THE COVARIANCE BETWEEN THE SURPLUS PRIOR TO AND AT RUIN IN THE CLASSICAL RISK MODEL

BY

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Abstract

For the classical model of risk theory, we consider the covariance between the surplus prior to and at ruin, given that ruin occurs. A general expression for this covariance is given when the initial surplus u is zero, and we show that the covariance (and hence the correlation coefficient) between these two variables is positive, zero or negative according to the equilibrium distribution of the claim size distribution having a coefficient of variation greater than, equal to, or less than one. For positive values of u, the formula for the covariance may not always lead to explicit results and we thus also study its asymptotic behaviour. Our results are illustrated by a number of examples.

Keywords

Ruin probability; Renewal equation; Deficit at ruin; Surplus prior to ruin; Reliability classes; Coefficient of variation.

1. Introduction

In recent years, interest in actuarial risk theory has focused on the expected discounted penalty function, more widely known as the Gerber-Shiu function. The extensive research which followed the introduction of this function by Gerber and Shiu in 1998 has shown the advantages of the simultaneous study of a number of quantities in risk theory. In their original work, Gerber and Shiu (1998) studied three such quantities, in the context of the classical model of risk theory: the time until ruin, the surplus prior to ruin and the deficit at the time of ruin. Here we concentrate on the last two of these three quantities and, more specifically, we find expressions for their covariance. Technically, unless ruin is certain, both the surplus prior to ruin and the deficit at ruin are defective random variables. However, conditioning on the event that ruin occurs, both variables are proper and, apart from our main result (Theorem 2.1) which gives an exact expression for their covariance, we obtain various conditions under

Astin Bulletin 42(2), 631-653. doi: 10.2143/AST.42.2.2182811

which the covariance (and hence the correlation coefficient of these two variables) is positive, zero or negative.

We consider the classical risk model with surplus process $\{U(t) : t \geq 0\}$ where the surplus, $U(t)$, at time t is given by

$$U(t) = u + ct - \sum_{i=1}^{N(t)} Y_i,$$

where $u \geq 0$ is the initial surplus, $c > 0$ is the premium rate, and the Y_i's represent the sizes of claims. These claims are assumed to be independent identically distributed random variables with distribution function (d.f.) P, density function p and they arrive to an insurer according to a Poisson process $\{N(t) : t \geq 0\}$ with intensity λ. The k-th moment of the claim size distribution is $\mu_k = \int_0^\infty x^k \, dP(x)$ for $k = 1, 2, \ldots$. Throughout the paper we assume that $\mu_3 < \infty$. We assume further that the claims are independent of the claim-arrivals process.

Ruin occurs if $U(t) < 0$ for some $t > 0$. Let $\psi(u)$ denote the probability of ruin with initial capital u, i.e.

$$\psi(u) = \mathbb{P}(U(t) < 0 \text{ for some } t > 0 \mid U(0) = u).$$

We assume that the premium loading factor $\theta = (c - \lambda\mu_1) / (\lambda\mu_1)$ associated with the surplus process is positive. Under that assumption, it holds in particular that $\psi(u) < 1$ for all u (see for example Rolski *et al.* (1999, p. 162)). Let $T = \inf\{t : U(t) < 0\}$ be the time of ruin. Then $\psi(u) = \mathbb{P}(T < \infty \mid U(0) = u)$, and so T is a defective random variable. At the time of ruin, the (modulus) of the deficit at ruin is $|U(T)|$, while we denote by $U(T-)$ the insurer's surplus immediately prior to ruin. Intuitively, one expects that these two variables are somehow related and a first naive argument might be that the larger the surplus prior to ruin, the smaller the value of $|U(T)|$. However, both $U(T-)$ and $|U(T)|$ are defective random variables, so that we cannot speak about their covariance, or correlation coefficient, unless we condition on the event that ruin occurs. In this case, we define the (proper) random variables $V(T-) = U(T-) \mid T < \infty$ and $V(T) = |U(T)| \| T < \infty$ and the main focus in the present study is to consider the covariance between $V(T-)$ and $V(T)$. Note that both the time of ruin, T, as well as $V(T-)$ and $V(T)$ and their defective counterparts, depend on the initial surplus u, even though this dependence is suppressed in the notation. However, since we consider the covariance of $V(T-)$ and $V(T)$ as a function of u, we make this dependence explicit by defining the function

$$C(u) := Cov_u(V(T-), V(T)) = E_u(V(T-)\, V(T)) - E_u(V(T-))\, E_u(V(T)). \quad (1)$$

Here the subscript u denotes that the expectation (or covariance) is considered with respect to the conditional measure, i.e. given that ruin occurs with initial capital u.

It is clear that the calculation of $C(u)$ relies on quantities which are special cases of the Gerber-Shiu function, for which a vast literature is available in recent years. However, in Theorems 2.1 and 2.2 below we give two alternative representations for this function, which in particular allow a study of the covariance structure in terms of reliability classifications.

Note that the intuitive reasoning mentioned above for the correlation between $U(T-)$ and $|U(T)|$ is no longer credible when we consider $V(T-)$ and $V(T)$. Instead, one expects that the covariance, and correlation structure between these two random variables might be influenced by the presence of reliability (or ageing) properties in the claim size distribution P or the ladder height distribution P_1 associated with the surplus process. In fact, for the classical model we consider here, it is well-known that the distribution of ladder heights (i.e. the sizes of the drops in the surplus, given that a drop occurs) coincides with the equilibrium distribution associated with P, which has density $\bar{P}(x)/\mu_1$; here, and in the sequel, $\bar{F} = 1 - F$ denotes the tail of a distribution F on $[0, \infty)$.

The problem we address in the present paper seems not to have been studied in detail for the classical risk model, although Li and Garrido (2002) study the covariance of the surplus prior to and at the time of ruin for a discrete-time surplus process. More precisely, for this model Li and Garrido (2002) consider the covariance under the assumption that the initial surplus is zero ($u = 0$); the premium income per unit time is assumed to be $c = 1$, while claim sizes $Y_1, Y_2, \ldots$, follow a discrete distribution P which has an n-th order moment μ_n. For this model, they found that

$$Cov(U(T-1), |U(T)| \,|\, T < \infty,\ U(0) = 0) = \frac{\mu_{(3)}}{6\mu_1} - \frac{\mu_{(2)}^2}{4\mu_1^2},$$

where $\mu_{(n)} := E[Y(Y-1) \ldots (Y-n+1)]$, $n \geq 1$, is the n-th factorial moment of Y; here Y is a random variable which has the same distribution as the Y_i. According to Li and Garrido (2002), this suggests that a sufficient condition for $U(T-1)$ and $|U(T)|$ to be positively (negatively) correlated is that the equilibrium distribution of P has a decreasing (increasing) failure rate (i.e. it is DFR, resp. IFR). Recall that a d.f. F is said to be a DFR (IFR) distribution if $\bar{F}(x+y)/\bar{F}(x)$ is nondecreasing (nonincreasing) in x for any $y \geq 0$. If F is absolutely continuous with density f, then it is DFR (IFR) when the failure rate $h_F(x) = f(x)/\bar{F}(x)$ is nonincreasing (nondecreasing). Further, in view of the duality between the classical model of risk theory and a single-server queueing system (see, for example, Rolski *et al.* (1999, Chapter 5)), we note that the results of Boxma (1984) are also relevant here, although they are not used in the sequel.

The paper is organized as follows: the main result, along with its relation with the HNWUE (HNBUE) reliability classes and the coefficient of variation of the distribution P_1, are given in the next section. Section 3 contains an asymptotic result for the covariance between $V(T)$ and $V(T-)$. Using the above covariance, we also give a characterization of the claim size distribution.

Section 4 deals with the correlation coefficient between $V(T)$ and $V(T-)$ for the case where the initial surplus is zero. Examples are given to illustrate our results.

2. The covariance of the surplus prior to and at ruin

We defined above $P_1(x)$ to be the equilibrium distribution of the claim size distribution $P(x)$ and μ_k for $k = 1, 2, 3, \ldots$ as the k-order moment of $P(x)$. By Lin and Willmot (2000, relations (4.6, 5.3, 5.2)) we have

$$E_u(|V(T)|) = \frac{1}{\psi(u)} \int_u^{\infty} \psi(x)dx - \frac{\mu_2}{2\mu_1\theta}, \tag{2}$$

$$E_u(V(T-)) = \frac{1}{\theta\psi(u)} \left\{ \int_0^u \psi(u-x)\,x\,dP_1(x) + \int_u^{\infty} x\,dP_1(x) \right\} - \frac{\mu_2}{2\mu_1\theta} \tag{3}$$

and, finally,

$$E_u(V(T-)\,V(T)) = \frac{1}{\theta\psi(u)} \left\{ \int_0^u \psi(u-x)\,x\overline{P}_1(x)\,dx + \int_u^{\infty} x\overline{P}_1(x)\,dx \right\} - \frac{\mu_3}{6\mu_1\theta}. \tag{4}$$

We also consider the distributions $G_1(x)$ and $G_2(x)$ with tails

$$\overline{G}_1(u) = \frac{\int_u^{\infty} x\,dP_1(x)}{\int_0^{\infty} x\,dP_1(x)} \Rightarrow \int_u^{\infty} x\,dP_1(x) = \frac{\mu_2}{2\mu_1}\,\overline{G}_1(u) \tag{5}$$

and

$$\overline{G}_2(u) = \frac{\int_u^{\infty} x\,dP_2(x)}{\int_0^{\infty} x\,dP_2(x)} \Rightarrow \int_u^{\infty} x\,dP_2(x) = \frac{\mu_3}{3\mu_2}\,\overline{G}_2(u) \tag{6}$$

here $P_2(x)$ is the equilibrium distribution of $P_1(x)$.

Note that, for $i = 1, 2$, the distribution G_i above is known as the *length-biased distribution* associated with P_i. Such distributions arise naturally in sampling when the probability of selecting an individual from a population depends on its magnitude. Another important area of occurrence is renewal theory, where G_i is the limiting distribution of the total lifetime in a renewal process with inter-arrival distribution P_i, see e.g. Feller (1971, p. 371). In a context similar to ours, the distributions G_1, G_2 have been used by Lin and Willmot (2000, Section 5).

Let $\delta(u) = 1 - \psi(u)$ be the probability of non-ruin in the model. In the next theorem we use the tail of the equilibrium distribution associated with $\delta(u)$, namely

$$\psi_1(u) = \frac{\int_u^\infty \psi(t)\,dt}{\int_0^\infty \psi(t)\,dt} = \frac{2\mu_1\theta}{\mu_2}\int_u^\infty \psi(t)\,dt. \tag{7}$$

Theorem 2.1. *For any $u \geq 0$, the covariance of the surplus prior to and at the time of ruin, given that ruin occurs, is given by*

$$C(u) = \frac{1}{\psi(u)}\left\{\frac{\mu_3}{6\mu_1}K_2(u) - \frac{\mu_2^2}{4\mu_1^2}a(u)K_1(u)\right\}, \tag{8}$$

where

$$a(u) = \frac{1}{\theta}\left[\frac{\psi_1(u)}{\psi(u)} - 1\right] \tag{9}$$

and $K_1(u)$, $K_2(u)$ are two non-negative functions which satisfy the following defective renewal equations

$$K_i(u) = \frac{1}{1+\theta}\int_0^u K_i(u-x)\,dP_1(x) + \frac{1}{1+\theta}\overline{G}_i(u) \tag{10}$$

for $i = 1, 2$. In particular, when the initial surplus is zero, we have

$$C(0) = \frac{\mu_3}{6\mu_1} - \frac{\mu_2^2}{4\mu_1^2}. \tag{11}$$

Proof. Inserting relations (2), (3) and (4) in (1), we obtain that

$$C(u) = \frac{1}{\theta\psi(u)}[I_1(u) - I_2(u) + I_3(u)], \tag{12}$$

where

$$I_1(u) = \int_u^\infty x\overline{P}_1(x)dx - \frac{\mu_3}{6\mu_1}\psi(u),$$

$$I_2(u) = \frac{\mu_2}{2\mu_1\theta}\frac{\psi_1(u)}{\psi(u)}\int_u^\infty xdP_1(x) - \frac{\mu_2^2}{4\mu_1^2\theta}\psi_1(u) - \frac{\mu_2}{2\mu_1\theta}\int_u^\infty xdP_1(x) + \frac{\mu_2^2}{4\mu_1^2\theta}\psi(u)$$

and

$$I_3(u) = \int_0^u \psi(u-x)\,x\overline{P}_1(x)dx - \frac{\mu_2}{2\mu_1\theta}\left[\frac{\psi_1(u)}{\psi(u)} - 1\right]\int_0^u \psi(u-x)xdP_1(x).$$

We now simplify each of the functions $I_i(u)$ for $i = 1, 2, 3$ in turn. First, note that

$$I_1(u) = \frac{\mu_2}{2\mu_1}\int_u^\infty x dP_2(x) - \frac{\mu_3}{6\mu_1}\psi(u)$$

and then we obtain from (6) that

$$I_1(u) = \frac{\mu_3}{6\mu_1}\,[\overline{G}_2(u) - \psi(u)].$$

For the function $I_2(u)$, we derive that

$$\begin{aligned} I_2(u) &= \frac{\mu_2}{2\mu_1\theta}\int_u^\infty x dP_1(x)\left[\frac{\psi_1(u)}{\psi(u)} - 1\right] - \frac{\mu_2^2}{4\mu_1^2\theta}[\psi_1(u) - \psi(u)] \\ &= \frac{\mu_2^2}{4\mu_1^2}\,a(u)\,[\overline{G}_1(u) - \psi(u)], \end{aligned} \tag{13}$$

using (5) and (9) in the second step. Finally, for the function $I_3(u)$ we get

$$I_3(u) = \frac{\mu_2}{2\mu_1}\int_0^u \psi(u-x)\,x dP_2(x) - \frac{\mu_2}{2\mu_1}a(u)\int_0^u \psi(u-x)\,x dP_1(x).$$

By (5) and (6) we see that

$$x dP_1(x) = \frac{\mu_2}{2\mu_1}dG_1(x) \quad \text{and} \quad x dP_2(x) = \frac{\mu_3}{3\mu_2}dG_2(x),$$

respectively. Thus,

$$I_3(u) = \frac{\mu_3}{6\mu_1}\int_0^u \psi(u-x)dG_2(x) - \frac{\mu_2^2}{4\mu_1^2}a(u)\int_0^u \psi(u-x)\,dG_1(x).$$

Substituting the functions I_1, I_2, I_3 from above into (12), we deduce that

$$\begin{aligned} C(u) = \frac{1}{\theta\psi(u)}\Bigg\{ &\frac{\mu_3}{6\mu_1}\left[\int_0^u \psi(u-x)dG_2(x) + \overline{G}_2(u) - \psi(u)\right] \\ &- \frac{\mu_2^2}{4\mu_1^2}a(u)\left[\int_0^u \psi(u-x)\,dG_1(x) + \overline{G}_1(u) - \psi(u)\right]\Bigg\}. \end{aligned} \tag{14}$$

Integrating by parts the integral $\int_0^u \psi(u-x)\,dG_1(x)$ and keeping in mind that $\delta'(u) = -\psi'(u)$ for $u > 0$, we obtain

$$\int_0^u \psi(u-x)\,dG_1(x) = \psi(u) - \frac{1}{1+\theta}\,\overline{G}_1(u) + \int_0^u \delta'(u-x)\,\overline{G}_1(x)\,dx.$$

Similarly, for the first integral on the right of (14), we derive that

$$\int_0^u \psi(u-x)\,dG_2(x) = \psi(u) - \frac{1}{1+\theta}\,\overline{G}_2(u) + \int_0^u \delta'(u-x)\,\overline{G}_2(x)\,dx.$$

Using the last two equations, (14) yields after a straightforward computation,

$$C(u) = \frac{1}{\theta\psi(u)}\left\{\frac{\mu_3}{6\mu_1}\int_{0^-}^u \overline{G}_2(u-x)\,d\delta(x) - \frac{\mu_2^2}{4\mu_1^2}\,a(u)\int_{0^-}^u \overline{G}_1(u-x)\,d\delta(x)\right\}.$$

By setting

$$K_i(u) = \frac{1}{\theta}\int_{0^-}^u \overline{G}_1(u-x)\,d\delta(x),\ i = 1,2, \tag{15}$$

we have that the functions K_i above are for $i = 1,2$, the solutions of the defective renewal equations given in (10), and this completes the proof of the first part. Equation (11) then follows immediately on noting that $\psi(0) = K_1(0) = K_2(0) = (1+\theta)^{-1}$ and $a(0) = 1$. □

We now make a few remarks about the function $K_1(u)$ in (15). First, we note that $K_1(u) \geq \psi(u)$ for any $u \geq 0$. To see this, note that the tail of the length-biased distribution of P_1 is always larger than the tail of P_1, i.e.

$$\overline{G}_1(u) \geq \overline{P}_1(u),$$

see, e.g. Gupta and Keating (1986). Inserting this bound in (15), we derive that

$$K_1(u) = \frac{1}{\theta}\int_{0^-}^u \overline{G}_1(u-x)\,d\delta(x) \geq \frac{1}{\theta}\int_{0^-}^u \overline{P}_1(u-x)\,d\delta(x) = \psi(u),$$

as asserted.

Let now for an integrable function f on $[0,\infty)$, a function $T_0 f$ be defined by

$$(T_0 f)(u) = \int_u^\infty f(x)\,dx.$$

The operator T_0 is a special case of the Dickson-Hipp operator, see Dickson and Hipp (2001). Note that the compound distribution $\delta(x)$ can be written as

$$\delta(x) = \frac{\theta}{1+\theta}\sum_{k=0}^{\infty}\left(\frac{1}{1+\theta}\right)^k P_1^{*k}(x),$$

where $P_1^{\star k}$ denotes the k–th Lebesgue–Stieltjes convolution power of P_1; for $k=0$, $P_1^{\star 0}$ is the indicator function on the nonnegative half-line. We also consider the usual convolution between two integrable functions f, g on $[0,\infty)$, which we denote by $f * g$, and it is defined by $(f*g)(x) = \int_0^x f(x-t)g(t)\,dt$.

In view of (15), the last equation gives

$$K_1(u) = \frac{1}{1+\theta}\,\overline{G}_1(u) + \sum_{k=1}^{\infty}\left(\frac{1}{1+\theta}\right)^{k+1}\int_0^u \overline{G}_1(u-x)\,dP_1^{\star k}(x).$$

Assume that $p_1(x)$ is the density of $P_1(x)$. Define also

$$\tilde{p}_1(x) = xp_1(x), \qquad x \geq 0.$$

From (5) we see that

$$\overline{G}_1(u) = \frac{\int_u^{\infty}\tilde{p}_1(x)\,dx}{\frac{\mu_2}{2\mu_1}} = \frac{2\mu_1}{\mu_2}\,T_0\tilde{p}_1(u).$$

Combining the last two results, after some straightforward calculus we get that $K_1(u)$ can be written as follows:

$$\begin{aligned} K_1(u) &= \frac{1}{1+\theta}\overline{G}_1(u) + \sum_{k=1}^{\infty}\left(\frac{1}{1+\theta}\right)^{k}\cdot\frac{2\mu_1}{\mu_2}\,T_0\big(\tilde{p}_1 * p_1^{*k}\big)(u) - \frac{2\mu_1}{\mu_2\theta}\,\psi(u) \\ &= \frac{1}{1+\theta}\overline{G}_1(u) + T_0\left[\sum_{k=1}^{\infty}\left(\frac{1}{1+\theta}\right)^{k}\cdot\frac{2\mu_1}{\mu_2}\big(\tilde{p}_1 * p_1^{*k}\big)(u)\right] - \frac{2\mu_1}{\mu_2\theta}\,\psi(u) \end{aligned} \tag{16}$$

(note we can interchange the translation transform T_0 with Σ). We mention further that from formula (7) in Borovkov and Dickson (2008), the convolution $(\tilde{p}_1 * p_1^{*k})(u)$ is explicitly available for any k; in particular, we have that

$$(\tilde{p}_1 * p_1^{*k})(u) = \frac{u}{k+1}\,p_1^{*(k+1)}(u).$$

We also note that a formula analogous to (16) holds for $K_2(u)$, when $\tilde{p}_1(x)$ is replaced by $\tilde{p}_2(x) = xp_2(x)$, where p_2 is a density for the distribution P_2.

As already mentioned in Section 1, it is reasonable to expect that the sign of the covariance between $V(T)$ and $V(T-)$ might be affected by the presence of an ageing property by the equilibrium distribution P_1 associated with the claim size distribution P. In fact, we shall need the following definition.

Definition 2.1. *A distribution F supported on $[0,\infty)$ is HNWUE (HNBUE), we call this as 'harmonic new worse (better) than used in expectation', if for all $x \geq 0$,*

$$\int_x^\infty \overline{F}(t)dt \geq (\leq) \mu_F e^{-x/\mu_F},$$

where μ_F is the mean of F.

The HNWUE (HNBUE) classes were introduced by Rolski (1975) and have further been studied by Klefsjö (1981,1982). They seem to be the largest among the commonly used ageing classes of distributions. In particular, if a distribution is DFR (IFR), then it is also HNWUE (HNBUE). For further details on these, and various other reliability classes, we refer e.g. to Willmot an Lin (2001) or Lai and Xie (2006).

An immediate application of Theorem 2.1 is the following, which shows that when the initial surplus is zero, membership of P_1 in the HNWUE or HNBUE class determines the sign of the covariance between $V(T)$ and $V(T-)$.

Corollary 2.1. *If P_1 is HNWUE (HNBUE), then* $C(0) \geq (\leq) 0$.

Proof. Let P_1 be HNWUE (HNBUE). Then

$$\int_x^\infty \overline{P}_1(t)dt \geq (\leq) \frac{\mu_2}{2\mu_1} e^{-2\mu_1 x/\mu_2}.$$

Integrating with respect to x over the interval $[0, \infty)$, we obtain

$$\frac{\mu_3}{3\mu_1} \geq (\leq) \frac{\mu_2^2}{2\mu_1^2},$$

and the result is now obvious in view of (11). □

Remark 2.1. Note that the ageing condition in the above result is imposed on the equilibrium distribution P_1 rather than the original claim size distribution P. It is very easy to find examples where P_1 is in the HNWUE or HNBUE class. In particular, if P has a decreasing (increasing) failure rate, so does P_1, hence they are both HNWUE (HNBUE) distributions. Indeed, if P is DFR (IFR) distribution and $h_{P_1}(x)$ is the failure rate of the equilibrium distribution, then

$$\int_0^\infty \frac{\overline{P}(x+t)}{\overline{P}(x)} dt = \frac{\overline{P}_1(x)}{p_1(x)} = \frac{1}{h_{P_1}(x)}$$

is nondecreasing (nonincreasing), so P_1 is also DFR (IFR) distribution. For more details, see Willmot and Lin (2001, Chapter 2).

Despite providing some insight into the way that the variables $V(T)$, $V(T-)$ are correlated, a shortcoming of Corollary 2.1 is that it works only in one direction. There exist distributions which are neither HNWUE nor HNBUE, and for such distributions Corollary 2.1 cannot be used. Instead, we give below a result which has an 'if and only if' form, and this is based on the notion of the coefficient of variation (CV) of the equilibrium distribution P_1. We denote this by CV_{P_1}; if μ_{P_1}, σ_{P_1} are, respectively, the mean and standard deviation of a random variable having distribution P_1, then one can verify that

$$CV_{P_1} = \frac{\sigma_{P_1}}{\mu_{P_1}} = \frac{\sqrt{\dfrac{\mu_3}{3\mu_1} - \dfrac{\mu_2^2}{4\mu_1^2}}}{\dfrac{\mu_2}{2\mu_1}}. \tag{17}$$

Note that when a distribution is HNWUE (HNBUE), then its coefficient of variation is greater than (less than) or equal to one, see Bhattacharjee and Sengupta (1996).

We further introduce the function

$$b(u) = \frac{a(u)K_1(u)}{K_2(u)}, \quad u \geq 0,$$

where the functions $a(u)$ and $K_i(u)$ for $i = 1, 2$ are as in Theorem 2.1. Note in particular that $b(0) = 1$. Using the above notation, Theorem 2.1 admits the following equivalent representation.

Theorem 2.2. *For any $u \geq 0$, the function $C(u)$ is given by the formula*

$$C(u) = \frac{\mu_2^2}{4\mu_1^2} \frac{K_2(u)}{\psi(u)} \left[\frac{1}{2}(CV_{P_1}^2 + 1) - b(u) \right]. \tag{18}$$

Moreover, for $u = 0$ we have

$$C(0) = \frac{\mu_2^2}{8\mu_1^2}(CV_{P_1}^2 - 1). \tag{19}$$

The following is now an immediate deduction from the above theorem.

Corollary 2.2. *For an arbitrary (fixed) $u \geq 0$, it holds that $b(u) \leq (\geq) \frac{1}{2}(CV_{P_1}^2 + 1)$ if and only if $C(u) \geq (\leq) 0$.*

Moreover, equation (19) gives the aforementioned characterization for the sign of the covariance when $u = 0$, in terms of the coefficient of variation CV_{P_1}.

Corollary 2.3. *It holds that $CV_{P_1} \geq 1$ $(0 \leq CV_{P_1} \leq 1)$* if and only if $C(0) \geq (\leq) 0$.

We note that the Cauchy-Schwarz inequality implies that $\mu_2^2 \leq \mu_1\mu_3$, which means that

$$\frac{\mu_3}{3\mu_1} - \frac{\mu_2^2}{4\mu_1^2} \geq \frac{\mu_2^2}{12\mu_1^2},$$

see Rao and Feldman (2001). Thus, by (17) we get that CV_{P_1} is always greater than or equal to $\sqrt{3}/3$. This means that, in fact, the covariance $C(0)$ is non-positive for $\sqrt{3}/3 \leq CV_{P_1} \leq 1$.

In the present context, an interesting question is whether the condition $C(0) = 0$ implies that the claim size distribution is exponential. In view of Corollary 2.3, the answer is negative, since a unit value for the coefficient of variation does not characterize the exponential distribution, even if we restrict attention to absolutely continuous distributions (see for example Bhattacharjee and Sengupta (1996)). When we only consider HNBUE and HNWUE distributions, however, we have the following useful result.

Lemma 2.1. (Basu and Bhattacharjee, 1984). *Suppose that a distribution F is either HNBUE or HNWUE. Then F is exponential if and only if its coefficient of variation is equal to 1.*

In fact, Basu and Bhattacharjee (1984) gave the result only for the HNBUE case, however it is easy to see that it holds in the form given above. The above discussion (see also Example 2.1, below) and (19) yield the following characterization, keeping in mind that $K_2(u) > 0$ for any $u \geq 0$.

Proposition 2.1. *Suppose that the equilibrium distribution P_1 associated with the claim size distribution P is either HNBUE or HNWUE. Then, the following are equivalent:*

(i) $C(0) = 0$.
(ii) $CV_{P_1} = 1$.
(iii) *P_1 is exponential distribution.*
(iv) $C(u) = 0$ *for any* $u \geq 0$.

Note that the claim size distribution P is an exponential distribution with mean μ_1 if and only if the equilibrium distribution P_1 is also exponential with the same mean. Thus, in particular, we see that $C(0) = 0$ if and only if the claim sizes are exponential.

We close this section with an example, illustrating the case of exponential claim sizes.

Example 2.1. (Exponential) If the claim size distribution function is $P(x) = 1 - e^{-\xi x}$, $x \geq 0$, $\xi > 0$, then

$$\overline{P}(x) = \overline{P}_1(x) = \overline{P}_2(x) = e^{-\xi x},$$

while it is well-known (see for example Rolski *et al.* (1999, Chapter 5)) that the probability of ruin is given by

$$\psi(u) = \frac{1}{1+\theta} e^{-\theta\xi u/(1+\theta)}, \quad u \geq 0.$$

Consequently, we get that $a(u) = 1$ for any $u \geq 0$, while

$$\overline{G}_1(x) = \overline{G}_2(x) = (\xi x + 1) e^{-\xi x}, \quad x \geq 0,$$

which is the tail of a Gamma$(2, \xi)$ distribution. Moreover, from the last expression we obtain that we have that $K_1(u) = K_2(u)$ and this, in turn, yields

$$b(u) = \frac{a(u) K_1(u)}{K_2(u)} = 1.$$

By (8) it then follows that $C(u) = 0$ for any $u \geq 0$.

3. An asymptotic result when the adjustment coefficient exists

The expression for the function $C(u)$ in Theorem 2.1 may not always be easy to obtain explicitly. In this section, we give an asymptotic result for $C(u)$ as the initial capital $u \to \infty$. This is given in terms of the adjustment coefficient, R, for the risk model, defined as the unique positive solution (provided that such a solution exists) of the Lundberg equation,

$$\int_0^\infty e^{Rt} dP_1(t) = 1 + \theta.$$

If the adjustment coefficient exists, then the well-known Cramér-Lundberg asymptotic result [see, for example, Rolski *et al.* (1999, p. 172)] gives that

$$D := \lim_{u\to\infty} e^{Ru}\psi(u) = \frac{\int_0^\infty e^{Rt}\overline{P}_1(t)dt}{\int_0^\infty t e^{Rt} dP_1(t)} = \frac{\theta}{R\int_0^\infty t\, e^{Rt} dP_1(t)}, \tag{20}$$

under the condition that $\int_0^\infty t e^{Rt} dP_1(t) < \infty$.

Recall that $\overline{P}_2(t) = 1 - P_2(t)$ is the tail of the equilibrium distribution associated with P_1 and let $\overline{P}_3(t) = 1 - P_3(t)$ be the tail of the equilibrium distribution associated with P_2. Then the following is easily checked using integration by parts.

Lemma 3.1. *For the functions* $\overline{P}_2(t)$, $\overline{P}_3(t)$, *it holds that*

$$\int_0^\infty e^{Rt}\,\overline{P}_2(t)\,dt \;=\; \frac{2\mu_1\theta - \mu_2 R}{\mu_2 R^2} \tag{21}$$

and

$$\int_0^\infty e^{Rt}\,\overline{P}_3(t)\,dt \;=\; \frac{6\mu_1\theta - 3\mu_2 R - \mu_3 R^2}{\mu_3 R^3}. \tag{22}$$

We shall also need the next result, which follows from the last lemma.

Lemma 3.2. *For the functions* $\overline{G}_1(t)$, $\overline{G}_2(t)$ *defined in* (5) *and* (6), *it holds that*

$$\int_0^\infty e^{Rt}\,\overline{G}_1(t)\,dt \;=\; \frac{2\mu_1\theta - \mu_2 R}{\mu_2 R^2} + \frac{2\mu_1}{\mu_2}\int_0^\infty e^{Rt}\,t\overline{P}_1(t)\,dt, \tag{23}$$

and

$$\int_0^\infty e^{Rt}\,\overline{G}_2(t)\,dt \;=\; \frac{6\mu_1\theta - 3\mu_2 R - \mu_3 R^2}{\mu_3 R^3} + \frac{3\mu_2}{\mu_3}\int_0^\infty e^{Rt}\,t\overline{P}_2(t)\,dt. \tag{24}$$

Proof. By the definition of the functions $\overline{G}_1(t)$ and $\overline{G}_2(t)$, integration by parts yields that

$$\overline{G}_1(t) \;=\; \overline{P}_2(t) + \frac{2\mu_1}{\mu_2}\,t\overline{P}_1(t)$$

and

$$\overline{G}_2(t) \;=\; \overline{P}_3(t) + \frac{3\mu_2}{\mu_3}\,t\overline{P}_2(t),$$

respectively. Multiplying by e^{Rt} and integrating over $(0,\infty)$, the result follows by (21) and (22). □

Our main result in this section concerns the asymptotic behaviour of $C(u)$, which depends on the asymptotics of the functions

$$\tilde{K}_i(u) \;=\; e^{Ru}K_i(u), \qquad i = 1, 2.$$

Since for $i = 1, 2$, the functions $K_i(u)$ satisfy the renewal equations (10), the limit of $\tilde{K}_i(u)$ as $u \to \infty$ can be obtained from Theorem V.7.1. in Asmussen (2003) provided we show that the functions $e^{Ru}\overline{G}_i(u)$ are directly Riemann integrable (d.R.i.), see Asmussen (2003) for further details on this.

Using an argument as in Embrechts *et al*. (1997, pp. 31-32), we see that if

$$\int_0^\infty x e^{Rx} dG_i(x) < \infty, \tag{25}$$

then $e^{Ru}\overline{G}_i(u)$ is a d.R.i. function for $i = 1,2$.

We now state the following result.

Theorem 3.1. *Assume that* (25) *holds for* $i = 1,2$. *Then*

$$\begin{aligned}\lim_{u\to\infty} C(u) &= \frac{1}{2\theta\mu_1 R}\left\{\frac{6\mu_1\theta - 3\mu_2 R - \mu_3 R^2}{3R} - \frac{2\mu_1\mu_2\theta - \mu_2^2 R}{2\mu_1}\left(\frac{2\mu_1}{\mu_2 R} - \frac{1}{\theta}\right)\right\}\\ &\quad + \frac{R\mu_2}{2\theta\mu_1}\int_0^\infty e^{Rt} t\left[\overline{P}_2(t) - \left(\frac{2\mu_1}{\mu_2 R} - \frac{1}{\theta}\right)\overline{P}_1(t)\right]dt.\end{aligned} \tag{26}$$

Proof. By (8) we obtain

$$\lim_{u\to\infty} C(u) = \frac{1}{\lim_{u\to\infty} e^{Ru}\psi(u)}\left\{\frac{\mu_3}{6\mu_1}\lim_{u\to\infty} e^{Ru}K_2(u) - \frac{\mu_2^2}{4\mu_1^2}\lim_{u\to\infty} a(u)\lim_{u\to\infty} e^{Ru}K_1(u)\right\}. \tag{27}$$

First we note that, under the assumptions of the theorem, we have that $\int_0^\infty t e^{Rt} dP_1(t) < \infty$, so that the Cramér-Lundberg result in (20) holds. Further, the functions $e^{Ru}\overline{G}_1(u)$ and $e^{Ru}\overline{G}_2(u)$ are d.R.i., so that Theorem V.7.1. in Asmussen (2003) applies. More precisely, and if we define for simplicity two constants D_1, D_2 by

$$D_i = \lim_{u\to\infty} e^{Ru}K_i(u)$$

for $i = 1,2$, then we obtain, using also the formulae from Lemma 3.2, that

$$D_1 = \frac{\int_0^\infty e^{Rt}\overline{G}_1(t)dt}{\int_0^\infty t e^{Rt} dP_1(t)} = \frac{\dfrac{2\mu_1\theta - \mu_2 R}{\mu_2 R^2} + \dfrac{2\mu_1}{\mu_2}\int_0^\infty e^{Rt} t\overline{P}_1(t)dt}{\int_0^\infty t e^{Rt} dP_1(t)}, \tag{28}$$

and

$$D_2 = \frac{\int_0^\infty e^{Rt}\overline{G}_2(t)dt}{\int_0^\infty t e^{Rt} dP_1(t)} = \frac{\dfrac{6\mu_1\theta - 3\mu_2 R - \mu_3 R^2}{\mu_3 R^3} + \dfrac{3\mu_2}{\mu_3}\int_0^\infty e^{Rt} t\overline{P}_2(t)dt}{\int_0^\infty t e^{Rt} dP_1(t)}. \tag{29}$$

Next, applying l'Hôpital's rule and using the equation (20) we see that

$$\lim_{u\to\infty} e^{Ru}\int_u^\infty \psi(t)\,dt \;=\; \frac{D}{R}.$$

Furthermore, by (7) and (9) we obtain

$$\begin{aligned}\lim_{u\to\infty} a(u) \;&=\; \lim_{u\to\infty}\frac{1}{\theta}\left[\frac{\psi_1(u)}{\psi(u)}-1\right]\\ &=\; \frac{1}{\theta}\left[\frac{2\mu_1\theta}{\mu_2}\lim_{u\to\infty}\frac{e^{Ru}\int_u^\infty \psi(t)\,dt}{e^{Ru}\psi(u)}-1\right].\end{aligned}$$

From the last two expressions we deduce, after a little algebra, that

$$\lim_{u\to\infty} a(u) \;=\; \frac{2\mu_1}{\mu_2 R}-\frac{1}{\theta}.$$

Inserting this result into (27) yields

$$\lim_{u\to\infty} C(u) \;=\; \frac{1}{D}\left\{\frac{\mu_3}{6\mu_1}D_2-\frac{\mu_2^2}{4\mu_1^2}D_1\left(\frac{2\mu_1}{\mu_2 R}-\frac{1}{\theta}\right)\right\}. \tag{30}$$

By substituting the constants D, D_1, D_2 from (20), (28) and (29) into the last expression, the result follows after some routine calculations. □

We now present an example to illustrate the asymptotic result of Theorem 3.1.

Example 3.1. (Mixture of two exponentials) We assume that the claim size distribution P is a mixture of two exponential distributions, so that its tail is given by

$$\overline{P}(x) \;=\; q e^{-b_1 x}+(1-q)\,e^{-b_2 x}, \qquad x\ge 0,\;\; 0<q<1,\;\; b_1,b_2>0.$$

Then,

$$\overline{P}_1(x) \;=\; q_1 e^{-b_1 x}+(1-q_1)\,e^{-b_2 x}$$

and

$$\overline{P}_2(x) \;=\; q_2 e^{-b_1 x}+(1-q_2)\,e^{-b_2 x}$$

where

$$q_1 \;=\; \frac{q b_2}{q b_2+(1-q)\,b_1} \;\text{ and }\; q_2 \;=\; \frac{q_1 b_2}{q_1 b_2+(1-q_1)\,b_1}.$$

Moreover, the probability of ruin has the form

$$\psi(u) = C_1 e^{-r_1 u} + C_2 e^{-r_2 u},$$

where r_1, r_2 are the roots of Lundberg's equation and C_1, C_2 are positive constants [see, for example, Dufresne and Gerber (1989)]. Consequently, we derive that

$$\psi_1(u) = \frac{C_1 r_2}{C_1 r_2 + C_2 r_1} e^{-r_1 u} + \frac{C_2 r_1}{C_1 r_2 + C_2 r_1} e^{-r_2 u},$$

while the tails of the distributions $G_1(x), G_2(x)$ are given respectively by

$$\overline{G}_1(x) = q_2 (1 + b_1 x) e^{-b_1 x} + (1 - q_2)(1 + b_2 x) e^{-b_2 x}$$

and

$$\overline{G}_2(x) = q_3 (1 + b_1 x) e^{-b_1 x} + (1 - q_3)(1 + b_2 x) e^{-b_2 x},$$

where

$$q_3 = \frac{q_2 b_2}{q_2 b_2 + (1 - q_2) b_1}.$$

This shows that each of G_1, G_2 is a mixture of two Gamma distributions with parameters $(2, b_1)$ and $(2, b_2)$ and respective weights q_2 and $1 - q_2$ in the former case and q_3 and $1 - q_3$ in the latter.

To illustrate the above, we now consider a standard example, see e.g. Gerber *et al.* (1987). More precisely, we take $\theta = 2/5$, $b_1 = 3$, $b_2 = 7$ and $q = 1/2$. Then, $r_1 = 1$ ($r_1 = R$ is the adjustment coefficient), $r_2 = 6$, $C_1 = 24/35$ and $C_2 = 1/35$. We assume that the tail of the claim size distribution is given by

$$\overline{P}(x) = \frac{1}{2} e^{-3x} + \frac{1}{2} e^{-7x}.$$

We then obtain that

$$\overline{P}_1(x) = \frac{7}{10} e^{-3x} + \frac{3}{10} e^{-7x},$$

$$\overline{P}_2(x) = \frac{49}{58} e^{-3x} + \frac{9}{58} e^{-7x},$$

while for the functions $\overline{G}_1(x)$, $\overline{G}_2(x)$, we derive that

$$\overline{G}_1(x) = 0.844828 (1 + 3x) e^{-3x} + 0.155172 (1 + 7x) e^{-7x}$$

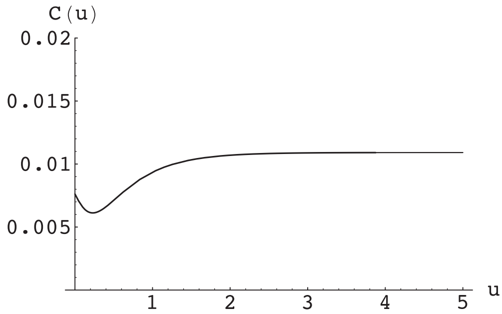

FIGURE 1: The covariance $C(u)$ when $0 \le u \le 5$.

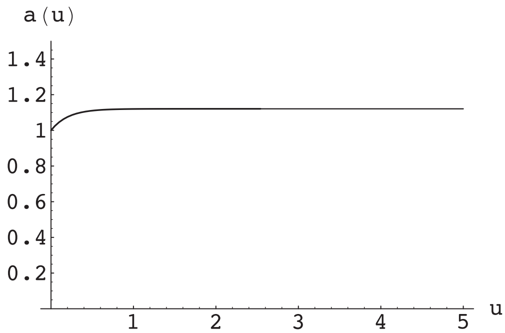

FIGURE 2: The function $a(u)$ when $0 \le u \le 5$.

and

$$\overline{G}_2(x) = 0.927027(1+3x)e^{-3x} + 0.072973(1+7x)e^{-7x}.$$

Moreover, from (9) we obtain that $\lim_{u\to\infty} a(u) = 1.12069$, while for the constants D, D_1 and D_2 defined in (20), (28) and (29), we find that $D = 0.685714$ $(= 24/35)$, $D_1 = 1.9064$, $D_2 = 2.03166$. Plugging these quantities into (26), or equivalently into (30), we deduce that

$$\lim_{u\to\infty} C(u) = 0.0109127.$$

Figures 1 and 2 present respectively the functions $C(u)$ and $a(u)$ for this example. Notice in particular that the function $C(u)$ first decreases and then increases, while it takes positive values for all $u \ge 0$; the function attains its minimum value for $u = 0.22944$.

In the previous section we gave a characterization of the exponential distribution under the assumption that the distribution P_1 is either HNBUE or HNWUE. Using a result by Sundt and dos Reis (2007), another simple characterization can be given which does not use that assumption.

More specifically, Sundt and dos Reis (2007) gave the following characterization of the distribution P of claim sizes in the classical risk model, assuming that the Cramér-Lundberg approximation (20) holds exact.

Lemma 3.3. [Sundt and dos Reis, 2007]**.** *Suppose that there exists* a $k < 1$ *such that*

$$\psi(u) = ke^{-Ru}, \quad u \geq 0.$$

Then the claim size distribution P is an exponential d.f. with mean $\mu_1 = [(1+\theta)R]^{-1}\theta$ *and* $k = (1+\theta)^{-1}$.

Theorem 3.2. *The claim size distribution is exponential with mean* $\mu_1 = [(1+\theta)R]^{-1}\theta$ *if and only if* $a(u) = 1$ *for any* $u \geq 0$.

Proof. The 'only if' part follows by Example 2.1. For the converse, we assume $a(u) = 1$, or equivalently,

$$\frac{1}{\theta}\left[\frac{\psi_1(u)}{\psi(u)} - 1\right] = 1.$$

Substituting $\psi_1(u) = 2\theta\mu_1 \int_u^\infty \psi(t)\,dt/\mu_2$, after a little algebra we arrive at the differential equation

$$\left(\int_u^\infty \psi(t)\,dt\right)' + \frac{2\theta\mu_1}{\mu_2(1+\theta)}\left(\int_u^\infty \psi(t)\,dt\right) = 0,$$

whose solution is

$$\int_u^\infty \psi(t)\,dt = \frac{\mu_2}{2\mu_1\theta}\, e^{-\frac{2\mu_1\theta}{\mu_2(1+\theta)}u}.$$

Taking the derivative with respect to u, we derive that

$$\psi(u) = \frac{1}{1+\theta}e^{-Ru}, \text{ where } R = \frac{2\mu_1\theta}{\mu_2(1+\theta)}.$$

By Lemma 3.3 the claim size distribution is an exponential d.f. with mean $\mu_1 = [(1+\theta)R]^{-1}\theta$. □

4. The correlation coefficient for $u = 0$ and numerical examples

We now consider the correlation coefficient between $V(T-)$ and $V(T)$ for the case where the initial surplus $u = 0$. We write $\rho(0)$ for this correlation coefficient.

When $u = 0$, the distribution of the surplus prior to ruin is the same as the distribution of the deficit at the time of ruin, namely

$$\Pr(|U(T)| \leq x \,|\, T < \infty, U(0) = 0) = \Pr(U(T-) \leq x \,|\, T < \infty, U(0) = 0) = P_1(x),$$

see Gerber and Shiu (1997). Thus, by (11) and (19), the correlation $\rho(0)$ is

$$\rho(0) = \frac{C(0)}{\sigma_{P_1}^2} = \frac{\dfrac{\mu_3}{6\mu_1} - \dfrac{\mu_2^2}{4\mu_1^2}}{\dfrac{\mu_3}{3\mu_1} - \dfrac{\mu_2^2}{4\mu_1^2}} = \frac{CV_{P_1}^2 - 1}{2CV_{P_1}^2} < \frac{1}{2}. \tag{31}$$

Example 4.1. (Mixture of two exponentials) We assume that the claim size d.f. is a mixture of two exponential distributions, so that it has a density

$$p(x) = qb_1e^{-b_1x} + (1-q)b_2e^{-b_2x}, \;\; x \geq 0, b_1, b_2 > 0, \; 0 < q < 1.$$

Then, the moments μ_1, μ_2, μ_3 are given by

$$\mu_n = q\frac{n!}{b_1^n} + (1-q)\frac{n!}{b_2^n}, \qquad \text{for} \;\; n = 1, 2, 3.$$

From (31), we see after a little algebra that the correlation coefficient is in this case

$$\rho(0) = \rho(0; q, b_1, b_2) = \frac{q(1-q)(b_2 - b_1)^2}{b_1 b_2 [qb_2 + (1-q)b_1]^2}.$$

For $b_1 \neq b_2$, we observe that $\rho(0) > 0$. We expect this result since, as we have seen in Example 3.1, the equilibrium distribution of P is also a mixture of two exponentials, namely P_1 is DFR, so it is HNWUE. Corollary 2.1 yields $C(0) \geq 0$, so $\rho(0) \geq 0$. For $b_1 = b_2$, we have $\rho(0) = 0$ (the case of exponential). In Figure 3 we consider a mixture of two exponential distributions with $q = 1/3$, $b_1 \in (0, 50]$ and $b_2 = 3$. We observe that for $b_1 = 3$ we have $\rho(0) = 0$ (exponential case).

Example 4.2. (Gamma). Let the claim size d.f. be a Gamma(a, b) with $a, b > 0$ and density

$$p(x) = \frac{b^a}{\Gamma(a)} x^{a-1} e^{-bx}, \;\; x \geq 0.$$

Then, the moments μ_1, μ_2, μ_3 are given by

$$\mu_1 = \frac{a}{b}, \;\; \mu_2 = \frac{a(a+1)}{b^2} \;\; \text{and} \;\; \mu_3 = \frac{a(a+1)(a+2)}{b^3}.$$

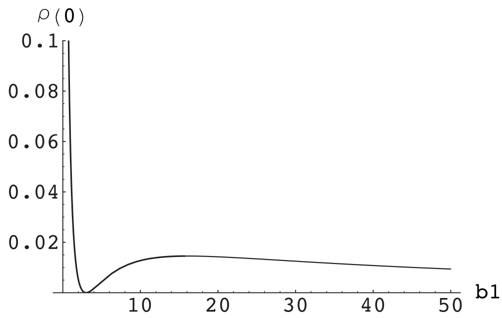

FIGURE 3: Correlation coefficient for a mixture of two exponential distributions with $q = 1/3$, $b_1 \in (0, 50]$ and $b_2 = 3$.

By (31), after some computations, the correlation is

$$\rho(0) = \rho(0; a) = \frac{1 - a}{a + 5},$$

independent of b. In Figure 4 (left part), we observe that $\lim_{a \to 0+} \rho(0) = sup_{a>0}\, \rho(0) = 0.2$, $\lim_{a \to \infty} \rho(0) = -1$ and $\rho(0) = 0$ for $a = 1$ (exponential case). Moreover, we have $\rho(0) > 0$ for $0 < a < 1$ (DFR case, see also Remark 2.1), whereas $\rho(0) < 0$ for $a > 1$ (IFR case). Further, the correlation as a function of a, is decreasing and convex. Finally, the coefficient of variation is given by

$$CV_{P_1}(a) = \sqrt{\frac{a + 5}{3(a + 1)}}.$$

The right part of Figure 4 shows that $CV_{P_1}(a)$ is also decreasing and convex with $\lim_{a \to \infty} CV_{P_1}(a) = \sqrt{3}/3$.

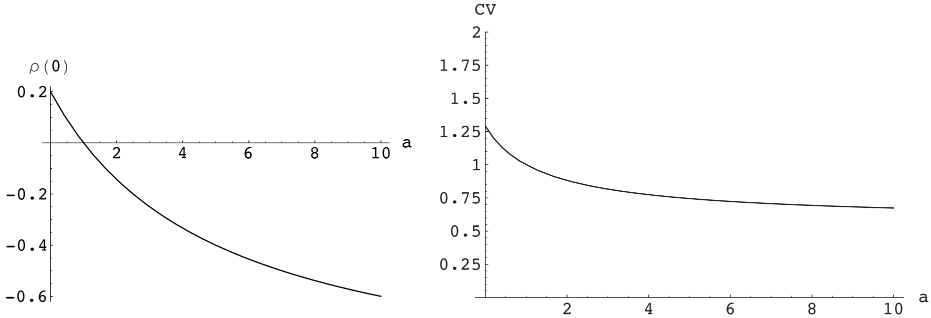

FIGURE 4: Correlation and CV_{P_1}, in the case where the claim size distribution is Gamma with shape parameter $a \in (0, 10]$.

Example 4.3. (Mixture of two Gamma densities with a common scale parameter, see Lai and Xie (2006, p. 49)). Suppose that the claim size d.f. is a mixture of two Gamma distributions with density function

$$p(x) = q\frac{b^{a_1}}{\Gamma(a_1)}x^{a_1-1}e^{-bx} + (1-q)\frac{b^{a_2}}{\Gamma(a_2)}x^{a_2-1}e^{-bx},$$
$$x \geq 0,\ a_1, a_2,\ b > 0,\ 0 < q < 1.$$

Then, the moments μ_1, μ_2, μ_3 are given by

$$\mu_1 = q\frac{a_1}{b} + (1-q)\frac{a_2}{b},\ \ \mu_2 = q\frac{a_1(a_1+1)}{b^2} + (1-q)\frac{a_2(a_2+1)}{b^2}$$

and

$$\mu_3 = q\frac{a_1(a_1+1)(a_1+2)}{b^3} + (1-q)\frac{a_2(a_2+1)(a_2+2)}{b^3}.$$

Using again (31), some straightforward calculations show that the correlation coefficient is

$$\rho(0) = \rho(0; q, a_1, a_2) = \frac{2A - 3B^2}{4A - 3B^2},$$

where

$$A = [qa_1 + (1-q)a_2]\,[qa_1(a_1+1)(a_1+2) + (1-q)a_2(a_2+1)(a_2+2)]$$

and

$$B = qa_1(a_1+1) + (1-q)a_2(a_2+1).$$

We observe that the correlation is independent of b and $\lim_{q\to 1-}\rho(0) = (1-a_1)/(a_1+5)$, the case of a Gamma distribution. Figure 5 shows the value of $\rho(0)$

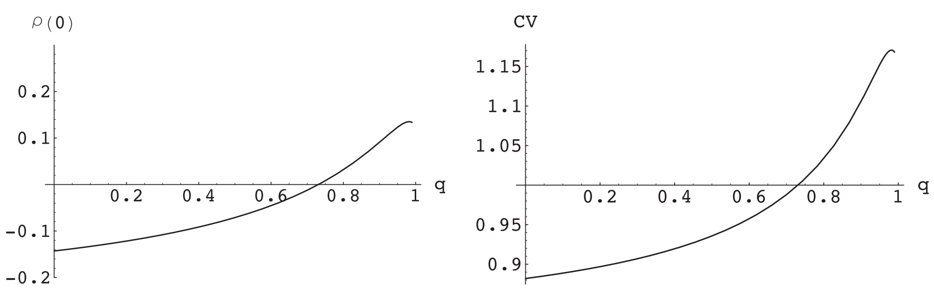

FIGURE 5: Correlation and CV_{P_1}, for mixture of two Gamma distributions with a common scale parameter b, $a_1 = 1/3$, $a_2 = 2$ and $q \in (0,1)$.

(left part) and CV_{P_1} (right part), as a function of the weight q in the mixture of the two Gamma densities for $a_1 = 1/3$, $a_2 = 2$. As expected, we see that $\rho(0) < 0$ when $CV_{P_1} < 1$ and $\rho(0) > 0$ when $CV_{P_1} > 1$. Moreover, for $q_0 = (57 - 3\sqrt{10})/65 \cong 0.730972$ we have $\rho(q_0) = 0$ and $CV_{P_1}(q_0) = 1$.

Acknowledgement

We thank both referees for some useful suggestions which improved the presentation of our results.

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