I(x)\, dF_X(x). \tag{13}$$

As a remark, if there is no effect from the reinsurer's risk constraint on Problem 13, its optimal solution is a stop-loss insurance I^* with the optimal deductible labelled as d^*, see Cheung et al. [13]. For $a_2 \le d^*$, the reinsurer's risk constraint is automatically satisfied by I^*, and hence the same I^* serves as the optimal solution of Problem 12. In the following, we consider the case where $a_2 > d^*$.

If $C \ge a_2 - d^*$, since the reinsurer's risk constraint is again naturally satisfied by I^* (as $I^*(a_2) = (a_2 - d^*)^+ \le C$), I^* remains the optimal solution of Problem 13. For each $k \in [0, a_2]$, we define a reinsurance contract $\hat{I}_k$ by:

$$\hat{I}_k(x) \triangleq x - (x-k)^+ + (x - a_2)^+, \ \ x \ge 0. \tag{14}$$

Lemma 1. *There exists a unique* $K \in (0, a_2 - d^*]$ *such that the equation* $(1+\theta)$ $\mathbb{E}[\hat{I}_K(x)] = P$ *holds.*

Proof. Define a function $e_1 : [0, a_2] \to [0, \infty]$ by $e_1(k) = \mathbb{E}[\hat{I}_k(X)]$. It is clear that $e_1(k)$ is continuous and strictly increasing. In addition, we also have

$$e_1(0) = \mathbb{E}\big[(X - a_2)^+\big] < \mathbb{E}\big[(X - d^*)^+\big] = \frac{P}{1+\theta}$$

and

$$e_1(a_2) = \mathbb{E}[X] > \mathbb{E}\big[(X - d^*)^+\big] = \frac{P}{1+\theta}.$$

Thus, there exists a unique $K \in (0, a_2)$ such that $e_1(K) = \mathbb{E}[\hat{I}_K(X)] = \frac{P}{1+\theta}$. For if $K > a_2 - d^*$, we have $\hat{I}_K(x) > I^*(x)$ for any $x > 0$, but this simply implies that $\mathbb{E}[\hat{I}_K(X)] > \mathbb{E}[I^*(X)] = \frac{P}{1+\theta}$, which is in conflict with the choice of K. Therefore, we conclude that $0 < K \le a_2 - d^*$. □

If $C < K$, there is simply no feasible reinsurance that can satisfy both the budget constraint and the reinsurer's risk constraint at the same time, and hence $\mathcal{I}_1 = \emptyset$. The only non-trivial case left is $K \le C \le a_2 - d^*$. For each $d \in [0, a_2 - C]$, we also define a contract I_d by

$$I_d(x) = (x-d)^+ - (x - d - C)^+ + (x - a_2)^+, \quad x \ge 0.$$

Theorem 2. *For any* $C \in [K, a_2 - d^*]$*, there exists a unique* $d^{**} \in [0, d^*)$ *such that* $I_{d^{**}}$ *is the optimal solution of Problem 13.*

Proof. The proof is divided into two steps. Firstly, we show that there exists $d^{**} \in [0, d^*)$ such that $I_{d^{**}} \in \mathcal{I}_1$. To this end, define a continuous and strictly decreasing function $e_2 : [0, d^*] \to [0, \infty)$ by $e_2(d) = \mathbb{E}[I_d(X)]$. Note that,

$$e_2(0) = \mathbb{E}[I_0(X)] \geq \mathbb{E}[I_K(X)] = \frac{P}{1+\theta}$$

and

$$e_2(d^*) = \mathbb{E}[I_{d^*}(X)] < \mathbb{E}[I^*(X)] = \frac{P}{1+\theta}.$$

By continuity, there exists a unique $d^{**} \in [0, d^*)$ such that $I_{d^{**}}$ satisfies the budget constraint. It is also clear that $I_{d^{**}}(a_2) = C$, i.e. the reinsurer's risk constraint is satisfied.

Secondly, we show that $I_{d^{**}}$ is the optimal solution of Problem 13. For each $I \in \mathcal{I}_1$, define

$$\tau_I \triangleq \sup\{x : I(x) \geq I_{d^{**}}(x)\},$$

which is well-defined since $I(a_2) \leq C$. It then follows that $I(x) \geq I_{d^{**}}(x)$ for all $x \in [0, \tau_I]$, and $I(x) \leq I_{d^{**}}(x)$ for all $x \in [\tau_I, \infty]$. Consider the following cases:

Case 1. For $a_1 < \tau_I \leq \infty$, $I(x) \geq I_{d^{**}}(x)$ for all $x \in [0, a_1] \subseteq [0, \tau_I]$, and we have

$$\begin{aligned}
&AV@R_{\alpha_1}(X - I(X)) - AV@R_{\alpha_1}\left(X - I_{d^{**}}(X)\right) \\
&= \frac{1}{\alpha_1}\int_{a_1}^{\infty} I_{d^{**}}(x) - I(x)dF_X(x) \\
&= \frac{1}{\alpha_1}(\mathbb{E}[I_{d^{**}}(X)] - \mathbb{E}[I(X)]) - \frac{1}{\alpha_1}\int_0^{a_1} I_{d^{**}}(x) - I(x)dF_X(x) \geq 0.
\end{aligned}$$

Case 2. For $0 < \tau_I \leq a_1$, $I(x) \leq I_{d^{**}}(x)$ for all $x \in [a_1, \infty] \subseteq [\tau_I, \infty]$, and we have

$$AV@R_{\alpha_1}(X - I(X)) - AV@R_{\alpha_1}(X - I_{d^{**}}(X)) = \frac{1}{\alpha_1}\int_{a_1}^{\infty} I_{d^{**}}(x) - I(x)dF_X(x) \geq 0.$$

Therefore, $AV@R_{\alpha_1}(I(X)) \geq AV@R_{\alpha_1}(I_{d^{**}}(X)$ for all $I \in \mathcal{I}_1$, and so $I_{d^{**}}$ is the minimizer. □

In comparison with the result in Cheung et al. [13], in which a stop-loss insurance is optimal, Theorem 2 reveals that one extra layer is required in the optimal solution of Problem 12 in the presence of the reinsurer's risk constraint $V@R_{\hat{\alpha}_2}(I(X)) \leq C$. Similar difference can be observed in the same optimal reinsurance decision problem under Wang's premium principle subject to reinsurer's risk constraint, as shown below in the next subsection.

4.2. Free Wang's Premium Problem

In this subsection, Wang's premium principle will be adopted without a budget constraint. According to the formulation in Problem 11, the reinsurer's risk constraint can be regarded as the condition that imposes an upper bound, which is equal to the sum of the threshold level of acceptable loss and the premium charged, on the indemnity at loss of amount b; while the insurer takes care of the AV@R risk measure of his terminal wealth at the level a. Without loss of generality, we assume that $g'(0) = +\infty$ (see Footnote 1) and $\theta = 0$.

Theorem 3. *Under Wang's premium principle, the optimal reinsurance of Problem 10 is either:*

1) a cap insurance or a double insurance layer if $a < b$;

2) a cap insurance if $b \leq a$.

Remark 1. *In most practical considerations, a risk sharing between a reinsurer and an insurer is viable because the reinsurer has a higher level of risk tolerance, i.e. being less risk averse, than that of the insurer; indeed, most reinsurers possess a relatively larger capacity of business than that of a common insurer, which in turn enhances the stability of its wealth. With less volatility of their wealth, reinsurers can normally afford less stringent risk management, than that of insurer, in order to foster more business opportunities. Mathematically, this observation can be expressed by $b \leq a$.*

Proof. Firstly, we convert Problem 10 into its Lagrangian dual form: $\max_{\lambda>0} H(\lambda)$ where

$$H(\lambda) \triangleq \min_{I\in\mathcal{I}}\left(\frac{1}{\alpha}\int_a^\infty x - I(x)dF_X(x) + P_I + \lambda\big(I(b) - L - P_I\big)\right). \tag{15}$$

For each $\lambda > 0$, define a functional:

$$\begin{aligned}
H_\lambda(I) &\triangleq \frac{1}{\alpha}\int_a^\infty -I(x)dF_X(x) + (1-\lambda)P_I + \lambda I(b) \\
&= \lambda I(b) + \int_0^a (1-\lambda)g' \circ S_X(x)I(x)dF_X(x) \\
&\quad + \int_a^\infty \left((1-\lambda)g' \circ S_X(x) - \frac{1}{\alpha}\right) I(x)dF_X(x),\ I \in \mathcal{I}.
\end{aligned} \tag{16}$$

By convexity, let I_λ be the minimizer of H_λ on $\mathcal{I}$, then we obtain an alternative expression:

$$H(\lambda) = \frac{1}{\alpha}\int_a^\infty x dF_X(x) - \lambda L + H_\lambda(I_\lambda).$$

For any fixed λ and feasible reinsurance I_0, we next seek for a modification, in a certain standard and parametric form, of I_0 such that its corresponding H_λ

is smaller than $H_\lambda(I_0)$. If it can be done, the optimal solution I_λ must be in the same form. By substituting this parametric form into (15), optimal parameters can then be determined. All the technical details of this derivation is included in Sections 6 A.3 and A.4. To complete the proof, we turn to the determination of optimal value of λ by using the first order condition as follows.

Case 1. For $a < b$, we obtain from Section 6 A.3 an explicit form of $H(\lambda)$. Indeed, for $\lambda \geq 1$,

$$H(\lambda) = \frac{1}{\alpha}\int_a^\infty x dF_X(x) - \lambda L + \int_b^\infty (1-\lambda) g \circ S_X(x) - \frac{1}{\alpha} S_X(x) dx$$

and hence

$$H'(\lambda) = -L - \int_b^\infty g \circ S_X(x) dx < 0,$$

so the maximum point must lie in the interval $(0, 1)$ since $\lim_{\lambda\to\infty} H(\lambda) = -\infty$. In Section 6 A.3, we deduce that the form of the optimal reinsurance for $\lambda \in (0, 1)$ is either a double insurance layer or a cap insurance. Finally, Kuhn-Tucker condition implies that the reinsurer's risk constraint holds at the boundary, that is to say,

$$I^*(b) = P_{I^*} + L = \int_0^\infty g' \circ S_X(x) I^*(x) dF_X(x) + L.$$

Case 2. For $b \leq a$, a similar argument yields that the maximum point also lies in the interval $(0, 1)$ because $H'(\lambda)$ is negative for $\lambda \geq 1$. According to the result in Section 6 A.4, we conclude that the optimal insurance is a cap insurance with the reinsurer's risk constraint holds at the boundary because of Kuhn-Tucker's condition, that is to say,

$$I^*(b) = P_{I^*} + L = \int_0^\infty g' \circ S_X(x) I^*(x) dF_X(x) + L.$$

□

Now we illustrate the result of Theorem 3 by two numerical examples which show that both double insurance layer and cap insurance could be a plausible optimal solution. Suppose that the loss X faced by the insurer follows an exponential distribution, and the distortion function is a power function:

$$S_X(x) = \begin{cases} e^{-mx}, & \text{when } x \geq 0; \\ 1, & \text{when } x < 0; \end{cases}$$

and $g(x) = x^k$ for some $m > 0$ and $k \in (0, 1)$. We first have

$$AV@R_\alpha(X - I(X) + P) = -\frac{1}{\alpha}\int_a^\infty x - I(x) de^{-mx} + \int_0^\infty e^{-mkx} I'(x) dx,$$

and the reinsurer's risk constraint can be rewritten as

$$I(b) = P + L = \int_0^\infty e^{-mkx} I'(x)dx + L.$$

For any chosen values for the parameters L, a, b, m, and k, one can identify the optimal reinsurance by comparing the respective minimal values of $AV@R_\alpha(X - I(X) + P)$ among all double insurance layers and among all cap insurances.

Example 1. Let $L = 30$, $a = 1$, $b = 50$, $m = 1$ and $k = 0.5$.

(i) For double insurance layer, we have

$$I(x) = x - (x - d_1)^+ + (x - b)^+ - (x - d_2)^+, \; x \geq 0,$$

for some $a \leq d_1 \leq b \leq d_2$. Then

$$\begin{aligned} AV@R_\alpha(X - I(X) + P) &= -\frac{1}{\alpha}\int_a^\infty xde^{-mx} - I(a) - \frac{1}{\alpha}\int_a^\infty d^{-mx} I'(x)dx + P \\ &= a + \frac{1}{m} - a - \frac{1}{\alpha}\left(\int_a^{d_1} + \int_b^{d_2}\right) e^{-mx}dx + P \\ &= \frac{1}{m} + \frac{1}{m\alpha}(e^{-md_1} - \alpha + e^{-md_2} - e^{-mb}) + P, \end{aligned}$$

and the reinsurer's risk constraint is

$$\begin{aligned} I(b) &= d_1 \\ &= P + L \\ &= \int_0^\infty e^{-mkx} I'(x)dx + L \\ &= \left(\int_0^{d_1} + \int_b^{d_2}\right) e^{-mkx}dx + L \\ &= -\frac{1}{mk}(e^{-mkd_1} - 1 + e^{-mkd_2} - e^{-mkb}) + L. \end{aligned}$$

Writing d_2 in terms of d_1 via

$$e^{-mkd_2} = mk(L - d_1) + 1 - e^{-mkd_1} + e^{-mkb}$$

gives

$$\begin{aligned} &AV@R_\alpha(X - I(X) + P) \\ &= \frac{1}{m} + \frac{1}{m\alpha}(e^{-md_1} - \alpha + e^{-md_2} - e^{-mb}) + d_1 - L \end{aligned}$$

$$= \frac{e^{ma}}{m}(e^{-d_1} + e^{-md_2} - e^{-mb}) + d_1 - L$$

$$= \frac{e^{ma}}{m}\left[e^{-d_1} - e^{-mb} + (mk(L-d_1) + 1 - e^{-mkd_1} + e^{-mkb})^{1/k}\right]d_1 - L,$$

which is a function in d_1. When $d_1 = 31.2642$, $AV@R_\alpha(X - I(X) + P)$ can achieve its minimum value 1.63212.

(ii) For cap insurance, we have

$$I(x) = x - (x-d)^+, \ x \geq 0,$$

for some $d > 0$. The reinsurer's risk constraint becomes

$$0 = I(b) - P - L = \begin{cases} b - L - \frac{1}{mk}(1 - e^{-mkd}), & \text{when } b \leq d; \\ d - L - \frac{1}{mk}(1 - e^{-mkd}), & \text{when } b > d; \end{cases}$$

For $b = 50$ and $L = 30$, $b - L - \frac{1}{mk}(1 - e^{-mkd}) > 0$ for all $d > 0$, and so the reinsurer's risk constraint can never be satisfied. Thus, $b > d$ and $d = L + \frac{1}{mk}(1 - e^{-mkd})$. Solving this equation yields that $d \approx 32$, while

$$\begin{aligned} AV@R_\alpha(X - I(X)) + P) &= -\frac{1}{\alpha}\int_a^\infty x de^{-mx} - I(a) - \frac{1}{\alpha}\int_a^\infty S_X(x)I'(x)dx + P \\ &= a + \frac{1}{m} - a - \frac{1}{\alpha}\int_a^d e^{-mx}dx + d - L \\ &= \frac{1}{m} + \frac{1}{\alpha m}\left(e^{-md} - e^{-ma}\right) + d - L \\ &= \frac{1}{m}e^{-m(d-a)} + d - L > 2. \end{aligned}$$

Therefore, the double insurance layer is better than the cap insurance under the present setting, that is to say, the optimal solution can only be a double insurance layer.

Example 2. Let $L = 195$, $a = 1$, $b = 200$, $m = 0.1$ and $k = 0.5$.

(i) For double insurance layer, again, we have

$$AV@R_\alpha(X - I(X) + P)$$

$$= \frac{e^{ma}}{m}\left[e^{-d_1} - e^{-mb} + \left(mk(L-d_1) + 1 - e^{-mkd_1} + e^{-mkb}\right)^{1/k}\right] + d_1 - L,$$

which achieves its minimum value of 10.9515 at $d_1 = 196.902$.

(ii) For cap insurance, observe that when $d \leq b$,

$$d = L + P = L - (e^{-mkd} - 1)$$

This equation leads to the solution $d \approx 215 > 200 = b$ which is absurd; and thus, $d > b$. From the reinsurer's risk constraint

$$b - L - \frac{1}{mk}(1 - e^{-mkd}),$$

we find that $d = 14.384$, and so

$$\begin{aligned} AV@R_\alpha(X - I(X) + P) &= \frac{1}{m} + \frac{1}{\alpha m}(e^{-md} - e^{-ma}) + b - L \\ &= 7.72683 < 10.9515. \end{aligned}$$

Thus, the optimal solution has to be a cap insurance.

5. Conclusion

In this paper, we studied the optimal reinsurance decision problem using the risk measure $AV@R$ of the retained loss of the insurer as the minimization objective. Under the budget constraint with premium being calibrated by Wang's premium principle, we first showed that the optimal reinsurance for an insurer must be either an insurance layer or a cap insurance. Secondly, by incorporating the reinsurer's risk constraint on his own potential terminal loss, we obtained two new results: (i) under fixed premium charged and calibrated under actuarial pricing principle, we established the optimality of a stop-loss insurance with a layer in the middle; (ii) under free premium calibrated under Wang's premium principle, either a double-insurance-layer or a cap insurance can serve as an optimal solution, depending on the values of the models parameters. Future work on the optimal reinsurance decision problems include the investigation of different premium pricing principles (such as the Dutch premium principle and the variance premium principle) and agents' constraints (general risk measures adopted by both parties).

6. Appendix

A.1. *Supplement to the proof of Proposition 1. We now solve for Problem 4:*

$$\min_{I \in \mathcal{I}_a(I_0)} \quad \frac{1}{\alpha}\int_a^\infty x - I(x)\,dF_X(x)$$

$$\text{such that} \quad \int_a^\infty I(x) g' \circ S_X(x)\,dF_X(x) \leq P_1.$$

Its equivalent Lagrangian dual problem is

$$\max_{\lambda \geq 0} D_1(\lambda),$$

where

$$\begin{aligned} D_1(\lambda) &\triangleq \min_{I \in \mathcal{I}_a(I_0)} \left\{ \frac{1}{\alpha} \int_a^\infty x - I(x) dF_X(x) + \lambda \left(\int_a^\infty I(x) g' \circ S_X(x) dF_X(x) - P_1 \right) \right\} \\ &= \min_{I \in \mathcal{I}_a(I_0)} D_1^\lambda(I) - \lambda P_1 + \frac{1}{\alpha} \int_a^\infty x dF_X(x), \end{aligned}$$

where

$$D_1^\lambda(I) \triangleq \int_a^\infty \left(\lambda g' \circ S_X(x) - \frac{1}{\alpha} \right) I(x) dF_X(x).$$

For each $\lambda > 0$, denote the minimizer in (17) as I_λ. Since $\psi^\lambda(x) \triangleq \lambda g' \circ S_X(x) - \frac{1}{\alpha}$ is non-decreasing, $x_0^\lambda \triangleq \sup\{x : \psi^\lambda(x) \leq 0\}$ is well-defined.

Case 1. For $x_0^\lambda \leq a$, $\psi^\lambda(x)$ is always positive when $x > a$, and therefore, $I_\lambda(x) \equiv I_0(a)$ for all $x \geq a$.

Case 2. For $a < x_0^\lambda < \infty$, $\psi^\lambda(x)$ is positive when $x > x_0^\lambda$ and non-positive otherwise. For each $d \in [a, x_0^\lambda]$, define

$$\mathcal{I}_{a, x_0^\lambda}(I_0, d) \triangleq \left\{ I \in \mathcal{I}_a(I_0) : I(x_0^\lambda) = d - a + I_0(a) \right\}.$$

Then $D_1^\lambda(I)$ achieves its minimum in the set $\mathcal{I}_{a, x_0^\lambda}(I_0, d)$ at

$$x \mapsto I(x) = (x - a + I_0(a))^+ - (x - d)^+.$$

Therefore, there exists $d^\lambda \in [a, x_0^\lambda]$ such that

$$I_\lambda(x) = (x - a + I_0(a))^+ - (x - d^\lambda)^+, \ x \geq 0.$$

Case 3. If $x_0^\lambda = \infty$, then $\psi^\lambda(x) \leq 0$ for all $x > 0$. In this case,

$$I_\lambda(x) = (x - a + I_0(a))^+, \ x \geq 0.$$

Therefore, I_λ is always a generalized insurance layer for all values of λ, that is to say, the optimal solution of Problem 4 must be in the form $I(x) = (x - a + I_0(x))^+ - (x - d_2)^+$ for some $0 < a \leq d_2 \leq \infty$. □

A.2. *Supplement to the proof of Theorem 2. We now solve for the problem*

$$\min_{\substack{0 \leq d_1 \leq a \\ a \leq d_2 \leq \infty}} L_\lambda(d_1, d_2).$$

Using Equation (6), we have

$$L_\lambda(d_1, d_2) = d_1 + \int_{d_1}^{a} \lambda g \circ S_X(x)dx + \int_{a}^{d_2} \lambda g \circ S_X(x) - \frac{1}{\alpha} S_X(x)dx.$$

Our objective is to find out the minimizer $(d_1^\lambda, d_2^\lambda)$ of L_λ. To this end, we first consider the first partial derivatives of L_λ:

1) $\partial L_\lambda / \partial d_1 = 1 - \lambda g \circ S_X(d_1)$ is continuous and non-decreasing in d_1, and

$$1 - \lambda g \circ S_X(0) \le \frac{\partial L_\lambda}{\partial d_1} \le 1 - \lambda g(\alpha); \tag{17}$$

2) $\partial L_\lambda / \partial d_2 = S_X(d_2)\, \phi^\lambda(d_2)$ where

$$\phi^\lambda(x) \triangleq \frac{\lambda g \circ S_X(x)}{S_X(x)} - \frac{1}{\alpha}$$

is non-decreasing; indeed,

$$\frac{d}{dx}\phi^\lambda(x) = \frac{\lambda F_X'(x)}{S_X(x)} \left(\frac{g \circ S_X(x)}{S_X(x)} - g' \circ S_X(x) \right) \ge 0,$$

since the concavity of g implies that

$$\frac{g(z)}{z} \ge g'(z) \text{ for any } z \in (0,1).$$

Moreover,

$$\frac{1}{\alpha}(\lambda g(\alpha) - 1) \le \phi^\lambda(d_2) \le \lambda g'(0) - \frac{1}{\alpha}.$$

Define $x_1^\lambda \triangleq \sup\{x : \phi^\lambda(x) \le 0\}$. Since g is concave and non-decreasing, $g \circ S_X(0) \ge g(\alpha)$ and $\alpha g'(0) \ge g(\alpha)$. It follows that:

$$d_1^\lambda = \begin{cases} a, & \text{when } \lambda > \frac{1}{g(\alpha)}; \\ S_X^{-1} \circ g^{-1}(\frac{1}{\lambda}), & \text{when } \frac{1}{g \circ S_X(0)} \le \lambda \le \frac{1}{g(\alpha)}; \\ 0, & \text{otherwise}, \end{cases} \tag{18}$$

and

$$d_2^\lambda = \begin{cases} a, & \text{when } \lambda > \frac{1}{g(\alpha)}; \\ x_1^\lambda, & \text{when } \frac{1}{\alpha g'(0)} \le \lambda \le \frac{1}{g(\alpha)}; \\ \infty, & \text{otherwise}. \end{cases} \tag{19}$$

□

A.3. *Supplement to case 1 in the proof of Theorem 10.* For $a < b$, we now solve for the problem $\min_{I \in \mathcal{I}} H_\lambda(I)$. According to the definition of H_λ (c.f. (16)),

$$H_\lambda(I) = \lambda I(b) + \int_0^a (1-\lambda) g' \circ S_X(x) I(x) dF_X(x) + \int_a^\infty \Big((1-\lambda) g' \circ S_X(x) - \frac{1}{\alpha} \Big) I(x) dF_X(x).$$

It is sufficient to find the minimizer I_λ of H_λ in the set $\mathcal{I}$ for each $\lambda > 0$.

Firstly for $\lambda \geq 1$, for any given $I_0 \in \mathcal{I}$, denote

$$\mathcal{I}_{a,b}(I_0) \triangleq \{ I \in \mathcal{I} : I(a) = I_0(a) \text{ and } I(b) = I_0(b) \}.$$

Since both $(1-\lambda) g \circ S_X(x) \leq 0$ and $(1-\lambda) g' \circ S_X(x) - \frac{1}{\alpha} < 0$ for all $x \geq 0$, by using a simple geometric approach, we could choose a modification I^* of I_0 from the set $\mathcal{I}_{a,b}(I_0)$ yet with a smaller value of H_λ:

$$I^*(x) = x - (x - I_0(a))^+ + (x-a)^+ - (x - I_0(b) + I_0(a) - a)^+ + (x-b)^+.$$

This modification is valid for all $I_0 \in \mathcal{I}$, a simple compactness argument implies that there exist $d_1 \in [0, a]$ and $d_2 \in [a, b]$ such that

$$I_\lambda(x) = x - (x-d_1)^+ + (x-a)^+ - (x-d_2)^+ + (x-b)^+.$$

It follows that

$$\widehat{H}_\lambda(d_1^\lambda, d_2^\lambda) \triangleq \min_{\substack{0 \leq d_1 \leq a \\ a \leq d_2 \leq b}} \widehat{H}_\lambda(d_1, d_2) = \min_{I \in \mathcal{I}} H_\lambda(I),$$

where

$$\widehat{H}_\lambda(d_1, d_2) = \lambda(d_1 + d_2 - a) - d_1 + (1-\lambda) \int_0^{d_1} g \circ S_X(x) dx + \Big(\int_a^{d_2} + \int_b^\infty \Big) \Big((1-\lambda) g \circ S_X(x) - \frac{1}{\alpha} S_X(x) \Big) dx.$$

Now,

$$\frac{\partial \widehat{H}_\lambda}{\partial d_1} = (\lambda - 1)(1 - g \circ S_X(d_1)) \geq 0,$$
$$\frac{\partial \widehat{H}_\lambda}{\partial d_2} = -\frac{1}{\alpha} S_X(d_2) - (\lambda - 1) g \circ S_X(d_2) + \lambda \geq 0.$$

Here, the first inequality follows directly from $\lambda \geq 1$, and the second inequality holds because $\partial \widehat{H}_\lambda / \partial d_2$ is non-decreasing in d_2 and is non-negative at $d_2 = a$. Thus, $d_1^\lambda = 0$ and $d_2^\lambda = a$, and $I_\lambda(x) = (x-b)^+$.

Secondly, for $0 < \lambda < 1$, the argument as the same as in the case that $\lambda \geq 1$. First of all, for every $\lambda \in (0,1)$, define three auxiliary functions on $\mathbb{R}^{+}$ and their respective roots as:

1) $\mu^{\lambda}(x) \triangleq (1-\lambda)\, g' \circ S_X(x) - \frac{1}{\alpha}$ and $y_0^{\lambda} \triangleq \sup\{y : \mu^{\lambda}(y) \leq 0\}$;

2) $\eta^{\lambda}(x) \triangleq \frac{S_X(x)}{1 - g \circ S_X(x)}\left(\frac{1}{\alpha} - \frac{g \circ S_X(x)}{S_X(x)}\right) - \lambda$ and $y_1^{\lambda} \triangleq \sup\{y : \eta^{\lambda}(y) \geq 0\}$;

3) $\gamma^{\lambda}(x) \triangleq (1-\lambda)\frac{g \circ S_X(x)}{S_X(x)} - \frac{1}{\alpha}$ and $y_2^{\lambda} \triangleq \sup\{y : \gamma^{\lambda}(y) \leq 0\}$.

It can be easily checked that:

1) μ^{λ} is non-decreasing (being a composition of two non-increasing functions), $\mu^{\lambda}(a) < 0$ and $\lim_{x \to +\infty} \mu^{\lambda}(x) = +\infty$, so $a < y_0^{\lambda} < +\infty$.
2) η^{λ} firstly decreases from positive infinity to zero and then becomes negative. Moreover, $\eta^{\lambda}(a) = 1 - \lambda > 0$. Hence $a < y_1^{\lambda} < +\infty$.
3) γ^{λ} is non-decreasing, $\gamma^{\lambda}(y_1^{\lambda}) = -\lambda / S_X(y_1^{\lambda}) < 0$ and

$$\gamma^{\lambda}(y_0^{\lambda}) = (1-\lambda)\left(\frac{g \circ S_X(y_0^{\lambda})}{S_X(y_0^{\lambda})} - g' \circ S_X(y_0^{\lambda})\right) \geq 0.$$

Hence $a < y_1^{\lambda} < y_2^{\lambda} \leq y_0^{\lambda} < +\infty$.

For any $I_0 \in \mathcal{I}$, denote

$$\mathcal{I}_{a,b,y_0^{\lambda}}(I_0) \triangleq \left\{I \in \mathcal{I}_{a,b}(I_0) : I(y_0^{\lambda}) = I_0(y_0^{\lambda})\right\}.$$

By using a simple geometric approach, we can again choose a modification I^{*} of I_0 from the set $\mathcal{I}_{a,b,y_0^{\lambda}}(I_0)$ such that $I^{*}(x) \geq I_0(x)$ when $a \leq x \leq y_0^{\lambda}$, and $I^{*}(x) \leq I_0(x)$ otherwise. Therefore, we conclude that the optimal form of I_{λ} is:

1) if $y_0^{\lambda} \leq b$, there exist $d \in [0,a]$, $d_1 \in [a, y_0^{\lambda}]$ and $d_2 \in [y_0^{\lambda}, b]$ such that $I_{\lambda}(x) = (x-d)^{+} - (x-d_1)^{+} + (x-d_2)^{+} - (x-b)^{+}$;

2) if $b < y_0^{\lambda} < \infty$, there exist $d \in [0,a]$, $d_1 \in [a,b]$ and $d_2 \in [b, y_0^{\lambda}]$ such that $I_{\lambda}(x) = (x-d)^{+} - (x-d_1)^{+} + (x-b)^{+} - (x-d_2)^{+}$.

Define

$$\Phi \triangleq \left\{(d, d_1, d_2) : 0 \leq d \leq a \leq d_1 \leq \min\{y_0^{\lambda}, b\} \leq d_2 \leq \max\{y_0^{\lambda}, b\}\right\},$$

we then have:

$$\widehat{H}_{\lambda}(d^{\lambda}, d_1^{\lambda}, d_2^{\lambda}) \triangleq \min_{(d, d_1, d_2) \in \Phi} \widehat{H}_{\lambda}(d, d_1, d_2) = \min_{I \in \mathcal{I}} H_{\lambda}(I),$$

where, for each $(d, d_1, d_2) \in \Phi$,

$$\widehat{H}_\lambda(d, d_1, d_2) \triangleq \lambda(d_1 - d + (b - d_2)^+) - (a - d) + (1-\lambda)\int_d^a g \circ S_X(x)dx$$
$$+ \left(\int_a^{d_1} + \int_{\min\{d_2, b\}}^{\max\{d_2, b\}}\right)\left((1-\lambda)\, g \circ S_X(x) - \frac{1}{\alpha}\, S_X(x)\right) dx.$$

To minimize $\widehat{H}_\lambda$ in Φ, we first note the following facts:

1) $\partial \widehat{H}_\lambda / \partial d = (1-\lambda)(1 - g \circ S_X(d)) \geq 0$ for all $0 < d < a$ and thus $d^\lambda = 0$;
2) $\partial \widehat{H}_\lambda / \partial d_1 = -(1 - g \circ S_X(d_1))\eta^\lambda(d_1)$ is non-positive when $d_1 < y_1^\lambda$ and positive otherwise;
3) if $y_2^\lambda \leq y_0^\lambda \leq b$, then $\partial \widehat{H}_\lambda / \partial d_2 = -(1 - g \circ S_X(d_2))\eta^\lambda(d_2) < 0$; if $b < y_0^\lambda$, then $\partial \widehat{H}_\lambda / \partial d_2 = S_X(d_2))\gamma^\lambda(d_2)$ is non-positive when $d_2 \leq y_2^\lambda$ and positive otherwise.

It follows that, for each $\lambda \in (0,1)$, there exist $d_1^\lambda = \min\{y_1^\lambda, b\}$ and $d_2^\lambda = \min\{y_2^\lambda, b\}$ such that $I_\lambda(x) = x - (x - d_1^\lambda)^+ + (x - b)^+ - (x - d_2^\lambda)^+$. In particular, $I_\lambda(x) = x - (x - y_1^\lambda)^+ + (x - b)^+ - (x - y_2^\lambda)^+$ is a double insurance layer if $y_1^\lambda < b < y_2^\lambda$. □

A.4. *Supplement to case 2 in the proof of Theorem 10.* For $b \leq a$, we now solve for the problem $\min_{I \in \mathcal{I}} H_\lambda(I)$. The proof is essentially the same as that in A.3., to avoid redundancy, we here only outline some key ideas. Firstly, for $\lambda \geq 1$, we can again show that the minimizer is

$$I_\lambda(x) = x - (x - d_1)^+ + (x - b)^+ - (x - d_2)^+ + (x - a)^+,$$

for some $d_1 \in [0, b]$ and $d_2 \in [b, a]$. It follows that

$$\widehat{H}_\lambda(d_1^\lambda, d_2^\lambda) \triangleq \min_{\substack{0 \leq d_1 \leq b \\ b \leq d_2 \leq a}} \widehat{H}_\lambda(d_1, d_2) = \min_{I \in \mathcal{I}} H_\lambda(\lambda),$$

where

$$\widehat{H}_\lambda(d_1, d_2) = \lambda d_1 - (d_1 + d_2 - b) + (1-\lambda)\left(\int_0^{d_1} + \int_b^{d_2}\right) g \circ S_X(x)dx$$
$$+ \int_a^\infty (1-\lambda)\, g \circ S_X(x) - \frac{1}{\alpha}\, S_X(x)dx.$$

Now, $\partial \widehat{H}_\lambda / \partial d_1 = (\lambda - 1)(1 - g \circ S_X(d_1)) \geq 0$ and $\partial \widehat{H}_\lambda / \partial d_2 = (1-\lambda)\, g \circ S_X(d_2) - 1 < 0$. Thus $d_1^\lambda = 0$, $d_2^\lambda = a$ and $I_\lambda(x) = (x - b)^+$.

Secondly, for $0 < \lambda < 1$, we can again conclude that the minimizer is in the form:

$$I_\lambda(x) = (x - d)^+ - (x - b)^+ + (x - d_1)^+ - (x - d_2)^+,$$

for some $d \in [0,b]$, $d_1 \in [b,a]$ and $d_2 \in [a, y_0^\lambda]$. Thus

$$\widehat{H}_\lambda(d^\lambda, d_1^\lambda, d_2^\lambda) \triangleq \min_{\substack{0 \le d \le b \\ b \le d_1 \le a \\ a \le d_2 \le y_0^\lambda}} \widehat{H}_\lambda(d, d_1, d_2) = \min_{I \in \mathcal{I}} H_\lambda(I),$$

where

$$\begin{aligned} \widehat{H}_\lambda(d, d_1, d_2) = {} & \lambda(b-d) - (b-d+a-d_1) \\ & + (1-\lambda)\left(\int_d^b + \int_{d_1}^a\right) g \circ S_X(x)dx \\ & + \int_a^{d_2} \left((1-\lambda)\, g \circ S_X(x) - \frac{1}{\alpha} S_X(x)\right) dx. \end{aligned}$$

Now, $\partial \widehat{H}_\lambda / \partial d = (1-\lambda)(1 - g \circ S_X(d)) \ge 0$; (ii) $\partial \widehat{H}_\lambda / \partial d_1 = 1 - (1-\lambda)\, g \circ S_X(d_1) > 0$; (iii) $\partial \widehat{H}_\lambda / \partial d_2 = \gamma^\lambda(d_2)$ is non-decreasing and has a root of $y_2^\lambda \in (a, y_0^\lambda)$. Therefore, $d^\lambda = 0$, $d_1^\lambda = b$ and $d_2^\lambda = y_2^\lambda$. It follows that $I_\lambda(x) = x - (x - y_2^\lambda)^+$. $\square$

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References

[1] Acerbi, C. and Tasche, D. (2002) On the coherence of expected shortfall. *Journal of Banking and Finance* **26(7)**, 1487-1503.

[2] Arrow, K.J. (1963) Uncertainty and the Welfare Economics of Medical Care. *American Economic Review* **53**, 941-973.

[3] Artzner, P., Delbaen, F., Eber, J.M. and Heath, D. (1999) Coherent measures of risk. *Mathematical Finance* 9(3), 203-228.

[4] Balbás, A., Balbás, B. and Heras, A. (2009) Optimal reinsurance with general risk measures. *Insurance: Mathematics and Economics* **44(3)**, 374-384.

[5] Basak, S. and Shapiro, A. (2001) Value-at-Risk based risk management: Optimal policies and asset prices. *The Review of Financial Studies* **14**, 371-405.
[6] Blazenko, G. (1985) The design of an optimal insurance policy: Note. *The American Economic Review* **75**, 253-255.
[7] Borch, K. (1960) An attempt to determine the optimum amount of stop loss reinsurance. *Transactions of the 16th International Congress of Actuaries*, 597-610.
[8] Borch, K. (1975) Optimal insurance arrangements. *ASTIN Bulletin* **8**, 284-290.
[9] Buhlmann, H. and Jewell, W.S. (1979) Optimal risk exchange. *ASTIN Bulletin* **10**, 243-262.
[10] Cai, J. and Tan, K.S. (2007) Optimal retention for a stop-loss reinsurance under the VaR and CTE risk measures. *ASTIN Bulletin* **37(1)**, 93-112.
[11] Cai, J., Tan, K.S., Weng, C. and Zhang, Y. (2008) Optimal reinsurance under VaR and CTE risk measures. *Insurance: Mathematics and Economics* **43(1)**, 185-196.
[12] Cheung, K.C. (2010) Optimal reinsurance revisited – geometric approach. *ASTIN Bulletin* **40(1)**, 221-239.
[13] Cheung, K.C., Sung, K.C.J., Yam, S.C.P. and Yung S.P. (2010) Optimal reinsurance under general law-invariant risk measures. To appear in *Scandinavian Actuarial Journal*.
[14] Delbaen, F. (2000) Coherent risk measures. Scuola Normale Superiore di Pisa, Cattedra.
[15] Delbaen, F. (2002) Coherent risk measures on general probability spaces. *Advances in Finance and Stochastics*, Springer-Verlag, Berlin, 1-37.
[16] Denuit, M. and Vermandele, C. (1998) Optimal reinsurance and stop-loss order. *Insurance: Mathematics and Economics* **22**, 229-233.
[17] Dhaene, J., Vanduffel, S., Tang, Q., Goovaerts, M., Kaas, R. and Vyncke, D. (2006) Risk measures and comonotonicity: a review. *Stochastic Models* **22**, 573-606.
[18] Embrechts, P. (2000) Extreme value theory: Potential and limitations as an integrated risk management tool. *Derivatives Use, Trading and Regulation* **6**, 449-456.
[19] Föllmer, H. and Schied, A. (2002) Convex measures of risk and trading constraints. *Finance and Stochastics* **6(4)**, 429-447.
[20] Föllmer, H. and Schied, A. (2004) *Stochastic Finance: An Introduction in Discrete Time*. Walter de Gruyter, Berlin.
[21] Frittelli, M. and Rossaza Gianin, E. (2002) Putting order in risk measures. *Journal of Banking and Finance* **26**, 1473-1486.
[22] Gerber, H.U. (1979) *An Introduction to Mathematical Risk Theory.* S.S. Huebner Foundation for Insurance Education, Wharton School, University of Pennsylvania, Philadelphia.
[23] Guerra, M. and Centeno, M.L. (2008) Optimal reinsurance policy: The adjustment coefficient and the expected utility criteria. *Insurance: Mathematics and Economics* **42(2)**, 529-539.
[24] Gollier, C. (1987) The design of optimal insurance contracts without the nonnegativity constraint on claims. *The Journal of Risk and Insurance* **54**, 314-324.
[25] Heath, D. and Ku, H. (2004) Pareto equilibria with coherent measures of risk. *Mathematical Finance* **14(2)**, 163-172.
[26] Hull, J.C. (2010) Risk Management and Financial Institutions. Pearson, U.S.A.
[27] Kaluszka, M. (2004a) Mean-variance optimal reinsurance arrangements. *Scandinarian Actuarial Journal* **1**, 28-41.
[28] Kaluszka, M. (2004b) An extension of Arrow's result on optimality of a stop loss contract. *Insurance: Mathematics and Economics* **35(3)**, 527-536.
[29] Kaluszka, M. (2005) Optimal reinsurance under convex principles of premium calculation. *Insurance: Mathematics and Economics* **36(3)**, 375-398.
[30] Moore, K.S. and Young, V.R. (2006) Optimal insurance in a continuous-time model. *Insurance Mathematics and Economics* **39**, 47-68.
[31] von Neumann, J. and Morgenstern, O. (1944) Theory of Games and Economic Behavior. Princeton University Press, Princeton, NJ.
[32] Raviv, A. (1979) The design of an optimal insurance policy. *The American Economic Review* **69**, 84-96.
[33] Rockafellar, R.T. and Uryasev, S. (2001) Conditional Value-at-Risk for general loss distributions. ISE Dept., University of Florida.
[34] Sung, K.C.J., Yam, S.C.P., Yung, S.P. and Zhou, J.H. (2011) Behavioral Optimal Insurance. *Insurance: Mathematics and Economics* **49(3)**, 418-428.

[35] TAN, K.S., WENG, C. and ZHANG, Y. (2009) VaR and CTE Criteria for optimal quota-share and stop-loss reinsurance. *North American Actuarial Journal* **13(4)**, 459-482.
[36] WANG, S. (1996) Premium calculation by transforming the layer premium density. *ASTIN Bulletin* **26**, 71-92.
[37] WANG, S., YOUNG, V.R. and PANJER, H.H. (1997) Axiomatic characterization of insurance prices. *Insurance: Mathematics and Economics* **21(2)**, 173-183.
[38] YAARI, M.E. (1987) The dual theory of choice under risk. *Econometrica* **55**, 95-115.
[39] YAARI, S., YOUNG V.R. and PANJER, H.H. (1997) Axiomatic characterization of insurance prices. *Insurance: Mathematics and Economics* **21(2)**, 173-183.
[40] YOUNG V. R. (1999) Optimal insurance under Wang's premium principle. *Insurance: Mathematics and Economics* **25**, 109-122.

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