A MULTIVARIATE DISCRETE POISSON-LINDLEY DISTRIBUTION: EXTENSIONS AND ACTUARIAL APPLICATIONS

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ABSTRACT

This paper proposes multivariate versions of the continuous Lindley mixture of Poisson distributions considered by Sankaran (1970). This new class of distributions can be used for modelling multivariate dependent count data when marginal overdispersion is present. After discussing some of its properties, a general multivariate model with Poisson-Lindley marginals and with a flexible covariance structure is proposed. Several specific models as well as one that allows correlations of any sign are considered, and then some estimation methods are discussed. Finally, some illustrative examples are given for fitting and demonstrating the usefulness of these bivariate distributions.

KEYWORDS

Actuarial, Bonus-Malus, Frequency Distribution, Poisson-Lindley Distribution, Regression, Sarmanov-Lee Family.

1. INTRODUCTION

In this paper, we study a class of multivariate mixed Poisson distributions, extending the Poisson-Lindley distribution (Sankaran (1970)) from the univariate to the multivariate case. The particular case of the bivariate distribution is studied in detail.

The mixture approach is a suitable methodology for deriving new families of multivariate distributions which has received a lot of attention in the past. For example, Aitchison and Ho (1989) obtained the multivariate Poisson-Log normal distribution; Stein et al. (1987) proposed a reparameterization of the Sichel (Sichel (1971)) distribution (Poisson-inverse Gaussian distribution) and extended the result to the bivariate case; and Gómez-Déniz et al. (2008) derived the multivariate negative binomial-inverse Gaussian distribution. A multivariate version of the normal-inverse Gaussian distribution introduced by Barndorff-Nielsen (1992) has been developed by Protassov (2004) and Øigard and Hanssen

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(2002). Finite mixtures of multivariate Poisson distributions have been studied by Karlis and Meligkotsidou (2007), and more recently, Sarabia and Gómez-Déniz (2011) studied the multivariate Poisson-beta distribution, extending some results obtained originally by Holla and Bhattacharya (1965). Karlis and Xekalaki (1987) studied general properties of Poisson mixtures model with respect to the bivariate case. Lai (2006) presented results concerning the construction of bivariate discrete distributions, while Sarabia and Gómez-Déniz (2008) examined some recent methods for the construction of multivariate distributions (discrete and continuous) including the mixture models.

One of the advantages of the mixture methodology is that for these distributions, formulas for moments and correlations have simple closed-forms and computation is straightforward. The extension of a mixture to the multivariate case is usually simple as well, and the marginal distributions are also simple and as the same form as the departure distribution. Moreover, the simulation and Bayesian estimation of mixtures are quite direct. Since the introduction of simulation-based methods for inference (especially the Gibbs sampler in a Bayesian framework), complicated densities such as those arising from mixture modeling can be handled satisfactorily.

In this paper, we propose a multivariate Poisson-Lindley distribution constructed by mixing independent Poisson distributions with the Lindley distribution. The continuous Lindley distribution (Lindley (1958) and Sankaran (1970)) is not commonly used in statistical literature. This distribution is a continuous one which depends on a single parameter obtained from the convex sum of an exponential and a gamma distribution.

Sankaran (1970) introduced a Poisson mixture distribution derived by mixing the Poisson parameter using the Lindley distribution (Lindley (1958)). The pdf of the Lindley distribution is given by

$$\pi_\Lambda(\lambda) = \frac{\theta^2}{1+\theta}(1+\lambda)\,e^{-\theta\lambda}, \;\; \lambda > 0, \;\; \theta > 0. \tag{1}$$

One of the main advantages of this distribution (it can be written as a convex sum of an exponential and a gamma distribution) is its simple formulation since it only depends on one parameter and closed form expressions for moments, cumulative distribution function, failure rate and other characteristics can be easily obtained. See Ghitany et al. (2008) for more details.

The k-th raw moment of the distribution (1) is given by

$$E(\Lambda^k) = \frac{k!}{1+\theta}\,\frac{\theta+k+1}{\theta^k},$$

while the Laplace transform is

$$L(t) = E(e^{-t\Lambda}) = \frac{\theta^2}{1+\theta}\,\frac{\theta+t+1}{(\theta+t)^2}. \tag{2}$$

The resulting distribution was found to be more suitable for modeling empirical data provided in Sankaran (1970) than the negative binomial and Hermite distributions.

Henceforth, we shall use $\Lambda \sim \mathcal{L}(\theta)$ to indicate that the continuous random variable Λ follows a Lindley distribution in (1).

As stated above, we propose multivariate versions of the continuous Lindley mixture of Poisson distributions considered by Sankaran (1970). The new class of distributions can be used for modelling multivariate dependent count data. After discussing some of its properties, a general multivariate model with Poisson-Lindley marginals and with a flexible covariance structure is proposed. This model is obtained by using the Sarmanov family of distributions; see Sarmanov (1966) and Lee (1996). Several specific models, as well as one that allows correlations of any sign, are considered. Estimation methods are discussed and some results that are useful in actuarial statistics are then highlighted. Finally, some illustrative examples for bivariate frequency data are given, including a portfolio of automobile insurance contracts when the count variable material represents two kinds of claims, namely, material damage and bodily injury. For this particular example, bonus-malus premiums are computed.

The rest of this paper is organized as follows. In Section 2, we describe the basic properties of the univariate Poisson-Lindley distribution. A general multivariate model with a Poisson-Lindley marginal is proposed in Section 3, where some of the properties of the new model are discussed. The mean vector, the covariance matrix and a formula for computing multivariate probabilities are all presented, along with the regression function and some results concerning problems encountered in the insurance industry. Section 4 describes an extension of the latter model, in which the Lindley distribution is replaced by a bivariate distribution with Lindley marginal distributions. As a consequence, a model with a flexible covariance structure is obtained, which would accommodate negative correlation as well. Estimation methods are then discussed in Section 5. A numerical application for bivariate frequencies data are given in Section 6 after which some concluding remarks are made finally in Section 7.

2. The univariate Poisson-Lindley distribution

In this section, we describe some basic properties of the univariate Poisson-Lindley distribution proposed by Sankaran (1970), and studied recently by Ghitany and Al-Mutairi (2009).

Definition 1. *A univariate random variable X that follows the Poisson-Lindley distribution is defined by the stochastic representation*

$$\begin{aligned} X \mid (\Lambda = \lambda) &\sim \mathcal{Po}(\phi\lambda), \\ \Lambda &\sim \mathcal{L}(\theta), \end{aligned}$$

where $\phi > 0$.

In this case, the probability mass function is given by

$$\Pr(X=x) = p_x = \frac{\theta^2\phi^x}{1+\theta}\,\frac{\theta+\phi+x+1}{(\theta+\phi)^{x+2}}, \tag{3}$$

for $\phi > 0$, $\theta > 0$ and $x = 0, 1, \ldots$ By setting $\phi = 1$ in (3), we obtain the probability mass function of the discrete Poisson-Lindley distribution discussed by Sankaran (1970).

Henceforth, we shall use $f_{\phi,\theta}(x)$ to denote the probability mass function in (3).

The Laplace transform of this discrete distribution is given by

$$E(e^{-tX}) = \frac{\theta^2}{\theta+1}\,\frac{\theta+\phi e^t+2}{(\theta+\phi e^t+1)^2}$$

and the factorial moments are given by

$$\mu_{[k]}(X) = E[X(X-1)\ldots(X-k+1)] = k!\left(\frac{\phi}{\theta}\right)^k \frac{\theta+k+1}{\theta+1}, \quad k=1,2,\ldots \tag{4}$$

From expression (4) we can obtain the first two moments around the origin given by

$$E(X) = \phi\,\frac{\theta+2}{\theta(\theta+1)},$$

$$E(X^2) = \phi\,\frac{2\phi(\theta+3)+\theta(\theta+2)}{\theta^2(1+\theta)}.$$

Furthermore, the distribution is unimodal since the mixture of the Poisson distribution results in a unimodal distribution. In addition, from (3), it is simple to verify that

$$f_{\phi,\theta}(0) = \left(\frac{\theta}{\phi+\theta}\right)^2 \frac{\theta+\phi+1}{\theta+1},$$

$$f_{\phi,\theta}(x+1) = \frac{\phi(\theta+\phi+x+2)}{(\theta+\phi)(\theta+\phi+x+1)}\, f_{\lambda,\phi}(x), \quad x=0,1,\ldots, \tag{5}$$

$$f_{\phi+1,\theta}(x) = \left(\frac{\phi+1}{\phi}\right)^x \left(\frac{\theta+\phi}{\theta+\phi+1}\right)^{x+2} \frac{\theta+\phi+x+2}{\theta+\phi+x+1}\, f_{\phi,\theta}(x).$$

Some other properties of this distribution have been discussed by Ghitany and Al-Mutairi (2009).

3. The bivariate and multivariate model

In this section, we study a mixed multivariate discrete distribution and some of its more important properties.

3.1. The multivariate model

The basic bivariate model we present here is of the form

$$\Pr(X_1 = x_1, \ldots, X_d = x_d) = \int_0^\infty \Pr(X_1 = x_1, \ldots, X_d = x_d | \Lambda)\, dF_\Lambda(\lambda),$$

in which it is assumed that

(i) Conditionally on Λ, the random variables $X_1, \ldots, X_d$ are independent, and

(ii) The conditional distribution of X_i, $i = 1, \ldots, d$ is univariate Poisson with parameters $\lambda\phi_i$, $\lambda > 0$, $\phi_i > 0$, denoted by $X_i \sim \mathcal{P}o(\lambda\phi_i)$.

Observe that this kind of model is similar to a longitudinal or panel data model where, for example in insurance framework, all d contracts of the same insured can be observed over time. See for example Frees (2004) and Boucher et al. (2009).

When $F_\Lambda(\lambda)$ is the cumulative distribution function of the Lindley distribution, the multivariate Poisson-Lindley distribution is obtained, which is formalized in the following definition.

Definition 2. *A multivariate random variable $\boldsymbol{X} = (X_1, \ldots, X_d)^\top$ following a basic multivariate Poisson-Lindley distribution is defined by the following stochastic representation:*

$$X_i \,|\, (\Lambda = \lambda) \sim \mathcal{P}o(\lambda\phi_i), \;\; i = 1, 2, \ldots, d, \;\; \textit{independent},$$
$$\Lambda \sim \mathcal{L}(\theta),$$

where $\phi_i > 0$, $i = 1, 2, \ldots, d$.

Proposition 1. *The probability mass function of the multivariate Poisson-Lindley distribution is given by*

$$p_{\underline{x}}(\theta) = \frac{\beta(\theta)}{\beta(\theta + \sum_{i=1}^d \phi_i)} \frac{\prod_{i=1}^d \phi_i^{x_i}}{\prod_{i=1}^d x_i!} \mu'_{\sum x_i}(\theta + \sum_{i=1}^d \phi_i), \;\; x_1, \ldots, x_d = 0, 1, \ldots, \quad (6)$$

where $\mu'_{\sum x_i}(\theta + \sum_{i=1}^d \phi_i)$ is the raw moment of order $\sum_{i=1}^d x_i$ of

$$\pi(\lambda) = e^{-\lambda\varphi} h(\lambda)\beta(\varphi) \quad (7)$$

and $\theta + \sum_{i=1}^{d} \phi_i$ *is the parameter of the linear exponential family of distributions in (7).*

Proof. The proof is easily obtained by compounding and taking into account that the continuous Lindley distribution given in (1) is a member of (7) with $\varphi = \theta$, $h(\lambda) = 1 + \lambda$ and $\beta(\varphi) = \varphi^2/(1+\phi)$. □

From (6) and, based on the fact that

$$\begin{aligned} \mu'_{\sum x_i}(\theta - \sum_{i=1}^{d} \phi_i) &= \frac{(\theta + \sum \phi_i)^2}{1 + \theta + \phi_i} \int_0^\infty \lambda^{\sum x_i}(1+\lambda)\exp(-\lambda(\theta + \sum \phi_i))d\lambda \\ &= \frac{(\theta + \sum \phi_i)^2}{1 + \theta + \phi_i} \frac{(\sum x_i)\,!\,(\theta + \sum \phi_i + \sum x_i + 1)}{(\theta + \sum \phi_i)^{\sum x_i + 2}}, \end{aligned}$$

it is readily seen that the probability mass function of the multivariate discrete Poisson-Lindley distribution can be expressed as

$$\begin{aligned} \Pr(X_1 = x_1, \ldots, X_d = x_d) &= p_{x_1, x_2, \ldots, x_d} \\ &= \frac{\theta^2 (\sum_{i=1}^{d} x_i)\,!\prod_{i=1}^{d} \phi_i^{x_i}}{(1+\theta)\prod_{i=1}^{d} x_i\,!} \frac{\theta + 1 + \sum_{i=1}^{d}(x_i + \phi_i)}{(\theta + \sum_{i=1}^{d} \phi_i)^{2 + \sum_{i=1}^{d} x_i}}. \end{aligned} \quad (8)$$

In the next result, (8) is presented in terms of a univariate Poisson-Lindley distribution.

Proposition 2. *The joint probability mass function of* $(X_1, \ldots, X_d)$ *can be expressed as*

$$\Pr(X_1 = x_1, \ldots, X_d = x_d) = \frac{(\sum_{i=1}^{d} x_i)\,!\prod_{i=1}^{d} \phi_i^{x_i}}{\prod_{i=1}^{d} x_i\,!\,(\sum_{i=1}^{d} \phi_i)^{\sum_{i=1}^{d} x_i}} \Pr(X = x_1 + \ldots + x_d), \quad (9)$$

where X *is a univariate Poisson-Lindley variable with parameters* θ *and* $\phi_1 + \phi_2$.

Proof. The proof is direct. □

From the assumptions in (i) and (ii), we deduce that the marginal distributions are univariate Poisson-Lindley distributions as in (3) and that any subvector $(X_1, \ldots, X_s)$, for $s < d$, is again a basic multivariate Poisson-Lindley distribution of dimension s. The joint moment generating function is given by

$$M_{\mathbf{X}}(t_1, \ldots, t_d) = \frac{\theta^2}{1+\theta} \frac{1 + \theta + \sum_{i=1}^{d} \phi_i (1 - e^{t_i})}{(\theta + \sum_{i=1}^{d} \phi_i (1 - e^{t_i}))^2}. \quad (10)$$

Furthermore, it is a simple exercise to show that the probability generating function of the multivariate Poisson-Lindley distribution is related to the moment generating function of the Lindley distribution, and that it can be expressed as

$$G_{\mathbf{X}}(s_1, \ldots, s_d) = \exp\left\{\lambda \sum_{i=1}^{d} \phi_i (s_i - 1)\right\} = M_\Lambda\left(\sum_{i=1}^{d} \phi_i (s_i - 1)\right), \tag{11}$$

where $M_\Lambda(\cdot)$ is the moment generating function of Λ.

The moments can be obtained through conditional expectations or directly from (8) or (10). For example, we have

$$E(X_i) = \phi_i \frac{2+\theta}{\theta(1+\theta)}, \quad i = 1, 2, \ldots, d, \tag{12}$$

$$E(X_i^2) = \phi_i \frac{2\phi_i(3+\theta) + \theta(2+\theta)}{\theta^2(1+\theta)}, \quad i = 1, 2, \ldots, d, \tag{13}$$

$$\mathrm{Var}(X_i) = \phi_i \frac{\phi_i(\theta^2 + 4\theta + 2) + \theta^3 + 3\theta^2 + 2\theta}{\theta^2(1+\theta)^2}, \quad i = 1, 2, \ldots, d, \tag{14}$$

$$E(X_i X_j) = 2\phi_i \phi_j \frac{\theta + 3}{\theta^2(1+\theta)}, \quad i, j = 1, 2, \ldots, d, \ i \neq j. \tag{15}$$

As a result, we have the covariance as

$$\mathrm{Cov}(X_i, X_j) = \phi_i \phi_j \frac{\theta^2 + 4\theta + 2}{\theta^2(1+\theta)^2}, \quad i \neq j, \tag{16}$$

which is always positive, and so the model possesses only non-negative covariances.

3.2. The basic bivariate model

In particular, for $d = 2$, the probability mass function of the bivariate Poisson-Lindley distribution is given by

$$p_{x_1, x_2} = \frac{\beta(\theta)}{\beta(\theta - \phi_1 - \phi_2)} \frac{\phi_1^{x_1} \phi_2^{x_2}}{x_1!\, x_2!} \mu'_{x_1 + x_2}(\theta - \phi_1 - \phi_2),$$

which can be rewritten as

$$p_{x_1, x_2} = \frac{\theta^2 (x_1 + x_2)!\, \phi_1^{x_1} \phi_2^{x_2}}{(1+\theta) x_1!\, x_2!} \frac{x_1 + x_2 + \theta + \phi_1 + \phi_2 + 1}{(\theta + \phi_1 + \phi_2)^{x_1 + x_2 + 2}}, \tag{17}$$

with the corresponding probability generating function as

$$G_{\mathbf{X}}(s,t)) = \frac{\theta^2}{1+\theta}\,\frac{1+\theta+\phi_1(1-s)+\phi_2(1-t)}{(\theta+\phi_1(1-s)+\phi_2(1-t))^2}. \tag{18}$$

In this case, expression (9) reduces to

$$p_{x_1,x_2} = \binom{x_1+x_2}{x_1}\frac{\phi_1^{x_1}\phi_2^{x_2}}{(\phi_1+\phi_2)^{x_1+x_2}}\,\Pr(X = x_1+x_2),$$

where X is the univariate Poisson-Lindley variable with parameters θ and $\phi_1 + \phi_2$.

From (18), we can obtain the probability generating function of the random variables $Z = X_1 + X_2$ and $W = X_1 - X_2$

$$G_Z(s) = \frac{\theta^2}{1+\theta}\,\frac{1+\theta+(\phi_1+\phi_2)(1-s)}{(\theta+(\phi_1+\phi_2)(1-s))^2},$$

$$G_W(s) = \frac{\theta^2}{1+\theta}\,\frac{1+\theta+\phi_1+\phi_2-(\phi_1-\phi_2)s)}{(\theta+\phi_1+\phi_2-(\phi_1-\phi_2)s)^2},$$

respectively.

When $\phi_2 = 1$, we have the pgf in (18) to be of the homogeneous type (see Kemp, 1981 and Walhin and Paris, 2001). Therefore, by using the characterization given in Kocherlakota and Kocherlakota (1992), the distribution of X_2 given $X_1 + X_2$ is binomially distributed, i.e., $X_2|(X_1 + X_2) \sim \text{Bin}(X_1 + X_2, 1/(1 + \phi_1))$.

Remark 1. *Observe that when $\phi_1 = \phi_2$, the probability mass function in (17) is symmetric in its arguments. The distribution obtained here is different from the one reported by Arbous and Sichel (1954) even though it does have a similar appearance.*

The probability mass function in (17) satisfies the following recursion:

$$\begin{aligned} p_{x_1,x_2} &= \frac{\phi_1\phi_2(\theta+x_1+x_2+\phi_1+\phi_2+1)}{(\theta+\phi_1+\phi_2)(\theta+\phi_1+\phi_2+x_1+x_2)x_1}\,p_{x_1-1,x_2} \\ &\quad+\frac{\phi_1\phi_2(\theta+x_1+x_2+\phi_1+\phi_2+1)}{(\theta+\phi_1+\phi_2)(\theta+\phi_1+\phi_2+x_1+x_2)x_2}\,p_{x_1,x_2-1} \\ &\quad-\frac{\phi_1\phi_2(\theta+x_1+x_2+\phi_1+\phi_2+1)}{(\theta+\phi_1+\phi_2)^2(\theta+\phi_1+\phi_2+x_1+x_2-1)x_1x_2}\,p_{x_1-1,x_2-1}, \end{aligned} \tag{19}$$

for $x_1 = 1,2, \ldots, x_2 = 1,2, \ldots$, with

$$p_{0,0} = \frac{\theta^2}{1+\theta}\,\frac{\theta+\phi_1+\phi_2+1}{(\theta+\phi_1+\phi_2)^2},$$

$$p_{x_1,0} = \frac{\theta^2(x_1!+1)\phi_1^{x_1}}{(1+\theta)x_1!}\,\frac{\theta+x_1+\phi_1+\phi_2+1}{(\theta+\phi_1+\phi_2)^{x_1+2}},$$

$$p_{0,x_2} = \frac{\theta^2(x_2!+1)\phi_2^{x_2}}{(1+\theta)x_2!}\,\frac{\theta+x_1+\phi_1+\phi_2+1}{(\theta+\phi_1+\phi_2)^{x_2+2}}.$$

The above recurrence relation facilitates easy computation of the bivariate distribution.

3.3. Results in relation to insurance

In the framework of automobile insurance, it is common to consider bivariate discrete distributions when the yearly claim frequencies are separated in two types of policies (for example, material damage and bodily injury). See Partrat (1994), Walhin and Paris (2001)) and Bermúdez (2009).

The basic bivariate distribution is suitable to be applied in the collective risk model when both number of claims and size of a single claim are implemented into the model to build the aggregate claim size or aggregate losses.

3.3.1. *The aggregate losses*

We shall begin by considering the situation where both claims frequency and the size of a claim are relevant, then the quantity of interest is the aggregate claim size variable

$$g(x,y) = \sum_{x_1,x_2=0}^{\infty} p_{x_1,x_2} f_1^{*x_1}(x) f_2^{*x_2}(y), \tag{20}$$

which is the joint probability density function of

$$(X,Y) = \left(\sum_{i=0}^{X_1} U_i,\ \sum_{i=0}^{X_2} V_i,\right)$$

where all the severities U and V are mutually independent, and also independent of (X_1, X_2) with probability functions (discrete or continuous) $f_1(u)$, $f_2(v)$, respectively, with x_1 and x_2-fold convolutions $f_1^{*x_1}(x)$ and $f_2^{*x_2}(y)$.

Recursions for certain bivariate counting distributions and their compound distributions of the form (20) have been provided in the actuarial literature; see, for example, Hesselager (1996), Vernic (1999), Walhin and Paris (2000b), and

Sundt (2002), among others. Moreover, bivariate recursions are of interest in prediction problems involving the conditional $g(y\,|\,x)$ of Y, given that $X = x$ has been observed; see Hesselager (1996) for further discussion on this issue.

Further study in this respect is needed to obtain a recursive expression for the aggregate claim size using the expression in (19) can be obtained when claim sizes are assumed to be either continuous or discrete. In the discrete case, for example, the probability generating function of (X, Y) is given by $G_{X,Y}(s_1, s_2) = G_{X_1,X_2}(G_U(s_1), G_V(s_2))$, where $G_U(s_1)$ and $G_V(s_2)$ are the probability generating functions of the random variables U and V, respectively. Then, from Johnson et al. (1997), the following correspondence is known between probabilities and the probability generating function:

$$p_{x_1,x_2} = \frac{1}{x_1\,!\,x_2!}\left\{\frac{\partial^{x_1+x_2} G_{X_1,X_2}(s,t)}{\partial s^{x_1}\,\partial t^{x_2}}\right\}_{(0,0)}.$$

So, the required probabilities can be computed in a straightforward manner. For the continuous case, we make use of the moment generating function in a similar fashion. The model parameters can then be estimated by using moment estimators based on well-known expressions for the mean and variance (see Partrat (1994)).

Let us now assume that the random variables X_1 and X_2 represent two kinds of claims in the insurance setting, for example, bodily injury and material damage. In this case, $E(X_1)/E(X_2) = \phi_1/\phi_2$ can be interpreted as the ratio of the mean frequencies of the two kinds of claims, and the sample value of this ratio can be used readily for estimating the parameter θ of the model.

3.3.2. *The aggregate deductibles*

The fact that the probability generating function of the new model is expressed in an analytic way can also be utilized to obtain the probability generating function of the random variable $(X_1(d_1), X_2(d_2))$ for d_i that can be deduced in type i $(i = 1, 2)$ claim amounts. Here, $X_i(d_i)$ is the random variable corresponding to yearly frequency of type i claims exceeding d_i.

Partrat (1994) then showed that the probability generating function of the random variable $(X_1(d_1), X_2(d_2))$ is given by

$$\begin{aligned} G_{X_1(d_1),\,X_2(d_2)}(s_1,s_2) = {} & G_{X_1,X_2}((1 - F_1(d_1))\,s_1 \\ & + F_1(d_1),\,(1 - F_2(d_2))\,s_2 + F_2(d_2)), \end{aligned}$$

where F_1 and F_2 are the cumulative distribution functions of the random variables U and V, respectively, while the probability generating function of the random variable $X(d_1, d_2)$, being $X = X_1 + X_2$, is given by

$$G_{X(d_1,d_2)}(s_1,s_2) = G_{X_1,X_2}((1 - F_1(d_1))\,s_1 + F_1(d_1),\,(1 - F_2(d_2))\,s_2 + F_2(d_2)).$$

By extending Corollary 2 in Partrat (1994), we then have the following proposition.

Proposition 3. *For the bivariate discrete probability mass function in (17), the probability generating function of* $(X_1(d_1), X_2(d_2))$ *is given by*

$$\begin{aligned} G_{X_1(d_1),\, X_2(d_2)}(s_1, s_2) &= \int_0^\infty \exp\{\varsigma\phi_1[(s_1-1)\varsigma \\ &\quad + \frac{\phi_2}{\varphi_1}\frac{1-F_2(d_2)}{1-F_1(d_1)}(s_2-1)]\}\,\pi(\varsigma)d\varsigma, \end{aligned}$$

where $\varsigma = \lambda[1 - F_1(d_1)]$ *and*

$$\pi(\varsigma) = \frac{\theta^2}{1+\theta}\left(1 + \frac{\varsigma}{1-F_1(d_1)}\right)\exp\left\{-\frac{\theta\varsigma}{1-F_1(d_1)}\right\}.$$

Proof. From (11) with $d = 2$, we have $G_{X_1, X_2}((s_1, s_2) = M_\Lambda(\phi_1(s_1 - 1) + \phi_2(s_2 - 1))$, and so

$$\begin{aligned} G_{X_1(d_1),\, X_2(d_2)}(s_1,s_2) &= \int_0^\infty \exp\{\lambda[\phi_1[(1-F_1(d_1))s_1 + F_1(d_1) - 1] \\ &\quad + \phi_2[(1-F_2(d_2))s_2 + F_2(d_2) - 1]]\}\,\pi(\lambda)d\lambda \\ &= \int_0^\infty \exp\{\lambda\phi_1(1-F_1(d_1))[s_1 \\ &\quad + \frac{F_1(d_1) - 1 + \phi_2/\phi_1 F_2(d_2) - \phi_2/\phi_1}{1-F_1(d_1)} \\ &\quad + \frac{\phi_2}{\phi_1}s_2\frac{1-F_2(d_2)}{1-F_1(d_1)}]\}\pi(\lambda)d\lambda \\ &= \int_0^\infty \exp\{\lambda\phi_1(1-F_1(d_1))[(s_1-1) \\ &\quad + \frac{\phi_2}{\phi_1}\frac{1-F_2(d_2)}{1-F_1(d_1)}(s_2-1)]\}\,\pi(\lambda)d\lambda, \end{aligned}$$

from which, after the change of variable $\varsigma = \lambda(1 - F_1(d_1))$, the desired result follows. □

4. An extended bivariate model

The above introduced model has the advantage of simplicity but does possess two shortcomings. First, the parameters in the marginal distributions are not free, in the sense that all marginal distributions share the same parameters.

Second, the model only has non-negative correlations and is therefore not appropriate for modeling multivariate count data with negative correlations between pairs of variables. With the aim of overcoming both problems, we seek a flexible two-dimensional distribution with Lindley marginal distributions.

4.1. A bivariate model with flexible covariance

Although any kind of copula satisfying the requirement that the marginal distributions are Lindley distributions can be considered, the Farlie-Gumbel-Morgenstern (Farlie (1960)) and the Sarmanov (Sarmanov (1966)) families of distributions are good candidates, due to their easy formulation. We consider here the Sarmanov family of distributions, which has been studied by Sarmanov (1966), Lee (1996), Kotz et al. (2000), Sarabia and Castillo (2006) and Sarabia and Gómez-Déniz (2011), among others. Let $f_1(\lambda_1)$ and $f_2(\lambda_2)$ be univariate probability density functions with supports Λ_i, $i = 1, 2$, and let $h_i(z)$, $i = 1, 2$, be bounded nonconstant functions such that

$$\int h_i(z)\, f_i(z)dz = 0, \;\; i = 1,2.$$

Sarmanov (1966) then defined the following bivariate probability density function with given marginal distributions $f_1(\theta_1)$ and $f_2(\theta_2)$:

$$f(\lambda_1,\lambda_2) = f_1(\lambda_1) f_2(\lambda_2)\left[1 + \omega h_1(\lambda_1)\, h_2(\lambda_2)\right], \tag{21}$$

where ω is a real number such that $1 + \omega h_1(\lambda_1)h_2(\lambda_2) \geq 0 \; \forall \, (\lambda_1, \lambda_2)$. Lee (1996) studied some properties of this family and proposed a multivariate version showing some general methods to obtain the mixing functions $h_i(\lambda_i)$, $i = 1, 2$. When $f_i(\lambda_i)$, $i = 1, 2$, are defined on $(0, \infty)$ the procedure consists of taking $h_i(\lambda_i) = \exp(-\lambda_i) - L_i(1)$, being $L_i(t) = \int_0^\infty \exp(-t\lambda_i) f_i(\lambda_i)d\lambda_i$, the Laplace transform of $f_i(\lambda_i)$. In this case, the correlation coefficient is given by

$$\rho = \frac{\omega[-L_1'(1) - \mu_1 L_1(1)]\,[-L_2'(1) - \mu_2 L_2(1)]}{\sigma_1\, \sigma_2},$$

where $L'(t)$ represents the first derivative of $L(t)$ and ω satisfies $\omega_1 \leq \omega \leq \omega_2$, with

$$\omega_1 = -1/\max\{L_1(1)L_2(1), (1 - L_1(1))(1 - L_2(1))\},$$
$$\omega_2 = 1/\max\{L_1(1)(1 - L_2(1), L_2(1)(1 - L_1(1))\}$$

Therefore, we consider the model in (21) with Lindley marginal distributions, wherein

$$f_i(\lambda_i) \sim \mathcal{L}(\theta_i), \;\; i = 1,2,$$
$$h_i(\lambda_i) = e^{-\lambda_i} - L_i(1), \;\; i = 1,2.$$

From (2), we have

$$h_i(\lambda_i) = e^{-\lambda_i} - \frac{\theta_i^2(\theta_i+2)}{(1+\theta_i)^3}, \quad i = 1, 2.$$

Thus, we have the following result.

Proposition 4. *The probability mass function of the bivariate Poisson-Lindley distribution under the Sarmanov model is given by*

$$\begin{aligned} p_{x_1,x_2} &= f_{\phi_1,\theta_1}(x_1) f_{\phi_2,\theta_2}(x_2) \\ &\quad \times \left\{1 + \omega \prod_{i=1}^{2}\left[\left(\frac{\phi_i+\theta_i}{\phi_i+\theta_i+1}\right)^{x_i+2} \frac{\phi_i+\theta_i+x_i+2}{\phi_i+\theta_1+x_i+1} - h_i(\theta_i)\right]\right\}, \end{aligned} \tag{22}$$

where $f_{\phi_i,\theta_i}(x_i)$, $i = 1, 2$*, represents the probability mass function corresponding to the univariate case.*

Proof. We see directly that

$$\begin{aligned} p_{x_1,x_2} &= f_{\phi_1,\theta_1}(x_1) f_{\phi_2,\theta_2}(x_2) \\ &\quad + \omega \prod_{i=1}^{2}\left[\frac{\phi_i^{x_i}}{x_i!}\int_0^\infty \lambda_i^{x_i} e^{-(\phi_i+1)\lambda_i} f(\lambda_i)d\lambda_i - h_i(\theta_i) f_{\phi_i,\theta_i}(x_i)\right]. \end{aligned}$$

Performing some computation, we obtain

$$\int_0^\infty \lambda_i^{x_i} e^{-(\phi_i+1)\lambda_i} f(\lambda_i)d\lambda_i = \frac{\theta_i^2 x_i!}{1+\theta_i} \frac{\phi_i+\theta_i+x_i+2}{(\phi_i+\theta_i+1)^{x_i+2}}, \quad i = 1, 2,$$

and therefore,

$$\begin{aligned} p_{x_1,x_2} &= f_{\phi_1,\theta_1}(x_1) f_{\phi_2,\theta_2}(x_2) \\ &\quad + \omega \prod_{i=1}^{2}\left[\left(\frac{\phi_i}{\phi_i+1}\right)^{x_i} f_{\phi_i+1,\theta_i}(x_i) - h_i(\theta_i) f_{\phi_i,\theta_i}(x_i)\right]. \end{aligned}$$

Finally, by using (5) and performing simple computations, the desired result follows. □

In this extended model, the means, second order moments around the origin and variance are as in (12), (13) and (14), respectively, with θ being replaced by θ_i.

Following Lee (1996), we have $E(\Lambda_1 \Lambda_2) = \mu_1\mu_2 + \omega v_1 v_2$, where μ_i is the mean of the Lindley distribution and v_i is given by

$$v_i = \int_0^\infty \lambda_i h_i(\lambda_i) f(\lambda_i) d\lambda_i.$$

In the present case, $v_i = -\theta_i(4+\theta_i)/(1+\theta_i)^4$, and so after some simple computation, we obtain

$$\begin{aligned} E(X_1 X_2) &= \frac{\phi_1\phi_2}{(1+\theta_1)^4(1+\theta_2)^4}[(1+\theta_1)^3(1+\theta_2)^3(2+\theta_1)(2+\theta_2) \\ &\quad + 4\omega\theta_1^2\theta_2^2(4+\theta_1)(4+\theta_2)] \end{aligned}$$

and

$$\mathrm{Cov}(X_1 X_2) = \frac{\omega\phi_1\phi_2\theta_1\theta_2(4+\theta_1)(4+\theta_2)}{(1+\theta_1)^4(1+\theta_2)^4}. \tag{23}$$

Thus, the new model possesses a correlation of any sign, depending on the sign of the dependence parameter ω.

4.2. Regression

Let us consider two non-overlapping time periods consisting of n_1 and n_2 unit time intervals, and let X_1, and X_2 be random variables representing the numbers of intervals in each of the time periods in which at least one event occurs. The conditional distribution of the X_2 given that $X_1 = x_1$ may be constructed to obtain the conditional expectation of X_2.

The conditional probability mass function of X_1, on $X_2 = x_2$, is obtained directly from (22) as

$$\begin{aligned} \Pr(X_1 \mid X_2 = x_2) &= f_{\phi_1,\theta_1}(x_1) \\ &\quad \times \left\{1 + \omega\prod_{i=1}^{2}\left[\left(\frac{\phi_i+\theta_i}{\phi_i+\theta_i+1}\right)^{x_i+2}\frac{\phi_i+\theta_i+x_i+2}{\phi_i+\theta_1+x_i+1} - h_i(\theta_i)\right]\right\}. \end{aligned} \tag{24}$$

The following result gives the regression of X_1, on $X_2 = x_2$, under the Sarmanov model.

Proposition 5. *The conditional expectation of X_1, on $X_2 = x_2$, i.e., the regression of X_1, on $X_2 = x_2$, is given by*

$$E(X_1 \mid X_2 = x_2) = \frac{\phi_1}{1+\theta_1} \times \left\{1 + \frac{2}{\theta_1} - \omega\left[\left(\frac{\phi_2+\theta_2}{\phi_2+\theta_2+1}\right)^{x_2+2} \frac{\phi_2+\theta_2+x_2+2}{\phi_2+\theta_2+x_2+1}\right] \frac{\theta_1(4+\theta_1)}{(1+\theta_1)^3}\right\}.$$

Proof. By rewriting (24) in the form

$$\begin{aligned}\Pr(X_1 = x_1 \mid X_2 = x_2) = {} & f_{\phi_1,\theta_1}(x_1) \\ & + \omega\left[\left(\frac{\phi_2+\theta_2}{\phi_2+\theta_2+1}\right)^{x_2+2} \frac{\phi_2+\theta_2+x_2+2}{\phi_2+\theta_2+x_2+1} - h_2(\theta_2)\right] \\ & \times \left[\frac{\phi_1^{x_1}\theta_1^2}{1+\theta_1}\left(\frac{1}{\phi_1+\theta_1+1}\right)^{x_1+2}(\phi_1+\theta_1+x_1+2)\right. \\ & \left. - h_1(\theta_1) f_{\phi_1,\theta_1}(x_1)\right],\end{aligned}$$

multiplying by x_1 and summing, and using the identity

$$\sum_{x_1=0}^{\infty} x_1 \phi_1^{x_1} \frac{\phi_1+\theta_1+x_1+2}{(\phi_1+\theta_1+1)^{x_1+2}} = -\frac{\theta_1(4+\theta_1)}{\theta_1(1+\theta_1)^4} \tag{25}$$

to find $\sum_{x_1=0}^{\infty} x_1 \Pr(X_1 = x_1 \mid X_2 = x_2)$, we obtain the desired result after some algebraic manipulations. □

For the basic bivariate model proposed in subsection 3.2 we have For that, assumed that $X_i \sim \mathcal{P}o(\phi_i \lambda)$, $i = 1, 2$, and $\Lambda \sim \mathcal{L}(\theta)$, the conditional distribution of X_2 on X_1 is given by

$$\Pr(X_2 = x_2 \mid X_1 = x_1) = \frac{(x_1+x_2)!\,\phi_2^{x_2}}{x_1!\,x_2!} \frac{(x_1+x_2+\phi_1+\phi_2+\theta+1)(\phi_1+\theta)^{x_1+2}}{(x_1+\phi_1+\theta+1)(\phi_1+\phi_2+\theta)^{x_1+x_2+2}}.$$

It is straightforward to obtain the conditional expectation, which is the regression of X_2 on X_1,

$$E(X_2 \mid X_1 = x_1) = \frac{\phi_2}{\phi_1+\theta} \frac{(x_1+1)(x_1+\phi_1+\theta+2)}{x_1+\phi_1+\theta+1}. \tag{26}$$

This is clearly non-linear in x_1.

The regression of X_1 on X_2 may be obtained in an analogous manner.

5. Parameter Estimation

In this section, we develop some estimation methods for the two bivariate models proposed in the preceding sections. In particular, we pay special attention to the moment and the maximum likelihood methods.

5.1. Estimation in the basic bivariate model

In order to compute moment estimates in the bivariate case, we use expressions (12) and (16) to come up with the moments equations

$$E(X_1) = \bar{x}_1, \tag{27}$$

$$E(X_2) = \bar{x}_2, \tag{28}$$

$$\mathrm{Cov}(X_1, X_2) = s_{12}, \tag{29}$$

where $\bar{x}_i\,(i = 1,2)$ and s_{12} are the sample means and covariance, respectively. From (29) and after some simple algebra, we obtain the following expression for θ:

$$\hat{\theta} = \frac{-2\bar{x}_1\bar{x}_2 + 2s_{12} + \sqrt{2(\bar{x}_1^2\bar{x}_2^2 - \bar{x}_1\bar{x}_2 s_{12})}}{\bar{x}_1\bar{x}_2 - s_{12}}. \tag{30}$$

Now, (30) is used in (27) and (28) in order to obtain the moment estimators of the other two parameters, which are given by

$$\hat{\phi}_1 = \frac{(1+\hat{\theta})\hat{\theta}}{2+\hat{\theta}}\,\bar{x}_1,$$

$$\hat{\phi}_2 = \frac{(1+\hat{\theta})\hat{\theta}}{2+\hat{\theta}}\,\bar{x}_2.$$

Let us now consider the maximum likelihood method. Given a bivariate sample (x_{1j}, x_{2j}), $j = 1, 2, \ldots, n$, from a bivariate Poisson-Lindley distribution in (17), the likelihood function $\ell(\theta, \phi_1, \phi_2)$ is proportional to

$$\begin{aligned} \ell \equiv \ell(\theta,\phi_1,\phi_2) \propto\ & 2n\log\theta + n\bar{x}_1\log\phi_1 + n\bar{x}_2\log\phi_2 - n\log(1+\theta) \\ & + \sum_{j=1}^{n}\log(x_{1j} + x_{2j} + \phi_1 + \phi_2 + \theta + 1) \\ & - n(\bar{x}_1 + \bar{x}_2 + 2)\log(\theta + \phi_1 + \phi_2). \end{aligned} \tag{31}$$

Differentiation of (31) with respect to θ gives

$$\frac{\partial \ell}{\partial \theta} = \frac{2n}{\theta} - \frac{n}{1+\theta} + \sum_{j=1}^{n} \frac{1}{x_{1j} + x_{2j} + \phi_1 + \phi_2 + \theta + 1} - \frac{n(\bar{x}_1 + \bar{x}_2 + 2)}{\phi_1 + \phi_2 + \theta}, \tag{32}$$

and similarly differentiation of (31) with respect to ϕ_1 and ϕ_2 yields

$$\frac{\partial \ell}{\partial \phi_1} = \frac{n\bar{x}_1}{\phi_1} + \sum_{j=1}^{n} \frac{1}{x_{1j} + x_{2j} + \phi_1 + \phi_2 + \theta + 1} - \frac{n(\bar{x}_1 + \bar{x}_2 + 2)}{\phi_1 + \phi_2 + \theta}, \tag{33}$$

$$\frac{\partial \ell}{\partial \phi_2} = \frac{n\bar{x}_1}{\phi_2} + \sum_{j=1}^{n} \frac{1}{x_{1j} + x_{2j} + \phi_1 + \phi_2 + \theta + 1} - \frac{n(\bar{x}_1 + \bar{x}_2 + 2)}{\phi_1 + \phi_2 + \theta}. \tag{34}$$

Setting (32) equal to zero produces

$$\sum_{j=1}^{n} \frac{1}{x_{1j} + x_{2j} + \phi_1 + \phi_2 + \theta} = \frac{n(\bar{x}_1 + \bar{x}_2 + 2)}{\phi_1 + \phi_2 + \theta} + \frac{n}{1+\theta} - \frac{2n}{\theta}, \tag{35}$$

which can be used in the likelihood equations in (33) and (34) to obtain

$$\frac{\bar{x}_1}{\phi_1} + \frac{1}{1+\theta} - \frac{2}{\theta}, \tag{36}$$

$$\frac{\bar{x}_2}{\phi_2} + \frac{1}{1+\theta} - \frac{2}{\theta}. \tag{37}$$

Now, by setting (36) and (37) to zero, it is easy to derive that

$$\phi_1 = \phi_2 \frac{\bar{x}_1}{\bar{x}_2}, \tag{38}$$

$$\theta = \frac{\phi_2 - \bar{x}_2 + \sqrt{\phi_2^2 + 6\phi_2\bar{x}_2 + \bar{x}_2^2}}{2\bar{x}_2}. \tag{39}$$

Upon substituting (38) and (39) in (34), for example, we obtain an expression that only depends on the parameter ϕ_2 which can be solved, after setting to zero, numerically. Finally, with the value of ϕ_2 so determined, ϕ_1 and θ can be obtained from (38) and (39), respectively.

The second partial derivatives are given by

$$\frac{\partial^2 \ell}{\partial \theta^2} = -\frac{2n}{\theta^2} + \frac{n}{(1+\theta)^2} - \sum_{j=1}^{n} \frac{1}{(x_{1j} + x_{2j} + \phi_1 + \phi_2 + \theta + 1)^2} + \frac{n(\bar{x}_1 + \bar{x}_2 + 2)}{(\phi_1 + \phi_2 + \theta)^2},$$

$$\frac{\partial^2 \ell}{\partial \phi_i^2} = -\frac{n\bar{x}_i}{\phi_i^2} - \sum_{j=1}^{n} \frac{1}{(x_{1j}+x_{2j}+\phi_1+\phi_2+\theta+1)^2}$$
$$+ \frac{n(\bar{x}_1+\bar{x}_2+2)}{(\phi_1+\phi_2+\theta)^2},\ i=1,2,$$

$$\frac{\partial^2 \ell}{\partial \theta \partial \phi_i} = \frac{\partial^2 \ell}{\partial \phi_1 \partial \phi_2} = -\sum_{j=1}^{n} \frac{1}{(x_{1j}+x_{2j}+\phi_1+\phi_2+\theta+1)^2} + \frac{n(\bar{x}_1+\bar{x}_2+2)}{(\phi_1+\phi_2+\theta)^2},\ i=1,2.$$

These elements can be evaluated for the determination of the Fisher information matrix, from which the variance-covariance matrix of the MLEs can be readily computed.

5.2. Estimation in the Sarmanov model

The marginal means and variances, together with the expression in (23) and the cross factorial moment

$$E[X_i(X_i-1)X_j] = \phi_i^2\phi_j\left[\frac{2(\theta_i+3)(\theta_j+2)}{\theta_i^2\theta_j(1+\theta_i)(1+\theta_j)}\right.$$
$$\left. + \frac{2\omega(\theta_i+3)(2+3\theta_i)}{(1+\theta_i)^5}\frac{\theta_j(\theta_j+4)}{(1+\theta_j)^4}\right],\ i,j=1,2,\ i\neq j,$$

can be used for the estimation of the parameters of the extended model by the method of moments. The maximum likelihood estimates can also be obtained in this case through a numerical method or by direct numerical search for the global maximum of the log-likelihood surface.

6. Numerical application

In order to see how the proposed models fit bivariate count data, we have chosen a data set with positive correlation. This set of data was taken from Partrat (1994) and it also appears in Vernic (1997). The example is regarding the claims corresponding to a large automobile insurance portfolio in France, including 181038 liability policies issued during the year 1989, and where the yearly claim frequencies have been divided into material damage and bodily injury.

6.1. Fitting the data

We have fitted this data set by using the basic model proposed in this paper and then, we have calculated different premiums, showing that the new model

TABLE 1

DATA SET 2. BODILY INJURIES (X_2) AND MATERIAL DAMAGE (X_1); SEE PARTRAT (1994) AND VERNIC (1997). OBSERVED AND EXPECTED FREQUENCIES OBTAINED BY MOMENT METHOD, MLE (BASIC MODEL) AND MLE (SARMANOV MODEL)

X_2 \ X_1	0	1	≥ 2	Total
0	171345 171285 171351 171334	918 900.00 896.74 911.63	2 3.98 4.70 4.84	172265
1	8273 8408.6 8272.24 8287.93	73 74.53 86.58 73.44	0 0.45 0.68 0.55	8346
2	389 348.55 399.34 400.00	5 4.26 6.27 4.87	0 0.03 0.06 0.04	394
3	31 13.29 19.28 19.22	1 0.21 0.40 0.29	0 0.00 0.00 0.00	32
≥ 4	1 0.48 0.93 0.92	0 0.01 0.02 0.01	0 0.00 0.00 0.00	1
Total	180039	997	2	181038

is suitable for this purpose. The relevant data are given in Table 1, which also shows, from top to bottom, the observed and the expected frequencies obtained by the moment method (above) obtained by using expressions (27)-(29) and the maximum likelihood method (using expression (31)) for the basic model as well as the Sarmanov model (below). In this case the logarithm of the pmf given in (22) was used to obtain the parameter estimates. As in the mixed bivariate Poisson considered in Partrat (1994) and Walhin and Paris (2001), we have assumed (in the maximum likelihood method and the Sarmanov model) that $\phi_2 = 1$, i.e., this parameter is the ratio of the mean frequencies of X_1 and X_2. As we can see, this simplified model works well when there is a large proportion of (0,0) in the sample. The maximum likelihood estimates were computed by searching for the global maximum of the log-likelihood surface.

Table 2 includes the estimates (by moment and maximum likelihood method), the χ^2 statistics, the p-values, and the log-likelihood function values, indicating quite a reasonable fit. Ten categories were considered for computing the χ^2 goodness-of-fit-test by grouping the classes {(1,1), (3,0)} and {(2,2), (3,1), (3,2), (4,1),

TABLE 2

SUMMARY OF RESULTS OBTAINED BY MOMENT METHOD (MM) AND MAXIMUM LIKELIHOOD METHOD (MLE)

	Basic Bivariate Model		Sarmanov Model
	MM	MLE	MLE
$\hat{\theta}$	0.5173	181.846 (0.0976)	–
$\hat{\theta}_1$	–	–	14.6061 (0.6797)
$\hat{\theta}_2$	–	–	181.804 (31.7293)
$\hat{\phi}_1$	0.0159	9.2247 (0.0312)	0.70010 (31.7293)
$\hat{\phi}_2$	0.0017	0.9999 (0.6431)	1
$\hat{\omega}$	–	–	1956.71 (0.7352)
χ^2	9.99	3.86	5.32
p-value	0.0755	0.5697	0.2683
$L_{\max}$	–	–43143.10	–43141.80

(4,2)}. As we can see, both models perform better than the bivariate Poisson-Gamma and bivariate Poisson-inverse Gaussian distribution considered by Partrat (1994).

6.2. Computing bonus-malus premiums

In Europe, it is common to use bonus-malus systems (BMS) in order to compute automobile insurance premiums. In BMS, the customer's premium depends only on the number of claims presented in the past, irrespective of their size. The methodology of a BMS ensures that the premium increases with the number of claims and decreases with the period n in which the policyholder does not make a claim. We can compute bonus-malus premiums (BMP) under the net premium principle (see Gómez-Déniz et al. (2008) and the references therein) with the expression

$$BMP = 100\,\frac{E_{\pi(\lambda|\text{data})}[\delta(\lambda)]}{E_{\pi(\lambda)}[\delta(\lambda)]},$$

where $\delta(\lambda)$, the risk premium, is in our case

$$\delta(\lambda) = \sum_{x_1,x_2} x_1 x_2 \Pr\left(X_1 = x_1, X_2 = x_2 \,|\, \lambda\right) = (1+\phi_2)\lambda,$$

$\pi(\lambda)$ is the prior distribution (the Lindley distribution) and $\pi(\lambda|\text{data})$ is the posterior distribution after the data are observed.

Nevertheless, the model assumed in this form is not able to catch the difference between two kinds of claims in a posteriori rating. For that, and following the spirit of the paper Walhin and Paris (2000a), we consider here an alternative expression for computing the bonus-malus premium given by

$$BMP = 100\,\frac{E_{\pi(\lambda|\underline{X_1})}(\delta_1(\lambda))\,E(C_{X_1}) + E_{\pi(\lambda|\underline{X_2})}(\delta_2(\lambda))\,E(C_{X_2})}{E_{\pi(\lambda)}(\delta_1(\lambda))\,E(C_{X_1}) + E_{\pi(\lambda}(\delta_2(\lambda))\,E(C_{X_2})}.$$

Here, $E(C_{X_1})$ and $E(C_{X_2})$ are the expectations of the random variables corresponding to costs of material damage and bodily injury, respectively, $\delta_i(\lambda) = \sum_{x_i} x_i \Pr(X_i = x_i)$, $i = 1,2$ and $\underline{X_i}$, $i = 1,2$, the observed data separately in both kinds of claims. By assuming that $E(C_{X_2}) = \alpha E(C_{X_1})$, (40) reduces to

$$BMP = 100\,\frac{E_{\pi(\lambda|\underline{X_1})}(\delta_1(\lambda)) + \alpha E_{\pi(\lambda|\underline{X_2})}(\delta_2(\lambda))}{E_{\pi(\lambda)}(\delta_1(\lambda)) + \alpha E_{\pi(\lambda)}(\delta_2(\lambda))}. \tag{40}$$

Although the pair likelihood-prior is not conjugate in this case, the bonus-malus net premiums given in (40) can be obtained after some algebra as

$$\begin{aligned} BMP = {} & \frac{100(1+\theta)}{(\alpha+\phi_1)(2+\theta)} \left[\frac{2+n(\phi_1+\bar{x}_1)+\theta}{1+n(\phi_1+\bar{x}_1)+\theta}\,\frac{\phi_1(1+n\bar{x}_1)}{\theta+n\phi_1} \right. \\ & \left. + \frac{2+n(1+\bar{x}_2)+\theta}{1+n(1+\bar{x}_2)+\theta}\,\frac{1+n\bar{x}_2}{\theta+n} \right] \end{aligned} \tag{41}$$

where $\bar{x}_i = \sum_{i=1}^{n} x_i/n$, $i = 1,2$.

Expression in (41) was used with the maximum likelihood estimates provided above and $\alpha = 15$ and the obtained results are given in Tables 3 and 4. The

TABLE 3

BONUS-MALUS NET PREMIUMS FOR $n = 1$

$\bar{x}_1$	$\bar{x}_2$				
	0	1	2	3	4
0	97.810	134.04	170.27	206.49	242.72
1	159.38	195.62	231.84	268.07	304.29
2	220.96	257.19	293.42	329.64	365.87
3	282.53	318.76	354.98	391.21	427.43
4	344.09	380.32	416.55	452.78	489.00

TABLE 4

BONUS-MALUS NET PREMIUMS FOR $n = 5$

$\bar{x}_1$	$\bar{x}_2$				
	0	1	2	3	4
0	90.59	242.29	393.95	545.60	697.21
1	391.82	543.51	695.18	846.82	998.44
2	692.96	844.65	996.32	1147.97	1299.58
3	994.03	1145.73	1297.40	1449.04	1600.66
4	1295.04	1446.73	1598.40	1750.04	1901.66

first were computed for a sample of $n = 1$ period of time and the second for $n = 5$.

CONCLUDING REMARKS

In this paper, a new discrete multivariate probability distribution, namely, the multivariate Poisson-Lindley distribution, is proposed. This model admits a bivariate version which is a natural extension of the univariate Poisson-Lindley distribution and it allows correlations of any sign.

We have discussed some of its important probabilistic properties as well as the problem of estimation of its parameters.

From the numerical results presented here, it can be concluded that the new model proposed in this paper appears to be suitable for the count data sets analyzed here. Therefore, the new models may be competitors for other bivariate discrete distributions mentioned in the literature, such as the bivariate Poisson-inverse Gaussian, the symmetric bivariate negative binomial, and the bivariate negative binomial-inverse Gaussian distributions.

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REFERENCES

AITCHISON, J. and HO, C.H. (1989) The multivariate Poisson-Log normal distribution. *Biometrika*, **76**, 643-653.

ARBOUS, A.G. and SICHEL, H.S. (1954) New techniques for the analysis of absenteeism data. *Biometrika*, **41**, 77-90.

BARNDORFF-NIELSEN, P., BLAESILD, P. and SESHADRI, V. (1992) Multivariate distributions with generalized inverse Gaussian marginals and associated Poisson mixtures. *The Canadian Journal of Statistics*, **20**, 109-120.

BERMÚDEZ, L. (2009) A priori ratemaking using bivariate Poisson regression models. *Insurance*: *Mathematics and Economics*, 44, 135-141.

BOUCHER, J.P., DENUIT, M. and GUILLÉN, M. (2009) Number of accidents or number of claims? An approach with zero-inflated Poisson models for panel data. *The Journal of Risk and Insurance*, **76(4)**, 821-846.

FARLIE, D.J. (1960) The performance of some correlation coefficients for a general bivariate distribution. *Biometrika*, **47**, 307-323.

FREES, E.W. (2004) *Longitudinal and Panel Data. Analysis and Applications in the Social Sciences*. Cambridge University Press.

GHITANY, M.E. and AL-MUTAIRI, L. (2009) Estimation methods for the discrete Poisson-Lindley distribution. *Journal of Statistical Computation and Simulation*, **79**, 1-9.

GHITANY, M.E., ATIEH, B. and NADARAJAH, S. (2008) Lindley distribution and its application. *Mathematics and Computers in Simulation*, **78**, 493-506.

GÓMEZ-DÉNIZ, E., SARABIA, J.M. and CALDERÍN, E. (2008) Univariate and multivariate versions of the negative binomial-inverse Gaussian distributions with applications. *Insurance*: *Mathematics and Economics*, **42**, 39-49.

HESSELAGER, O. (1996) Recursions for certain bivariate counting distributions and their compound distributions. *Astin Bulletin*, **26**, 35-52.

HOLLA, M.S. and BHATTACHARYA, S.K. (1965) On a discrete compound Poisson distribution. *Annals of the Institute of Statistical Mathematics*, **15**, 377-384.

JOHNSON, N.L., KOTZ, S. and BALAKRISHNAN, N. (1997) *Discrete Multivariate Distributions*, John Wiley and Sons, New York.

KARLIS, D. and MELIGKOTSIDOU, L. (2007) Finite mixtures of multivariate Poisson distributions with application. *Journal of Statistical Planning and Inference*, **137**, 1942-1960.

KARLIS, D. and XEKALAKI, E. (2005) Mixed Poisson distributions. *International Statistical Review*, **73**, 35-58.

KEMP, A.W. (1981) Computer sampling from homogeneous bivariate discrete distributions. *ASA Proceedings of the Statistical Computing Section*, 173-175.

KOCHERLAKOTA, S. and KOCHERLAKOTA, K. (1992) *Bivariate Discrete Distributions*, Marcel Dekker, New York.

KOTZ, S., BALAKRISHNAN, N. and JOHNSON, N.L. (2000) *Continuous Multivariate Distributions*, *Vol*. 1: *Models and Applications*. John Wiley and Sons, New York.

LAI, C.D. (2006) Constructions of discrete bivariate distributions. In: *Advances on Distribution Theory*, *Order Statistics and Inference* (Eds., N. Balakrishnan, E. Castillo, J.M. Sarabia), pp. 29-58, Birkhäuser, Boston.

LEE, M.-L.T. (1996) Properties and applications of the Sarmanov family of bivariate distributions. *Communications in Statistics-Theory and Methods*, **25**, 1207-1222.

LINDLEY, D.V. (1958) Fiducial distributions and Bayes's theorem. *Journal of the Royal Statistical Society*, *Series B*, **20**, 102-107.

ØIGARD, T.A. and HANSSEN, A. (2002) The multivariate normal inverse Gaussian heavy-tailed distribution: simulation and estimation. In: *IEEE International Conference on Acoustics*, *Speech and Signal Processing*, **2**, 1489-1492.

PARTRAT, C. (1994) Compound model for two dependent kinds of claim. *Insurance*: *Mathematics and Economics*, **15**, 219-231.

PROTASSOV, R.S. (2004) EM-based maximum likelihood parameter estimation for multivariate generalized hyperbolic distributions with fixed λ. *Statistics and Computing*, **14**, 67-77.

SARABIA, J.M. and CASTILLO, E. (2006) Bivariate distributions based on the generalized three-parameter beta distribution. In: *Advances on Distribution Theory*, *Order Statistics and Inference*, (Eds., N. Balakrishnan, E. Castillo, J.M. Sarabia), pp. 85-110, Birkhäuser, Boston.

SARABIA, J.M. and GÓMEZ-DÉNIZ, E. (2011) Construction of multivariate distributions: a review of some recent results. *Statistics and Operational Research Transactions* (*SORT*), **32**, 3-36 (with discussion).

SARABIA, J.M. and GÓMEZ-DÉNIZ, E. (2011) Multivariate Poisson-Beta distributions with applications. *Communications in Statistics-Theory and Methods*, **40**, 1093-1108.

SANKARAN, M. (1970) The discrete Poisson-Lindley distribution. *Biometrics*, **26**, 145-149.

SARMANOV, O.V. (1966) Generalized normal correlation and two-dimensional Frechet classes. *Doklady*, *Soviet Mathematics*, **168**, 596-599.

Sichel, H.S. (1971) On a family of discrete distributions particularly suited to represent long-tailed frequency data. In: *Proceedings of the Third Symposium on Mathematical Statistics* (*Eds. N.F. Laubscher*), *Council for Scientific and Industrial Research*, pp. 51-97, Pretoria, South Africa.

Stein, G., Zucchini, W. and Juritz, J. (1987) Parameter estimation for the Sichel distribution and its multivariate extension. *Journal of the American Statistical Association*, **82**, 938-944.

Sundt, B. (2002) Recursive evaluation of aggregate claims distributions. *Insurance*: *Mathematics and Economics*, **30**, 297-322.

Thyrion, P. (1972) Quelques observations statistiques sur la variable, nombre de sinistres en assurance automobile. *Astin Bulletin*, **VI(3)**, 203-232.

Vernic, R. (1997) On the bivariate generalized Poisson distribution. *Astin Bulletin*, **27**, 23-31.

Vernic, R. (1999) Recursive evaluation of some bivariate compound distributions. *Astin Bulletin*, **29**, 315-325.

Walhin, J.F. and Paris, J. (2000a) A general family of bivariate mixed Poisson distributions. www.stat.ucl.ac.be/ISpub/dp/2000/dp0011.ps

Walhin, J.F. and Paris, J. (2000b) Recursive formulae for some bivariate counting distributions obtained by the trivariate reduction method. *Astin Bulletin*, **30**, 141-155.

Walhin, J.F. and Paris, J. (2001) The mixed bivariate Hoffmann distribution. *Astin Bulletin*, **31**, 123-138.

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