

Evaluation of credit risk of a portfolio with stochastic interest rate and default processes ¹

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Abstract. This paper proposes a new model for evaluating credit risk of a portfolio consisting of interest rate sensitive assets. Our model is distinguished from existing risk valuation models such as CreditMetricsTM or CREDITRISK⁺ by (1) the dynamics of the default-free interest rate as well as hazard rate processes of defaultable assets are described by stochastic differential equations; and (2) prices of individual assets are evaluated by the *single* risk-neutral valuation framework. It is then possible to evaluate not only credit risk but also market risk of the portfolio in a synthetic manner. It is shown that value at risk (VaR) of the portfolio is approximately evaluated as a closed form solution.

Keywords. portfolio VaR, credit risk, market risk, risk-neutral valuation, Cornish-Fisher expansion

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1 Introduction

In recent years, risk management of financial assets has become more important for investment institutions and corporations than ever, and a prominent tool for this has been value at risk (VaR). VaR provides the potential for significant loss in a portfolio of financial assets and VaR's popularity is based on aggregation of several components of market risk into a *single* number. However, the market risk involved in trading operations is only a small fraction of total risks to which a typical financial institution is exposed. Hence, the desire to calculate some risk index arises which measures credit risk, including market risk as well, of a portfolio in a synthetic manner. In this paper, we propose a new model for evaluating credit risk of a portfolio consisting of interest rate sensitive assets. By using the Cornish-Fisher expansion, it is shown that the portfolio's VaR is approximately evaluated as a closed form solution.

Pricing of an *individual* asset subject to credit risk has been extensively studied in the literature. We refer to Duffie and Singleton (1996) for the survey of such pricing models. Among them, Jarrow and Turnbull (1995) assumed that the payoffs upon default are expressed as an exogenous fraction of the claim and they showed that, under some regularity conditions, the price is given by the expected, discounted payoffs under the risk-neutral probability measure. Duffie and Singleton (1996) proposed another model in which the payoffs are discounted by an interest rate that is adjusted so as to reflect the effect of default risk. Jarrow, Lando and Turnbull (1997) developed a Markov chain model for the term structure of credit risk spreads in order to incorporate credit rating information into the valuation methodology given by Jarrow and Turnbull (1995). Lando (1994), Das and Tufano (1996) and Kijima and Komoribayashi (1997) extended the Markov chain model in various ways.

When evaluating credit risk of a portfolio, however, none of these pricing models can be used directly. An obvious reason is due to the lack of consideration of portfolio effects, i.e. a diversification benefit and concentration risk. In 1997, JP Morgan published their credit risk model, called CreditMetrics™ (1997), to calculate the distribution of future exposures in a portfolio, so that VaR with an arbitrary probability level can be obtained. Recently, Credit Suisse Financial Products followed this line and published CREDITRISK+ (1997) for the same purpose, but by a different methodology.

These two models are well-constructed with clever insights about credit risk; however, they have apparent drawbacks. For example, in CreditMetrics™ (1997), their calculated present values depend only on credit risk; especially, most importantly, interest rate risk is not incorporated explicitly. This means that risks other than credit risk have no impact on the valuation of a portfolio, and the prices of assets calculated in this way may not be

consistent with observed prices in the market. On the other hand, in CREDITRISK⁺ (1997), they assume that credit risk is due to default losses only. This assumption considerably simplifies the model so that they can calculate the loss distribution analytically, in contrast to CreditMetricsTM (1997) in which some Monte Carlo simulation method ought to be used. However, the assumption is inappropriate, for example, when the valuation needs to be consistent with observed market prices, when the specific risk in credit risk needs to be evaluated, and so on.

In this paper, we propose a new model to evaluate credit risk as well as market risk of a portfolio with a stochastic default-free interest rate process and stochastic default processes of defaultable assets. For the default-free interest rate process, we can use any non-arbitrage model in the finance literature. Default is formulated by a hazard rate process, the process of the conditional density of default at a specific time given no default before that time. We assume that the hazard rate processes follow a multi-dimensional diffusion process, thereby incorporating the correlation effect on defaults. Future prices of all assets are evaluated by the risk-neutral valuation framework and the distribution of future value of a portfolio is obtained accordingly. The advantage of our model is that credit risk as well as market risk can be evaluated in a synthetic manner and the portfolio effects are taken into consideration explicitly. Moreover, market price of risk appears in the valuation formula, so that the calculated present values in our model are consistent with observed market prices. By assuming a simple diffusion process for the hazard rate processes, it is possible to derive an approximated, closed form solution of the portfolio VaR.

This paper is organized as follows. In the next section, we summarize ideal features of credit risk valuation models for a portfolio. Based on them, we then describe the framework of our model in Section 3. The key issue here is the formulation of default processes. We use stochastic differential equations to describe the evolution of the hazard rate processes of defaultable assets. The joint default distribution is then constructed under the assumption of conditional independence. In Section 4, assuming a simple diffusion process for default processes, closed form solutions for prices of basic financial instruments such as bonds and swaps are obtained. Detailed derivations of the solutions are given in Appendix A. The estimation of risk premia adjustments is also an important issue for practical implementation of our model. This topic is also discussed in this section, where risk premia adjustments are determined so that the calculated present values are consistent with observed market prices of defaultable discount bonds. In Section 5, the Cornish-Fisher expansion is applied to obtain an approximate solution of the portfolio VaR. Section 6 concludes this paper.

2 Ideal features of risk valuation models

In this section, we summarize ideal features that our credit risk valuation model would be desired to possess.² Since it is important for financial institutions to manage all the assets that they possess in a synthetic manner, each asset should be evaluated based on the *single* non-arbitrage valuation framework. By non-arbitrage prices, we mean the prices by which arbitrage opportunities among all the assets are precluded. Not only present values but also future prices should be calculated in this way. Although this valuation paradigm may not apply for non-traded financial instruments such as loans, pricing in a unified manner of all types of assets is necessary to evaluate financial risks synthetically.

Beside the non-arbitrage valuation paradigm, we have the following important issues we should consider in our risk valuation model. First, because financial institutions deal with portfolios and because defaults of assets included in a portfolio are correlated statistically, it is important to consider the correlation effects between default processes of the assets. Correlation between the default-free interest rate and the credit risks will also be of importance, because we need to discount future cashflows with respect to the default-free interest rate. However, by technical reasons, we will assume in our model that the default-free interest rate process is independent of the default processes.

Second, it is desired that the calculated present values should reflect all financial risks, not merely credit risk, such as market risk and liquidity risk. As pointed out in a review of CreditMetricsTM (1997), the present value calculated by considering credit risk only may differ from the observed market price due to the ignorance of other risks. Such inconsistency should be avoided if not impossible.

Third, according to Moody's Investors Service (1995), it is observed that the shape of the term structure of default rates varies over credit ratings. For example, as Fons (1994) observed, assets with high credit ratings have increasing hazard rate (IHR) functions for default while assets with low credit ratings have decreasing hazard rate (DHR) functions.³ These empirical results should be incorporated for our credit risk valuation model.

Fourth, VaR seems to be recognized as a useful tool to evaluate market risk in financial institutions. Since we intend to synthesize all financial risks by our model, it would be preferred to use VaR as a risk index of the portfolio under consideration.

Finally, the computational issue should be of importance in practice. Ideally, the non-

²Artzner et al. (1998) discussed coherent measures of risks in an axiomatic way, where they present and justify a set of four desirable properties of measurement of them.

³These empirical findings can be explained theoretically by the notion of stochastic monotonicity in the Markov transition probability matrix of credit ratings; see Kijima (1998).

arbitrage price of each asset is given by a simple closed form solution. In general, valuation of credit risk of a large portfolio requires enormous amount of calculation. If all the prices are given by closed form solutions, we can reduce the computational time considerably. Some approximation technique would then be applied to evaluate the portfolio VaR with an arbitrary probability level.

3 The model framework

This section describes the framework of our model for evaluating credit risk of a portfolio. Because we employ the non-arbitrage valuation paradigm, it is important to distinguish the risk-neutral probability measure from the observed probability measure explicitly.⁴ In what follows, we shall denote the observed probability measure by P while the risk-neutral probability measure by \tilde{P} . Recall that the risk-neutral probability measure is needed only for pricing of financial assets. The question of how to obtain the risk-neutral probability measure will be discussed in the next section under some restricted situation. The probability space as well as the equipped filtration are constructed in the canonical way.

The proposed model consists of the following four components:

1. Generation of the stochastic structures for uncertainty,
2. Valuation of the present value of a portfolio,
3. Valuation of the future portfolio value, and
4. Calculation of the portfolio VaR.

In this section, we will explain each component by this order. Figure 1 depicts the relationship between these components. The input data in our model are the present values of defaultable assets, evaluated if necessary, and the parameters of the stochastic structures. Future scenarios can then be generated, by which the distribution of future price of the portfolio can be calculated in an obvious way. If the stochastic structures are simple enough, then we would expect to obtain the distribution analytically. In any case, however, we can calculate the portfolio VaR based on the distribution function.

(Figure 1 here)

⁴In most of market risk valuation models, the observed probability measure and the risk-neutral probability measure are assumed to be indistinguishable. This is so, because the risk horizon for market risk is quite short and the assumption will hold as a first, and good, approximation. However, the risk horizon for credit risk is much longer (typically more than 1 year) than the market risk counterpart (typically only few days). Hence, the risk-neutral probability measure should be distinguished from the observed probability measure explicitly in our credit risk valuation model.

3.1 The basic stochastic structures

In this subsection, we define two sets of stochastic processes, one is for the default-free interest rate process and the other for default processes, in terms of stochastic differential equations (hereafter abbreviated by SDE's).

As to the default-free interest rate process, we can use any non-arbitrage interest rate model in the finance literature for our purpose. For example, the Heath, Jarrow and Morton (hereafter HJM) model (1992) is a good candidate, because it incorporates all current information in the yield curve, and relies on markets being dynamically complete. Preferences are embedded into the observable term structure, and arbitrage opportunities among bonds of different maturities are precluded. Moreover, if we want the spot rate process to be Markovian for the sake of computability, then there are available restricted forms of volatility functions that have the desired Markovian property in the HJM framework. According to Ritchken and Sankarasubramanian (1995), if the volatility function $\gamma(t, T)$ of the instantaneous forward rate process in the one-factor HJM framework satisfies

$$\frac{\partial}{\partial T}\gamma(t, T) = -\kappa(T)\gamma(t, T), \quad T \geq t,$$

for some deterministic function $\kappa(T)$, then the spot rate process $r(t)$ under the risk-neutral measure follows the SDE

$$dr(t) = \left\{ \kappa(t)[f(0, t) - r(t)] + \frac{\partial}{\partial t}f(0, t) + \phi(t) \right\} dt + \sigma_f(r(t), t)d\tilde{z}_f(t), \quad (3.1)$$

so that it is Markovian, where $\tilde{z}_f(t)$ is the standard Wiener process under the risk-neutral probability measure, $f(t, T)$ the instantaneous forward rate at time t for date T , $\sigma_f(r, t)$ the volatility function depending on the spot rate level, and

$$\phi(t) = \int_0^t \gamma^2(s, t)ds = \int_0^t \sigma_f^2(r(s), s) e^{-2\int_s^t \kappa(x)dx} ds.$$

The extension of this result to the multi-factor case is given by Inui and Kijima (1998).

The construction of default processes is much involved. In our model, we assume that defaults are generated by hazard rate processes. To be more specific, let τ_j denote the default time of asset j , and let $h_j(t)$ be its hazard rate process. The hazard rate $h_j(t)$ represents the instantaneous rate that the default occurs at time t given no default before that time. That is, the hazard rate under the observed probability measure is defined by

$$h_j(t) = \lim_{dt \rightarrow 0} \frac{P\{t < \tau_j \leq t + dt | \tau_j > t\}}{dt}, \quad t \geq 0. \quad (3.2)$$

The hazard rate under the risk-neutral probability measure is defined similarly.

Suppose that we have a portfolio consisting of n defaultable assets, and define

$$\mathbf{h}(t) = (h_1(t), \dots, h_n(t)), \quad t \geq 0.$$

It is assumed that the hazard rate processes $h_j(t)$ under the observed probability measure follow the system of SDE's

$$dh_j(t) = \mu_j(\mathbf{h}(t), t)dt + \sigma_j(\mathbf{h}(t), t)dz_j(t), \quad j = 1, 2, \dots, n, \quad (3.3)$$

where μ_j and σ_j are the drift and the volatility functions of the hazard rate process $h_j(t)$, respectively, and $\mathbf{z}(t) = (z_1(t), \dots, z_n(t))$ is the n -dimensional Wiener process equipped with the usual filtration $\{\mathcal{F}_t\}$ generated from $z_f(t)$ and $\mathbf{z}(t)$. The process $\mathbf{h}(t)$ is therefore an Ito diffusion process.⁵ We note that the hazard rate must be non-negative, so that the functions μ_j and σ_j are required to satisfy some conditions under which the hazard rate processes $h_j(t)$ stay non-negative. The hazard rate processes $\tilde{h}_j(t)$ under the risk-neutral probability measure can be constructed by the usual change of measure.

It is well-known that the realization of the hazard rate processes $h_j(t)$ alone cannot determine the joint distribution of default times τ_j , since the joint distribution cannot be constructed from their marginal distributions except the independent case. Hence, a further assumption is necessary for our purpose. In our model, we assume that τ_j are *conditionally independent* given the realization of the underlying stochastic processes. That is, given $\mathbf{h}(t) = (h_1(t), \dots, h_n(t))$, the conditional event $\{t < \tau_j \leq t + dt | \tau_j > t\}$ happens independently according to the marginal probability $h_j(t)dt$; see (3.2). The default processes are then constructed completely by the hazard rate SDE's (3.3). Note that the conditional independence does not imply the ordinal independence and vice versa; see, e.g., page 55 of Stoyanov (1987).

The correlation between defaults of different assets is assumed to be driven by the correlation between the Wiener processes $z_j(t)$. Namely, we assume that

$$dz_j(t)dz_k(t) = \begin{cases} \rho_{jk}(t)dt, & \text{for } j \neq k, \\ dt, & \text{for } j = k, \end{cases} \quad (3.4)$$

for some deterministic functions $\rho_{jk}(t)$. It should be noted that the correlation structure (3.4) is invariant by the change of measure. That is, we have

$$d\tilde{z}_j(t)d\tilde{z}_k(t) = \begin{cases} \rho_{jk}(t)dt, & \text{for } j \neq k, \\ dt, & \text{for } j = k, \end{cases}$$

⁵In the reliability literature, several stochastic models have been constructed for systems subject to failure. See, e.g., Kijima, Li and Shaked (1998) for the survey of such models.

where $\tilde{z}(t) = (\tilde{z}_1(t), \dots, \tilde{z}_n(t))$ is the n -dimensional Wiener process under the risk-neutral probability measure \tilde{P} .

In our model, it is assumed that the default-free interest rate process is independent of the default processes. A justification of this assumption can be found in Jarrow, Lando and Turnbull (1997).⁶ This assumption is necessary only to simplify our later analyses. If we rely on a Monte Carlo simulation method throughout risk valuation, then it is a simple matter to introduce a dependence structure between the default-free interest rate and the default processes in the expense of computational efforts.

3.2 Valuation of present values

In this subsection, we consider an asset that obliges the issuer to pay some state-contingent cash at its maturity. If asset j is defaultable, then the issuer may not be able to pay the whole amount of payment; instead, the issuer will only pay some fraction X_j at some future time T_j after default, i.e. $T_j \geq \tau_j$. Then, provided that the state-contingent payment X_j and the time T_j of it are predetermined or determined by the underlying stochastic processes, the risk-neutral valuation method can be applied so that the time t price of asset j is given by

$$p_j(t) = \tilde{E}_t \left[\exp \left\{ - \int_t^{T_j} r(s) ds \right\} X_j \right] \quad \text{on } \{\tau_j > t\}, \quad (3.5)$$

where \tilde{E}_t denotes the conditional expectation operator given the history \mathcal{F}_t of the underlying stochastic processes up to time t under the risk-neutral probability measure \tilde{P} . The price of an asset with more complicated cashflows can be expressed as a linear combination or an integral of (3.5), because the expectation operator is linear. The present value of a portfolio is then equal to a sum of the present values of all assets included in the portfolio.

It should be noted that, even if the default-free interest rate process $r(t)$ is independent of the default processes, the random variables $e^{-\int_t^{T_j} r(s) ds}$ and X_j in expression (3.5) may not be so, because the payment instance T_j may depend on the default time. If T_j are always equal to some prespecified times, as assumed in Jarrow and Turnbull (1995), then the two random variables are mutually independent. In our model, we use this assumption later to obtain a closed-form solution for each asset price.

⁶In Jarrow, Lando and Turnbull (1997), they stated that the assumption appears to be a reasonable, first approximation for the observed probabilities in investment grade debts, but the accuracy of this approximation deteriorates for speculative grade debts.

3.3 The distribution of future portfolio value

Let \bar{t} denote the risk horizon under consideration, and suppose that the default-free interest rate $r(t)$ follows the SDE (3.1) while the hazard rate processes (3.3). Under the conditions, these processes are Markovian, and if the default time τ_j is later than the risk horizon \bar{t} then the future value of defaultable asset j at the risk horizon \bar{t} is given by (3.5), i.e.,

$$p_j(\bar{t}) = \tilde{E} \left[\exp \left\{ - \int_{\bar{t}}^{\tau_j} r(s) ds \right\} X_j \mid r(\bar{t}), h(\bar{t}) \right] \quad \text{on } \{\tau_j > \bar{t}\}.$$

If the event $\{\tau_j \leq \bar{t}\}$ happens, then $p_j(\bar{t})$ is given as a (random) recovery rate of the asset j . The event $\{\tau_j \leq \bar{t}\}$ can be constructed from the joint hazard rate process $h(t)$ and the conditional independence assumption. In any case, however, since the risk horizon \bar{t} is a future time epoch, the value $p_j(\bar{t})$ is a random variable depending on the realizations of $r(t)$ and $h(t)$. The distribution of the realizations is given in terms of the observed probability measure P . To explain this more explicitly, define the following region in the R^{n+1} -dimensional Euclidean space:

$$A(x) = \{(r(\bar{t}), h(\bar{t})) : p_j(\bar{t}) \leq x\}, \quad x \geq 0.$$

If the distribution function of $(r(\bar{t}), h(\bar{t}))$ is denoted by $F(r, h)$ under the observed probability measure P , then the distribution function of the random variable $p_j(\bar{t})$ is given by

$$P\{p_j(\bar{t}) \leq x\} = \int_{A(x)} dF(r, h), \quad x \geq 0,$$

where the integral means the Lebesgue-Stieltjes integral.

Suppose that the portfolio value at the risk horizon \bar{t} is given by

$$\pi(\bar{t}) = \sum_{j=1}^n w_j p_j(\bar{t}), \tag{3.6}$$

where w_j is the weight of asset j in the portfolio. Implicit in this equation is the assumption that we will not change the portfolio weights until the risk horizon \bar{t} . Since, in principle, the joint distribution of $(p_1(\bar{t}), \dots, p_n(\bar{t}))$ can be obtained from the joint distribution function $F(r, h)$ of $(r(\bar{t}), h(\bar{t}))$, we have enough data to determine the distribution function of $\pi(\bar{t})$. However, since it is in general very difficult to calculate the joint distribution function, and hence so is the distribution of $\pi(\bar{t})$ analytically, we would either need further assumptions on the underlying stochastic structures or employ some Monte Carlo simulation approach. The former case will be discussed in the next section. In the latter case, an appropriately formulated model generates scenarios, each corresponding to one possible path of the underlying stochastic processes that are correlated to each other, and enables one to calculate future values of defaultable assets as well as its portfolio for each scenario. Collecting these samples, we can then determine the distribution of future value of the portfolio.

3.4 Evaluation of VaR

As we have already mentioned, VaR is a prominent tool to evaluate financial risks of a portfolio for financial institutions. Let $G(x)$ denote the distribution function of future value of the portfolio. For an arbitrary probability level α , we define the number x_α to be the infimum satisfying

$$G(x_\alpha) = \alpha, \quad 0 < \alpha < 1.$$

The number x_α is called the 100α -percentile of $G(x)$. Then, VaR is defined to be the difference between the current portfolio value and x_α . Hence, the distribution function $G(x)$ of future value of the portfolio is enough to determine the desired VaR.

4 A simplified model

In the previous section, we describe the basic framework of our risk valuation model. However, as is easily seen there, the model seems too complicated to obtain analytical results unless further assumptions are imposed. In this section, we derive analytical expressions for the non-arbitrage prices of defaultable assets such as corporate bonds and swaps under some simplifying assumptions.

4.1 Specialized default processes

Suppose that the current time is 0 and denote the hazard rate at time t of asset j by $h_j(t)$, $t \geq 0$. In what follows, we assume that the hazard rate process $h_j(t)$ satisfies the following SDE under the observed probability measure P :

$$dh_j(t) = b_j(t)dt + \sigma_j dz_j(t), \quad t \geq 0, \quad (4.1)$$

where $b_j(t)$ is a deterministic function of time t and σ_j is non-negative constant.⁷ From the SDE (4.1), it follows that

$$h_j(t) = h_j(0) + \int_0^t b_j(s)ds + \sigma_j z_j(t), \quad t \geq 0, \quad (4.2)$$

which shows that $h_j(t)$ is normally distributed with

$$\text{mean } h_j(0) + \int_0^t b_j(s)ds \text{ and variance } \sigma_j^2 t.$$

⁷Davis and Mavroidis (1997) studied the same model for valuation and potential exposure calculation for defaultable swaps, but not for a portfolio.

Hence, even when $h_j(0) + \int_0^t b_j(s)ds \geq 0$, the hazard rate $h_j(t)$ becomes negative with positive probability.⁸

If we specify the function $b_j(t)$ such that

$$h_j(0) + \int_0^t b_j(s)ds = \lambda_j \gamma_j (t + m_j)^{\gamma_j - 1}, \quad (4.3)$$

where λ_j , γ_j and m_j are non-negative constants, then $E[h_j(t)]$ is the (delayed) hazard rate function of Weibull distributions with shape parameter γ_j and scale parameter λ_j . Weibull distributions are one of the well-studied distributions in survival analyses, and their advantage is that they can express various shapes of the term structure of hazard rates by the two parameters λ_j and γ_j . They are IHR if $\gamma_j > 1$ while DHR if $\gamma_j < 1$. If $\gamma_j = 1$, then they are called CHR (constant hazard rate) and must be exponential distributions. As to the statistical issues about inference of the parameters,⁹ we refer to Hoyland and Rausand (1994).

Using the specialized hazard rate processes (4.2), we can now derive the joint distribution of the default times τ_j as follows. Recall that, in order to evaluate the portfolio effects appropriately, we need the joint survival distribution $P\{\tau_1 > t_1, \dots, \tau_n > t_n\}$. By our earlier assumption, given \mathcal{F}_t where $t \geq \max_j t_j$, the survival probabilities are conditionally independent, i.e.,

$$P_t\{\tau_1 > t_1, \dots, \tau_n > t_n\} = \prod_{j=1}^n P_t\{\tau_j > t_j\}, \quad (4.4)$$

⁸Another possibility to model the hazard rate process $h_j(t)$ is to assume the *mean reverting process*

$$dh_j(t) = (b_j(t) - a_j h_j(t))dt + \sigma_j dz_j(t), \quad t \geq 0,$$

as suggested by Aonuma (1998). It is well known that

$$h_j(t) = e^{-a_j t} \left[h_j(0) + \int_0^t e^{a_j s} b_j(s) ds + \sigma_j \int_0^t e^{a_j s} dz_j(s) \right], \quad t \geq 0,$$

so that $h_j(t)$ is normally distributed with

$$\text{mean } e^{-a_j t} h_j(0) + \int_0^t e^{-a_j(t-s)} b_j(s) ds \text{ and variance } \frac{\sigma_j^2}{2a_j} (1 - e^{-2a_j t}).$$

But, since the variance does not grow linearly, the probability that the hazard rate $h_j(t)$ becomes negative should be smaller than that of the model (4.2). We shall study this model with an emphasis on statistical inference of the parameters in a separate paper.

⁹We may assume that every obligor belongs to a credit class whose members are statistically indistinguishable in their default processes, and that the parameters are given as a function of the class, $i = i(j)$ say, to which obligor j belongs. Perhaps, it would be appropriate that the classification of obligors is based on the credit rating, industries, and so on. In this paper, however, the parameters are written as a function of asset j , not of $i(j)$, for simplicity.

where P_t denotes the conditional probability measure given \mathcal{F}_t . It follows that

$$P\{\tau_1 > t_1, \dots, \tau_n > t_n\} = E \left[\prod_{j=1}^n P_t\{\tau_j > t_j\} \right].$$

But, by the definition of hazard rates, the conditional survival probabilities are given by

$$P_t\{\tau_j > t_j\} = \exp \left\{ - \int_0^{t_j} h_j(t) dt \right\}, \quad j = 1, 2, \dots, n, \quad (4.5)$$

where $h_j(t)$ are defined by (4.2). Collecting these information, it follows that

$$\begin{aligned} P\{\tau_1 > t_1, \dots, \tau_n > t_n\} &= E \left[\exp \left\{ - \sum_{j=1}^n \int_0^{t_j} h_j(t) dt \right\} \right] \\ &= \exp \left\{ - \sum_{j=1}^n B_j(0, t_j) + \frac{1}{2} \sum_{j=1}^n \sum_{k=1}^n \sigma_j c_{jk}(t_j, t_k) \sigma_k \right\}, \quad (4.6) \end{aligned}$$

where

$$\begin{aligned} B_j(u, v) &= \int_u^v \left[h_j(u) + \int_u^t b_j(s) ds \right] dt \\ &= (v - u)h_j(u) + \int_u^v (v - s)b_j(s) ds, \quad v > u, \end{aligned} \quad (4.7)$$

and, defining $a \wedge b = \min\{a, b\}$,

$$c_{jk}(t_j, t_k) = \int_0^{t_j \wedge t_k} (t_j - s)(t_k - s) \rho_{jk}(s) ds. \quad (4.8)$$

The detailed derivation of Equation (4.6) is given in Appendix A.1. Notice that the probability given by (4.6) may not define the joint survival probability. If, in particular, $\rho_{jk}(t)$ are constant, ρ_{jk} say, then (4.8) becomes, after some algebra, that

$$c_{jk}(t_j, t_k) = \rho_{jk} \frac{(t_j \wedge t_k)^2}{6} [3(t_j \vee t_k) - t_j \wedge t_k],$$

where $a \vee b = \max\{a, b\}$. The Weibull case (4.3) leads to

$$P\{\tau_1 > t_1, \dots, \tau_n > t_n\} = \exp \left\{ - \sum_{j=1}^n \lambda_j [(t_j + m_j)^{\gamma_j} - m_j^{\gamma_j}] + \frac{1}{2} \sum_{j=1}^n \sum_{k=1}^n \sigma_j c_{jk}(t_j, t_k) \sigma_k \right\}.$$

The joint distribution under the risk-neutral probability measure can be obtained similarly.

The marginal survival probabilities $P\{\tau_j > t_j\}$ can be obtained from (4.6) by putting $t_k = 0$ for $k \neq j$, and we have

$$P\{\tau_j > t_j\} = \exp \left\{ -B_j(0, t_j) + \frac{1}{6} \sigma_j^2 t_j^3 \right\}, \quad j = 1, 2, \dots, n, \quad (4.9)$$

since, from (4.8),

$$c_{jj}(t_j, t_j) = \int_0^{t_j} (t_j - s)^2 ds = \frac{1}{3} t_j^3.$$

In order for $P\{\tau_j > t_j\}$ in (4.9) to be the survival probability, it must hold that the function $B_j(t) - \sigma_j^2 t^3/6$ is non-decreasing in t and diverges as $t \rightarrow \infty$.¹⁰

4.2 The specialized default-free interest rate process

As to the default-free interest rate process $r(t)$ under the observed probability measure P , we assume the following simple mean reverting model (see Vasicek (1977)):

$$dr(t) = a(\bar{r} - r(t))dt + \sigma_f dz_f(t), \quad t \geq 0. \quad (4.10)$$

It is well known that $r(t)$ is normally distributed with

$$\text{mean } \bar{r} + \{r(0) - \bar{r}\}e^{-at} \text{ and variance } \frac{\sigma_f^2}{2a}(1 - e^{-2at}).$$

Hence, in this simplified model, the default-free interest rate process also becomes negative with positive probability.

For the pricing of default-free discount bonds, we want to employ the extended Vasicek model of Hull and White (1990), since the prices obtained by this model are consistent with the observed current prices in the market. As pointed out by Inui and Kijima (1998), this model coincides with the one-factor Markovian HJM model with $\sigma_f(r, t) = \sigma_f$ and $\kappa(t) = a$ in (3.1), i.e. these functions are constant. The resulting SDE under the risk neutral probability measure becomes

$$dr(t) = (\theta_f(t) - ar(t))dt + \sigma_f d\tilde{z}_f(t), \quad t \geq 0, \quad (4.11)$$

where

$$\theta_f(t) = af(0, t) + \frac{\partial}{\partial t}f(0, t) + \frac{\sigma_f^2}{2a}(1 - e^{-2at});$$

see Kijima and Nagayama (1994). The SDE (4.11) is consistent with (4.10) if we take the market price of risk as

$$\lambda(t) = \frac{a\bar{r} - \theta_f(t)}{\sigma_f}, \quad t \geq 0.$$

Note that, since $\theta_f(t)$ is a deterministic function of time t , the market price of risk $\lambda(t)$ is also a deterministic function of time t in this simplified model.

According to Hull and White (1990), the time t price of the default-free discount bond with maturity T is given by

$$p_0(t, T) = H_1(t, T)e^{-H_2(t, T)r(t)}, \quad 0 \leq t < T, \quad (4.12)$$

¹⁰In the Weibull case (4.3), this requirement implies that $\gamma_j \geq 3$, i.e. it must be the IHR case. However, for the practical use, all we need is that the function is non-decreasing in t before the maturity of asset j .

where

$$H_2(t, T) = \frac{1}{a}(1 - e^{-a(T-t)})$$

and

$$H_1(t, T) = \exp \left\{ \frac{1}{2} \int_t^T \sigma_f^2 H_2^2(u, T) du - \int_t^T \theta_f(u) H_2(u, T) du \right\}.$$

Since $r(t)$ is normally distributed under the observed probability measure, the future price $p_0(t, T)$ in (4.12) is log-normally distributed.

4.3 Pricing of defaultable discount bonds

In this subsection, we explain our valuation method to price defaultable discount bonds. In order to derive closed form solutions for the prices, we further impose the following assumptions:

- The recovery rate of discount bond j is constant and given by δ_j , $0 \leq \delta_j < 1$; and
- If the discount bond j defaults before the maturity T_j , the investor always receives the cash δ_j at the maturity T_j , regardless of the event $\{\tau_j \leq \bar{t}\}$ or $\{\tau_j > \bar{t}\}$.

In the following, we assume that $T_j \geq \bar{t} \geq t$ for all j unless stated otherwise. From the above assumptions, if $\tau_j > \bar{t}$ then the time t price of the defaultable discount bond with maturity T_j is given by (3.5), i.e.

$$p_j(t, T_j) = \tilde{E}_t \left[\exp \left\{ - \int_t^{T_j} r(s) ds \right\} \left\{ \delta_j 1_{\{\tau_j \leq T_j\}} + 1_{\{\tau_j > T_j\}} \right\} \right] \quad \text{on } \{\tau_j > \bar{t}\},$$

where 1_A denotes the indicator function meaning that $1_A = 1$ if the event A is true and $1_A = 0$ otherwise. Since the default-free interest rate process is independent of the default processes, it follows that

$$\begin{aligned} p_j(t, T_j) &= \tilde{E}_t \left[\exp \left\{ - \int_t^{T_j} r(s) ds \right\} \right] \tilde{E}_t \left[\delta_j 1_{\{\tau_j \leq T_j\}} + 1_{\{\tau_j > T_j\}} \right] \\ &= p_0(t, T_j) \left\{ \delta_j + (1 - \delta_j) \tilde{P}_t \{\tau_j > T_j\} \right\} \quad \text{on } \{\tau_j > \bar{t}\}, \end{aligned} \quad (4.13)$$

where \tilde{P}_t denotes the conditional probability measure given the history \mathcal{F}_t under the risk-neutrality and

$$p_0(t, T) = \tilde{E}_t \left[\exp \left\{ - \int_t^T r(s) ds \right\} \right]$$

is the time t price of the default-free discount bond with maturity T . In our model, the prices of default-free discount bonds are given by (4.12).

If default occurs before the risk horizon, on the other hand, then we evaluate its value as

$$\begin{aligned} p_j(t, T_j) &= \tilde{E}_t \left[\exp \left\{ - \int_t^{T_j} r(s) ds \right\} \delta_j \right] \\ &= \delta_j p_0(t, T_j) \quad \text{on } \{t < \tau_j \leq \bar{t}\}. \end{aligned} \quad (4.14)$$

4.4 Estimation of risk premia adjustments

Let $\tilde{h}_j(t)$ be the risk-adjusted hazard rate processes and let $h_j(t)$ be the observed hazard rate processes. Following Kijima (1998), we assume that there exist some risk premia adjustments $\ell_j(t)$ satisfying

$$\tilde{h}_j(t) = h_j(t) + \ell_j(t), \quad j = 1, 2, \dots, n. \quad (4.15)$$

In general, the risk premia adjustments $\ell_j(t)$ depend on the whole history; however, it will be assumed in this simplified model that $\ell_j(t)$ are deterministic functions of time t . The survival probability of τ_j under the risk-neutral probability measure \tilde{P} is then given by

$$\begin{aligned} \tilde{P}\{\tau_j > t\} &= E \left[\exp \left\{ - \int_0^t [h_j(s) + \ell_j(s)] ds \right\} \right] \\ &= L_j(0, t) P\{\tau_j > t\}, \end{aligned} \quad (4.16)$$

where

$$L_j(t, T) = e^{-\int_t^T \ell_j(s) ds}.$$

It follows from (4.9) that

$$\tilde{P}\{\tau_j > t\} = \exp \left\{ -B_j(0, t) + \frac{1}{6} \sigma_j^2 t^3 - \int_0^t \ell_j(s) ds \right\}, \quad j = 1, 2, \dots, n.$$

Risk premia adjustments $\ell_j(s)$ are determined so that the calculated present values are consistent with the current observed prices of discount bonds. Let $p_0(0, t)$ and $p_j(0, t)$ be the current prices of the default-free discount bond and defaultable discount bond j , respectively, with maturity t . From (4.13) and (4.16), we have

$$\frac{p_j(0, t)}{p_0(0, t)} = \delta_j + (1 - \delta_j) L_j(0, t) P\{\tau_j > t\}.$$

Solving this in terms of the risk premia adjustment $\ell_j(t)$ yields

$$\exp \left\{ - \int_0^t \ell_j(s) ds \right\} = \frac{1}{1 - \delta_j} \left\{ \frac{p_j(0, t)}{p_0(0, t)} - \delta_j \right\} \exp \left\{ B_j(0, t) - \frac{1}{6} \sigma_j^2 t^3 \right\}, \quad t \geq 0,$$

or

$$\ell_j(t) = -h_j(0) - \int_0^t b_j(s) ds + \frac{1}{2} \sigma_j^2 t^2 - \frac{\partial}{\partial t} \left[\log \left\{ \frac{p_j(0, t)}{p_0(0, t)} - \delta_j \right\} \right], \quad t \geq 0.$$

If, in particular, the Weibull case (4.3) is assumed, then we have

$$\ell_j(t) = -\lambda_j \gamma_j (t + m_j)^{\gamma_j - 1} + \frac{1}{2} \sigma_j^2 t^2 - \frac{\partial}{\partial t} \left[\log \left\{ \frac{p_j(0, t)}{p_0(0, t)} - \delta_j \right\} \right], \quad t \geq 0.$$

Risk premia adjustments $\ell_j(t)$ (or, equivalently, $L_j(t, T)$) adjust the difference between the observed probability measure and the risk-neutral probability measure (see (4.16)), and also risks other than the interest rate risk and credit risk (because they fit the current market prices).

4.5 The distribution of future price

With the risk premia adjustments $\ell_j(t)$, $t \geq 0$, at hand, we are now able to obtain the distribution of future price of defaultable discount bond j . The distribution of the default-free discount bond price was obtained earlier in Subsection 4.2.

Let the current time be zero and suppose that the event $\{\tau_j \leq \bar{t}\}$ happens, where \bar{t} is the risk horizon. Then, we have the value (4.14) with t being replaced by \bar{t} with probability

$$P\{\tau_j \leq \bar{t}\} = 1 - \exp\left\{-B_j(0, \bar{t}) + \frac{1}{6}\sigma_j^2\bar{t}^3\right\};$$

see (4.9). If, on the other hand, the event $\{\tau_j > \bar{t}\}$ occurs, then we need to consider the time \bar{t} price $p_j(\bar{t}, T_j)$ of discount bond j with maturity $T_j > \bar{t}$. To this end, we have from (4.2) that

$$h_j(t) = h_j(\bar{t}) + \int_{\bar{t}}^t b_j(s)ds + \sigma_j z_j^*(t - \bar{t}), \quad t > \bar{t}, \quad (4.17)$$

where $z_j^*(t - \bar{t}) = z_j(t) - z_j(\bar{t})$. The hazard rate process (4.17) should be compared with that in (4.2). Note that the increment $z_j^*(t - \bar{t})$ is independent of $\{z_j(u), u \leq \bar{t}\}$ and is normally distributed with mean 0 and variance $t - \bar{t}$. As in (4.16), the conditional survival probability under the risk-neutral probability measure is then given by

$$\begin{aligned} \tilde{P}_{\bar{t}}\{\tau_j > T_j\} &= E_{\bar{t}}\left[\exp\left\{-\int_{\bar{t}}^{T_j} [h_j(s) + \ell_j(s)]ds\right\}\right] \\ &= L_j(\bar{t}, T_j) \exp\left\{-B_j(\bar{t}, T_j) + \frac{1}{6}\sigma_j^2(T_j - \bar{t})^3\right\} \quad \text{on } \{\tau_j > \bar{t}\}, \end{aligned} \quad (4.18)$$

where $B_j(t, T_j)$ is defined in (4.7), i.e.

$$B_j(\bar{t}, T_j) = (T_j - \bar{t})h_j(\bar{t}) + \int_{\bar{t}}^{T_j} (T_j - s)b_j(s)ds.$$

Recall that $h_j(\bar{t})$ is normally distributed with

$$\text{mean } h_j(0) + \int_0^{\bar{t}} b_j(s)ds \text{ and variance } \sigma_j^2\bar{t}. \quad (4.19)$$

Hence, the survival probability $\tilde{P}_{\bar{t}}\{\tau_j > T_j\}$ in (4.18) is log-normally distributed and so, from (4.13), the price $p_j(\bar{t}, T_j)$ on the event $\{\tau_j > \bar{t}\}$ is given as a weighted sum of two log-normally distributed random variables.

Let $p_j(\bar{t})$ denote the future value of defaultable discount bond j at the risk horizon \bar{t} and suppose that we have the portfolio (3.6) consisting of discount bonds only. Then, from (4.13) and (4.14), it follows that

$$p_j(\bar{t}) = p_0(t, T_j) \left[\delta_j + (1 - \delta_j) \tilde{P}_{\bar{t}}\{\tau_j > T_j\} 1_{\{\tau_j > \bar{t}\}} \right], \quad (4.20)$$

whence

$$\pi(\bar{t}) = \sum_{j=1}^n w_j p_0(t, T_j) \left[\delta_j + (1 - \delta_j) \tilde{P}_{\bar{t}}\{\tau_j > T_j\} 1_{\{\tau_j > \bar{t}\}} \right].$$

Since the joint distribution of the default times τ_j and the correlation structure of the Wiener processes $z_j(t)$ are known, we can in principle evaluate the distribution function of $\pi(\bar{t})$. However, in order to do this, we need to consider all the combinations of defaults. The number of combinations grows exponentially as the number n of assets in the portfolio increases, which makes the exact valuation practically intractable even for a reasonably small size of portfolio. In the next section, we use the Cornish-Fisher expansion to obtain an approximation of the portfolio VaR in a closed form. For this purpose, we obtain here the mean and the variance of $\pi(\bar{t})$. Higher moments of $\pi(\bar{t})$ can be obtained similarly in the aid of the result given in Appendix A.2.

In order to simplify the expressions below, we introduce the following notation:

$$\begin{aligned} P_0^I(T) &= E[p_0(\bar{t}, T)]; \\ P_0^{II}(T_j, T_k) &= E[p_0(\bar{t}, T_j)p_0(\bar{t}, T_k)]. \end{aligned}$$

From (4.12), we obtain

$$P_0^I(T) = H_1(\bar{t}, T) \exp \left\{ -H_2(\bar{t}, T)E[r(\bar{t})] + \frac{1}{2}H_2^2(\bar{t}, T)V[r(\bar{t})] \right\},$$

since $r(\bar{t})$ is normally distributed with mean

$$E[r(\bar{t})] = r(0)e^{-a\bar{t}} + \bar{r}(1 - e^{-a\bar{t}})$$

and variance

$$V[r(\bar{t})] = \frac{\sigma_f^2}{2a}(1 - e^{-2a\bar{t}}).$$

Similarly, we have

$$\begin{aligned} P_0^{II}(T_j, T_k) &= H_1(\bar{t}, T_j)H_1(\bar{t}, T_k) \exp \{ -[H_2(\bar{t}, T_j) + H_2(\bar{t}, T_k)]E[r(\bar{t})] \} \\ &\quad \times \exp \left\{ \frac{1}{2}[H_2(\bar{t}, T_j) + H_2(\bar{t}, T_k)]^2 V[r(\bar{t})] \right\}. \end{aligned}$$

In particular, if $j = k$ then

$$P_0^{II}(T_j, T_j) = H_1^2(\bar{t}, T_j) \exp \{ -2H_2(\bar{t}, T_j)E[r(\bar{t})] + 2H_2^2(\bar{t}, T_j)V[r(\bar{t})] \}.$$

Higher moments of $p_0(\bar{t}, T)$ can be obtained similarly.

Next, recall that the default-free interest rate $r(t)$ is independent of the hazard processes $h_j(t)$. It follows from (4.20) that

$$E[p_j(\bar{t})] = P_0^I(T_j) \left\{ \delta_j + (1 - \delta_j)E \left[1_{\{\tau_j > \bar{t}\}} \tilde{P}_{\bar{t}}\{\tau_j > T_j\} \right] \right\}.$$

Note from (4.18) that

$$E \left[1_{\{\tau_j > \bar{t}\}} \tilde{P}_{\bar{t}} \{ \tau_j > T_j \} \right] = L_j(\bar{t}, T_j) E \left[E_{\bar{t}} \left[1_{\{\tau_j > \bar{t}\}} e^{-\int_{\bar{t}}^{T_j} h_j(s) ds} \right] \right].$$

But, since the process $h_j(t)$ is Markovian, given $\mathcal{F}_{\bar{t}}$, the past event $\{\tau_j > \bar{t}\}$ and the future random variable $e^{-\int_{\bar{t}}^{T_j} h_j(s) ds}$ are conditionally independent. It follows from (4.5) that

$$\begin{aligned} E_{\bar{t}} \left[1_{\{\tau_j > \bar{t}\}} e^{-\int_{\bar{t}}^{T_j} h_j(s) ds} \right] &= P_{\bar{t}} \{ \tau_j > \bar{t} \} E_{\bar{t}} \left[e^{-\int_{\bar{t}}^{T_j} h_j(s) ds} \right] \\ &= e^{-\int_0^{\bar{t}} h_j(s) ds} E_{\bar{t}} \left[e^{-\int_{\bar{t}}^{T_j} h_j(s) ds} \right], \end{aligned} \quad (4.21)$$

whence

$$\begin{aligned} E \left[1_{\{\tau_j > \bar{t}\}} \tilde{P}_{\bar{t}} \{ \tau_j > T_j \} \right] &= L_j(\bar{t}, T_j) E \left[E_{\bar{t}} \left[e^{-\int_0^{T_j} h_j(s) ds} \right] \right] \\ &= L_j(\bar{t}, T_j) Q_j(T_j). \end{aligned} \quad (4.22)$$

Here and hereafter, we denote

$$Q_j(T) = P \{ \tau_j > T \} = \exp \left\{ -B_j(0, T) + \frac{1}{6} \sigma_j^2 T^3 \right\}.$$

The mean $E[\pi(\bar{t})] = \sum_{j=1}^n w_j E[p_j(\bar{t})]$ can now be calculated by combining these results.

The calculation of the variance $V[\pi(\bar{t})]$ is much involved. For this purpose, the following result is useful which is a special case of the general result given in Appendix A.2:

$$\begin{aligned} E \left[1_{\{\tau_j > \bar{t}\}} 1_{\{\tau_k > \bar{t}\}} \tilde{P}_{\bar{t}} \{ \tau_j > T_j \} \tilde{P}_{\bar{t}} \{ \tau_k > T_k \} \right] \\ = L_j(\bar{t}, T_j) Q_j(T_j) L_k(\bar{t}, T_k) Q_k(T_k) e^{\sigma_j \eta_{jk} \sigma_k}, \end{aligned} \quad (4.23)$$

where

$$\eta_{jk} = \int_0^{\bar{t}} (T_j - s)(T_k - s) \rho_{jk}(s) ds.$$

The adjustment factors η_{jk} appear in (4.23) because of the correlation structure. Note the difference between η_{jk} and $c_{jk}(T_j, T_k)$ in (4.8). Now, in order to calculate $E[\pi(\bar{t})^2]$, we need to evaluate $E[p_j(\bar{t}) p_k(\bar{t})]$. It follows from (4.20) that

$$\begin{aligned} E[p_j(\bar{t}) p_k(\bar{t})] &= P_0^{II}(T_j, T_k) \left\{ \delta_j \delta_k + \delta_j (1 - \delta_k) L_k(\bar{t}, T_k) Q_k(T_k) + \delta_k (1 - \delta_j) L_j(\bar{t}, T_j) Q_j(T_j) \right. \\ &\quad \left. + (1 - \delta_j)(1 - \delta_k) L_j(\bar{t}, T_j) Q_j(T_j) L_k(\bar{t}, T_k) Q_k(T_k) e^{\sigma_j \eta_{jk} \sigma_k} \right\}. \end{aligned}$$

In particular, if $j = k$ then we have

$$E[p_j(\bar{t})^2] = P_0^{II}(T_j, T_j) \left[\delta_j^2 + 2\delta_j(1 - \delta_j) L_j(\bar{t}, T_j) Q_j(T_j) + (1 - \delta_j)^2 L_j^2(\bar{t}, T_j) Q_j^2(T_j) e^{\sigma_j^2 \eta_{jj}} \right],$$

where

$$\eta_{jj} = \bar{t} \left(T_j^2 - T_j \bar{t} + \frac{\bar{t}^2}{3} \right).$$

These are enough to calculate the variance since

$$V[\pi(\bar{t})] = \sum_{j=1}^n \sum_{k=1}^n w_j w_k E[p_j(\bar{t}) p_k(\bar{t})] - (E[\pi(\bar{t})])^2.$$

Higher moments of $\pi(\bar{t})$ can be obtained similarly.

4.6 Other defaultable assets

In this subsection, we provide valuation formulas to price other defaultable assets; namely, fixed and floating coupon bonds and interest rate swaps. Throughout this subsection, we only consider the event $\{\tau_j > t\}$.

4.6.1 Fixed rate bonds

Consider a defaultable coupon bond with fixed interest rate. The time t price is expressed as a linear combination of the discount bond price (4.13) with different maturities and face values, because a coupon bond is regarded as a portfolio of discount bonds. The price of the coupon bond j with coupon rate C , payment dates $\mathbf{T} = (T_1, T_2, \dots, T_m)$, and maturity T_m is given by

$$p_j(t, \mathbf{T}; C) = C \sum_{i=1}^m p_j(t, T_i) + p_j(t, T_m), \quad (4.24)$$

where $p_j(t, T_i)$ are given by (4.13).

4.6.2 Floating rate bonds

Consider a defaultable coupon bond with continuous floating interest rate

$$C(t) = \alpha r(t) + \beta, \quad t \geq 0, \quad (4.25)$$

where α and β are constant. We first consider a default-free bond whose outcome from the investment during the period $(T_{n-1}, T_n]$ with the floating interest rate $C(t)$ is redeemed at time T_n . The time t price of this bond is given by

$$\begin{aligned} p_0(t, T_{n-1}, T_n; \alpha, \beta) = & \exp \left\{ \beta(T_n - T_{n-1}) - K_1(t, T_{n-1}, T_n; \alpha) r(t) \right. \\ & \left. - K_2(t, T_{n-1}, T_n; \alpha) + \frac{1}{2} \sigma_f^2 K_3(t, T_{n-1}, T_n; \alpha) \right\}, \end{aligned} \quad (4.26)$$

where

$$\begin{aligned} K_1(t, t_1, t_2; \alpha) &= \alpha H_2(t, t_1) + (1 - \alpha) H_2(t, t_2), \\ K_2(t, t_1, t_2; \alpha) &= \alpha \int_{t_1}^{t_2} \theta_f(u) H_2(u, t_1) du + (1 - \alpha) \int_{t_1}^{t_2} \theta_f(u) H_2(u, t_2) du \end{aligned}$$

and

$$K_3(t, t_1, t_2; \alpha) = \int_{t_1}^{t_2} \{ \alpha H_2(u, t_1) + (1 - \alpha) H_2(u, t_2) \}^2 du + \int_{t_1}^{t_2} (1 - \alpha)^2 H_2^2(u, t_2) du$$

The derivation of (4.26) is given in Appendix A.3 with a more detailed expression for K_3 . The time t price of the corresponding defaultable bond j is then given by

$$p_j(t, T_{n-1}, T_n; \alpha, \beta) = p_0(t, T_{n-1}, T_n; \alpha, \beta) [\delta_j + (1 - \delta_j) \tilde{P}_t\{\tau_j > T_n\}].$$

We note that the volatility σ_f of the default-free interest rate appears on the price p_j only through the term $\exp\left\{\frac{1}{2}\sigma_f^2 K_3\right\}$.

The time t price of the coupon bond j with the floating interest rate $C(t)$ is expressed as

$$p_j(t, \mathbf{T}; \alpha, \beta) = \sum_{n=1}^m \{p_j(t, T_{n-1}, T_n; \alpha, \beta) - p_j(t, T_{n-1}, T_n; 0, 0)\} + p_j(t, T_{m-1}, T_m; 0, 0) \quad (4.27)$$

where $\mathbf{T} = (T_1, T_2, \dots, T_m)$ is the coupon payment dates and T_m is the maturity date. The first term on the right hand side of (4.27) corresponds to the coupon paid off at time T_n . From the definition, $p_j(t, T_{n-1}, T_n; 0, 0)$ is equal to the price of the defaultable discount bond $p_j(t, T_n)$, the latter being independent of T_{n-1} . In the case where the interest rate is fixed, i.e. $\alpha = 0$, and $\Delta T = T_n - T_{n-1}$ is constant, Equation (4.27) is reduced to (4.24) with $e^{\beta\Delta T} - 1$ being replaced by C .

4.6.3 Interest rate swaps

A plain vanilla interest rate swap is an exchange of the payoffs with a floating interest rate and those with a fixed interest rate. It is decomposed into a fixed rate bond and a floating rate bond; therefore the present value of the swap is equal to the sum of the present values of the two bonds.

Consider a plain vanilla swap with maturity T_m such that corporate A receives the fixed interest rate β_B while corporate B receives the floating interest rate $r(t) + \beta_A$. The present value of the swap evaluated from the standpoint of corporate A is given by

$$P_{sw}^A = p_B(t, \mathbf{T}; 0, \beta_B) - p_A(t, \mathbf{T}; 1, \beta_A),$$

where $p_B(t, \mathbf{T}; 0, \beta_B)$ is the time t price of the bond with the fixed interest rate β_B issued by corporate B, $p_A(t, \mathbf{T}; 1, \beta_A)$ is that of the bond with the floating interest rate $r(t) + \beta_A$ issued by corporate A, and $\mathbf{T} = (T_1, T_2, \dots, T_m)$ denotes the payment dates. The prices $p_B(t, \mathbf{T}; 0, \beta_B)$ and $p_A(t, \mathbf{T}; 1, \beta_A)$ can be derived from the pricing formula (4.27) of the floating rate bonds. Namely, we have

$$p_B(t, \mathbf{T}; 0, \beta_B) = \sum_{n=1}^m (e^{\beta_B(T_n - T_{n-1})} - 1)p_0(t, T_n)[\delta_B + (1 - \delta_B)\tilde{P}_t^B\{\tau_B > T_n\}] + p_0(t, T_m)[\delta_B + (1 - \delta_B)\tilde{P}_t^B\{\tau_B > T_m\}]$$

and

$$p_A(t, \mathbf{T}; 1, \beta_A) = \sum_{n=1}^m \{e^{\beta_A(T_n - T_{n-1})}p_0(t, T_{n-1}) - p_0(t, T_n)\}[\delta_A + (1 - \delta_A)\tilde{P}_t^A\{\tau_A > T_n\}] + p_0(t, T_m)[\delta_A + (1 - \delta_A)\tilde{P}_t^A\{\tau_A > T_m\}],$$

where δ_A and δ_B are the respective recovery rates of corporates A and B. The survival probabilities under the risk-neutral probability measure \tilde{P} are calculated from (4.16).

5 Approximation of portfolio VaR's

The closed form solutions obtained in the previous section suggest that the distribution of future value of a portfolio is a mixture of correlated, log-normally distributed random variables. This means that we cannot expect a simple analytical expression for the distribution of the future portfolio value. In this section, we derive some approximate expression for the portfolio VaR if the portfolio consists of many assets.

For the future portfolio value $\pi(\bar{t})$, we define

$$Y_n = \frac{\pi(\bar{t}) - n\mu}{\sqrt{n}\sigma},$$

where

$$\mu = \frac{E[\pi(\bar{t})]}{n}, \quad \sigma = \sqrt{\frac{V[\pi(\bar{t})]}{n}}. \quad (5.1)$$

The mean and the variance of $\pi(\bar{t})$ have been obtained in the previous section. Let $G_{\pi,n}(x)$ and $G_{Y,n}(x)$ denote the distribution functions of $\pi(\bar{t})$ and Y_n , respectively. For any α , $0 < \alpha < 1$, we have

$$G_{\pi,n}^{-1}(\alpha) = n\mu + \sqrt{n}\sigma G_{Y,n}^{-1}(\alpha), \quad (5.2)$$

where $G_{\pi,n}^{-1}(\alpha)$ and $G_{Y,n}^{-1}(\alpha)$ represent the 100α -percentiles of $\pi(\bar{t})$ and Y_n , respectively.

If the number n of assets included in a portfolio is large enough, we may employ the following Cornish-Fisher expansion (see, e.g., Equation (6) in Berger (1972)):

$$G_{Y,n}^{-1}(\alpha) = z_\alpha + \frac{\gamma_3}{6\sqrt{n}}(z_\alpha^2 - 1) + \frac{\gamma_4}{24n}(z_\alpha^3 - 3z_\alpha) - \frac{\gamma_3^2}{36n}(2z_\alpha^3 - 5z_\alpha) + o\left(\frac{1}{n}\right), \quad (5.3)$$

where z_α denotes the 100α -percentile of the standard normal distribution. This expansion is validated if $\pi(\bar{t})$ can be approximated by a sum of independent and identically distributed random variables X_i , $i = 1, 2, \dots, n$, with the mean μ and the standard deviation σ given in (5.1). The parameters γ_3 and γ_4 in (5.3) are the skewness and the excess kurtosis of X_1 , i.e.

$$\gamma_3 = \frac{E[(X_1 - \mu)^3]}{\sigma^3}, \quad \gamma_4 = \frac{E[(X_1 - \mu)^4]}{\sigma^4} - 3,$$

respectively. These parameters can be calculated from the third and the fourth moments of $\pi(\bar{t})$. More terms in the expansion (5.3), if necessary, can be evaluated from higher moments of $\pi(\bar{t})$ in a similar manner.

If the first term in (5.3) dominates the other terms, then (5.2) and (5.3) together yield

$$G_{\pi,n}^{-1}(\alpha) = n\mu + \sqrt{n}\sigma z_\alpha. \quad (5.4)$$

This is a special case where the central limit theorem is used for an approximation of $G_{Y,n}(x)$. The situation is validated, for example, if the number n is large enough and both γ_3 and γ_4 are small enough.

6 Concluding Remarks

In this paper, we propose a new model for evaluating credit risk and market risk of a portfolio consisting of interest rate sensitive assets in a synthetic manner, where a stochastic default-free interest rate process and stochastic default processes of defaultable assets play a central role. For the default-free interest rate process, we can use any non-arbitrage model in the finance literature. Defaults are formulated by hazard rate processes, which are assumed to follow a multi-dimensional diffusion process. Present and future prices of all the assets are evaluated by the *single* risk-neutral valuation framework, and the distribution of future value of the portfolio is obtained by assuming the conditional independence on default times. The calculated present prices are consistent with observed market prices through the risk premia adjustments. Also, the portfolio effects are taken into consideration explicitly in our model through the correlation among the default processes. In order to obtain closed form solutions for the asset prices, we provide a simplified model by imposing further assumptions. Finally, applying the Cornish-Fisher expansion, we derive an approximated expression of the portfolio VaR.

In the model framework described in Section 3, we assumed the following:

- The default times τ_j of defaultable assets are conditionally independent; and
- The default-free interest rate process is independent of the default processes.

The first assumption is necessary to construct the joint distribution of default times from the marginal distributions of each default time governed by the hazard rate process (3.3). Notice that the correlation between the hazard rate processes is transferred to the joint distribution under the assumption. Without this assumption, we must model the joint distribution directly; however, such a modeling would be very difficult because usually we have very little information about it. In contrast, the second assumption seems controversial, although it seems widely accepted in the literature partly because there is an evidence that defaults occur independently of the fundamentals of economy for firms with high credit ratings (see, e.g., Jarrow, Lando and Turnbull (1997)), and partly because it makes the derivation of pricing formulas considerably easier. The second assumption can be removed if we work on a Monte Carlo simulation throughout the risk valuation.

For our simplified model given in Section 4, we in addition assumed the following:

- The hazard rate processes follow a multi-dimensional Gaussian process (4.1);
- The recovery rate δ_j , $0 \leq \delta_j < 1$, is constant;

- If discount bond j defaults before the maturity T_j , the investor receives the cash δ_j at the maturity T_j regardless of the event $\{\tau_j \leq \bar{t}\}$ or $\{\tau_j > \bar{t}\}$; and
- The survival probabilities under the risk-neutral probability measure \bar{P} are given by (4.16), and the risk premia adjustments $\ell_j(t)$ are deterministic functions of time t .

In Section 4, we demonstrated that these assumptions lead to simple closed form solutions for asset prices. However, apparent drawbacks of the assumptions can be pointed out. The first assumption cannot rule out the possibility of negative hazard rates, which may then make the survival probability locally increasing. Recall that our pricing formula (4.13) includes a survival function as a major component. The second assumption may not be realistic because there are some evidences that a recovery rate fluctuates in time. The third and fourth assumptions are imposed only for the purpose of tractability and they are indeed artificial. Note that the risk premia adjustments $\ell_j(t)$ depend in general on the whole history. Further improvements on the simplified model would be of interest from both theoretical and practical points of view.

In Section 5, we employ the Cornish-Fisher expansion (5.3) to derive an approximate VaR for a large portfolio. This expansion is validated if the future portfolio value $\pi(\bar{t})$ can be approximated by a sum of independent, identically distributed random variables. Then, it would be expected that the Cornish-Fisher expansion might give us a good approximation of VaR for a portfolio consisting of many assets.

At present, some empirical studies as well as the model implementation are in progress. We will report them somewhere as soon as results come out.

A Proofs

In this appendix, we provide concise proofs of Equations (4.6), (4.23), and (4.26).

A.1 The joint survival probability

We have from (4.2), the conditional independence (4.4) and (4.5) that

$$\begin{aligned} P_t\{\tau_1 > t_1, \dots, \tau_n > t_n\} &= \exp \left\{ - \sum_{j=1}^n \int_0^{t_j} h_j(s) ds \right\} \\ &= \exp \left\{ - \sum_{j=1}^n B_j(0, t_j) + \sum_{j=1}^n \sigma_j \int_0^{t_j} z_j(s) ds \right\}. \end{aligned}$$

Now consider $I_j = \int_0^{t_j} z_j(s) ds$ and let $I = \sum_{j=1}^n \sigma_j I_j$. It is well-known that I_j are jointly, normally distributed. To obtain the covariance $c_{jk}(t_j, t_k) = \text{Cov}(I_j, I_k)$, we note that, upon

integration by parts,

$$\int_0^t z(s)ds = \int_0^t (t-s)dz(s). \quad (\text{A.1})$$

It follows from (3.4) that

$$\begin{aligned} c_{jk}(t_j, t_k) &= E \left[\int_0^{t_j} \int_0^{t_k} (t_j - t)(t_k - s) dz_j(t) dz_k(s) \right] \\ &= \int_0^{t_j \wedge t_k} (t_j - s)(t_k - s) \rho_{jk}(s) ds, \end{aligned}$$

where $a \wedge b = \min\{a, b\}$. Hence, I is normally distributed with mean 0 and variance

$$V[I] = \sum_{j=1}^n \sum_{k=1}^n \sigma_j c_{jk}(t_j, t_k) \sigma_k.$$

The moment generating function of I is given by

$$E[e^{-I}] = \exp \left\{ \frac{1}{2} V[I] \right\},$$

which proves Equation (4.6).

A.2 Relations between observed and risk-neutral survival probabilities

For $m = 1, 2, \dots, n$, let

$$1 \leq j(1) \leq j(2) \leq \dots \leq j(m) \leq n.$$

We prove the following general result:

$$\begin{aligned} &E \left[\prod_{k=1}^m 1_{\{\tau_{j(k)} > \bar{t}\}} \tilde{P}_{\bar{t}}\{\tau_{j(k)} > T_{j(k)}\} \right] \\ &= \prod_{k=1}^m L_{j(k)}(\bar{t}, T_{j(k)}) P\{\tau_{j(k)} > T_{j(k)}\} \exp \left\{ \frac{1}{2} \sum_{k \neq \ell} \sigma_{j(k)} \eta_{j(k), j(\ell)} \sigma_{j(\ell)} \right\}, \end{aligned} \quad (\text{A.2})$$

where

$$\eta_{jk} = \int_0^{\bar{t}} (T_{j(k)} - s)(T_{j(\ell)} - s) \rho_{j(k), j(\ell)}(s) ds. \quad (\text{A.3})$$

Note that the results (4.22) and (4.23) are special cases of (A.2) with $m = 1$ and $m = 2$, respectively.

From (4.18) and (4.21), we have

$$\begin{aligned} &E \left[\prod_{k=1}^m 1_{\{\tau_{j(k)} > \bar{t}\}} \tilde{P}_{\bar{t}}\{\tau_{j(k)} > T_{j(k)}\} \right] \\ &= \prod_{k=1}^m L_{j(k)}(\bar{t}, T_{j(k)}) E \left[\prod_{k=1}^m E_{\bar{t}} \left[1_{\{\tau_{j(k)} > \bar{t}\}} e^{-\int_{\bar{t}}^{T_{j(k)}} h_{j(k)}(s) ds} \right] \right] \\ &= \prod_{k=1}^m L_{j(k)}(\bar{t}, T_{j(k)}) E \left[\prod_{k=1}^m \exp \left\{ -\int_0^{\bar{t}} h_{j(k)}(s) ds - B_{j(k)}(\bar{t}, T_{j(k)}) + \frac{1}{6} \sigma_{j(k)}^2 (T_{j(k)} - \bar{t})^3 \right\} \right]. \end{aligned}$$

Recalling (4.7), we denote

$$J_j = \int_0^{\bar{t}} h_j(s) ds + (T_j - \bar{t})h_j(\bar{t}) + \int_{\bar{t}}^{T_j} (T_j - s)b_j(s) ds - \frac{1}{6}\sigma_j^2(T_j - \bar{t})^3.$$

It follows that

$$E \left[\prod_{k=1}^m 1_{\{\tau_{j(k)} > \bar{t}\}} \tilde{P}_{\bar{t}} \{\tau_{j(k)} > T_{j(k)}\} \right] = \prod_{k=1}^m L_{j(k)}(\bar{t}, T_{j(k)}) E \left[e^{-\sum_{k=1}^m J_{j(k)}} \right].$$

Since $\sum_{k=1}^m J_{j(k)}$ is normally distributed, we will calculate its mean and variance. To this end, we obtain from (4.19)

$$\begin{aligned} E[J_j] &= B_j(0, \bar{t}) + (T_j - \bar{t}) \left[h_j(0) + \int_0^{\bar{t}} b_j(s) ds \right] + \int_{\bar{t}}^{T_j} (T_j - s)b_j(s) ds - \frac{1}{6}\sigma_j^2(T_j - \bar{t})^3 \\ &= T_j h_j(0) + \int_0^{\bar{t}} (T_j - s)b_j(s) ds + \int_{\bar{t}}^{T_j} (T_j - s)b_j(s) ds - \frac{1}{6}\sigma_j^2(T_j - \bar{t})^3 \\ &= B_j(0, T_j) - \frac{1}{6}\sigma_j^2(T_j - \bar{t})^3. \end{aligned}$$

To calculate the variance, we consider

$$\begin{aligned} &E \left[\left(\int_0^{\bar{t}} z_j(s) ds + (T_j - \bar{t})z_j(\bar{t}) \right) \left(\int_0^{\bar{t}} z_k(s) ds + (T_k - \bar{t})z_k(\bar{t}) \right) \right] \\ &= E \left[\left(\int_0^{\bar{t}} (\bar{t} - s) dz_j(s) + (T_j - \bar{t}) \int_0^{\bar{t}} dz_j(s) \right) \left(\int_0^{\bar{t}} (\bar{t} - s) dz_k(s) + (T_k - \bar{t}) \int_0^{\bar{t}} dz_k(s) \right) \right], \end{aligned}$$

where the relation (A.1) is employed. It follows that, after some algebra,

$$\text{Cov}(J_j, J_k) = \sigma_j \sigma_k \int_0^{\bar{t}} (T_j - s)(T_k - s) \rho_{jk}(s) ds.$$

In particular, if $j = k$ then

$$V[J_j] = \sigma_j^2 \bar{t} \left(T_j^2 - T_j \bar{t} + \frac{\bar{t}^2}{3} \right).$$

Note that

$$\frac{1}{6}\sigma_j^2(T_j - \bar{t})^3 + \frac{1}{2}V[J_j] = \frac{1}{6}\sigma_j^2 T_j^3.$$

Hence,

$$\begin{aligned} E \left[\exp \left\{ - \sum_{k=1}^m J_{j(k)} \right\} \right] &= \exp \left\{ -E \left[\sum_{k=1}^m J_{j(k)} \right] + \frac{1}{2}V \left[\sum_{k=1}^m J_{j(k)} \right] \right\} \\ &= \exp \left\{ \sum_{k=1}^m \left(-E[J_{j(k)}] + \frac{1}{2}V[J_{j(k)}] \right) + \sum_{k < \ell} \text{Cov}(J_{j(k)}, J_{j(\ell)}) \right\} \\ &= \exp \left\{ \sum_{k=1}^m \left(-B_{j(k)}(0, T_{j(k)}) + \frac{1}{6}\sigma_{j(k)}^2 T_{j(k)}^3 \right) + \sum_{k < \ell} \text{Cov}(J_{j(k)}, J_{j(\ell)}) \right\}. \end{aligned}$$

The result (A.2) now follows in the aid of (4.9) and (A.3).

A.3 The pricing of floating rate bonds

Consider a default-free bond whose outcome from the investment during the period $(t_1, t_2]$ with the floating interest rate $C(t)$ in (4.25) is redeemed at time t_2 . The time t price of this bond is given by

$$p_0(t, t_1, t_2; \alpha, \beta) = \tilde{E}_t \left[\exp \left\{ \int_{t_1}^{t_2} C(s) ds - \int_t^{t_2} r(s) ds \right\} \right], \quad t \leq t_1 \leq t_2.$$

Assuming the SDE (4.11) for the default-free interest rate process $r(t)$ under the risk-neutral probability measure \tilde{P} , we have

$$r(s) = r(t)e^{-a(s-t)} + \int_t^s \theta_f(u)e^{-a(s-u)} du + \sigma_f \int_t^s e^{-a(s-u)} d\tilde{z}_f(u), \quad s \geq t.$$

Let $I(v, w) = \int_v^w r(s) ds$ where $w \geq v \geq t$. After some algebra, we obtain

$$\begin{aligned} I(v, w) &= r(t) \{H_2(t, w) - H_2(t, v)\} \\ &\quad + \int_t^w \theta_f(u) H_2(u, w) du - \int_t^v \theta_f(u) H_2(u, v) du \\ &\quad + \sigma_f \int_t^w H_2(u, w) d\tilde{z}_f(u) - \sigma_f \int_t^v H_2(u, v) d\tilde{z}_f(u). \end{aligned}$$

It follows that

$$\begin{aligned} p_0(t, t_1, t_2; \alpha, \beta) &= \exp\{\beta(t_2 - t_1)\} \tilde{E}_t [\exp\{\alpha I(t_1, t_2) - I(t, t_2)\}] \\ &= \exp\{\beta(t_2 - t_1) - K_1 r(t) - K_2\} \tilde{E}_t [\exp\{-\sigma_f X\}], \end{aligned}$$

where

$$\begin{aligned} K_1 &= \alpha H_2(t, t_1) + (1 - \alpha) H_2(t, t_2), \\ K_2 &= \alpha \int_t^{t_1} \theta_f(u) H_2(u, t_1) du + (1 - \alpha) \int_t^{t_2} \theta_f(u) H_2(u, t_2) du, \end{aligned}$$

and

$$X = \alpha \int_t^{t_1} H_2(u, t_1) d\tilde{z}_f(u) + (1 - \alpha) \int_t^{t_2} H_2(u, t_2) d\tilde{z}_f(u).$$

But, since X is normally distributed with mean 0 and variance

$$\begin{aligned} K_3 &= \int_t^{t_1} \{\alpha H_2(u, t_1) + (1 - \alpha) H_2(u, t_2)\}^2 du + \int_{t_1}^{t_2} (1 - \alpha)^2 H_2^2(u, t_2) du \\ &= \frac{(1 - \alpha)^2}{a^2} \left[t_2 - t - \frac{2}{a} \{1 - e^{-a(t_2-t)}\} + \frac{1}{2a} \{1 - e^{-2a(t_2-t)}\} \right] \\ &\quad + \frac{2\alpha(1 - \alpha)}{a^2} \left[t_1 - t - \frac{2}{a} \{1 - e^{-a(t_1-t)} + e^{-a(t_2-t_1)} - e^{-a(t_2-t)}\} \right] \\ &\quad + \frac{1}{2a} \{e^{-a(t_2-t_1)} - e^{-a(t_2+t_1-2t)}\} \\ &\quad + \frac{\alpha^2}{a^2} \left[t_1 - t - \frac{2}{a} \{1 - e^{-a(t_1-t)}\} + \frac{1}{2a} \{1 - e^{-2a(t_1-t)}\} \right], \end{aligned}$$

Equation (4.26) is now derived.

References

- [1] Aonuma, K. (1998), private communication.
- [2] Artzner, P., F. Delbaen, J.M. Eber and D. Heath (1998), "Definition of coherent measures of risk," Preprint.
- [3] Berger, G. (1972), "Integration of the normal power approximation," *ASTIN Bulletin*, 7, 90-95.
- [4] *CreditMetrics*TM (1997), JP Morgan.
- [5] *CREDITRISK*⁺ (1997), Credit Suisse Financial Products.
- [6] Das, S.R. and P. Tufano (1996), "Pricing credit-sensitive debt when interest rates, credit ratings and credit spreads are stochastic," *Journal of Financial Engineering*, 5, 161-198.
- [7] Davis, M. and T. Mavroidis (1997), "Valuation and potential exposure of default swaps," Technical Note RPD-18, Tokyo-Mitsubishi International PLC, London.
- [8] Duffie, D. and K. Singleton (1996), "Modeling term structures of defaultable bonds," Working paper, Stanford University.
- [9] Fons, J.S. (1994), "Using default rates to model the term structure of credit risk," *Financial Analysts Journal*, September-October, 25-32.
- [10] Heath D., R. Jarrow, and A. Morton (1992), "Bond pricing and the term structure of interest rates: A new methodology for contingent claims valuation," *Econometrica*, 60, 77-105.
- [11] Hoyland, A. and M. Rausand (1994), *System Reliability Theory; Models and Statistical Methods*, Wiley, New York.
- [12] Hull, J. and A. White, (1990), "Pricing interest-rate-derivative securities," *The Review of Financial Studies*, 3, 573-592.
- [13] Inui, K. and M. Kijima (1998), "A Markovian framework in multi-factor Heath-Jarrow-Morton models," *Journal of Financial and Quantitative Analysis*, to appear.
- [14] Jarrow, R.A., D. Lando and S.M. Turnbull (1997), "A Markov model for the term structure of credit risk spread," *Review of Financial Studies*, 10, 481-523.
- [15] Jarrow, R.A. and S.M. Turnbull (1995), "Pricing derivatives on financial securities subject to credit risk," *Journal of Finance*, 50, 53-86.
- [16] Kijima, M. (1998), "Monotonicities in a Markov chain model for valuing corporate bonds subject to credit risk," *Mathematical Finance*, to appear.
- [17] Kijima, M. and K. Komoribayashi (1997), "A Markov chain model for valuing credit risk derivatives," Preprint.
- [18] Kijima, M., H. Li and M. Shaked (1998), "Stochastic processes in reliability," in *Stochastic Processes: Theory and Methods*, Rao, C.R. and D.N. Shanbhag Ed., *Handbook in Statistics*, 18, to appear.
- [19] Kijima, M. and I. Nagayama. (1994), "Efficient numerical procedures for the Hull-White extended Vasicek model," *Journal of Financial Engineering*, 3, 275-292.
- [20] Lando, D. (1994), "Three essays on contingent claims pricing," PhD. Thesis, Department of Statistics, Cornell University.

- [21] Moody's Investors Service (1995), "Corporate bond defaults and default rates."
- [22] Ritchken, P. and L. Sankarasubramanian, (1995), "Volatility structures of forward rates and the dynamics of the term structure," *Mathematical Finance*, 5, 55-72.
- [23] Stoyanov, J. (1987), *Counterexamples in Probability*, Wiley, New York.
- [24] Vasicek, O.A. (1977), "An equilibrium characterization of the term structure," *Journal of Financial Economics*, 5, 177-188.

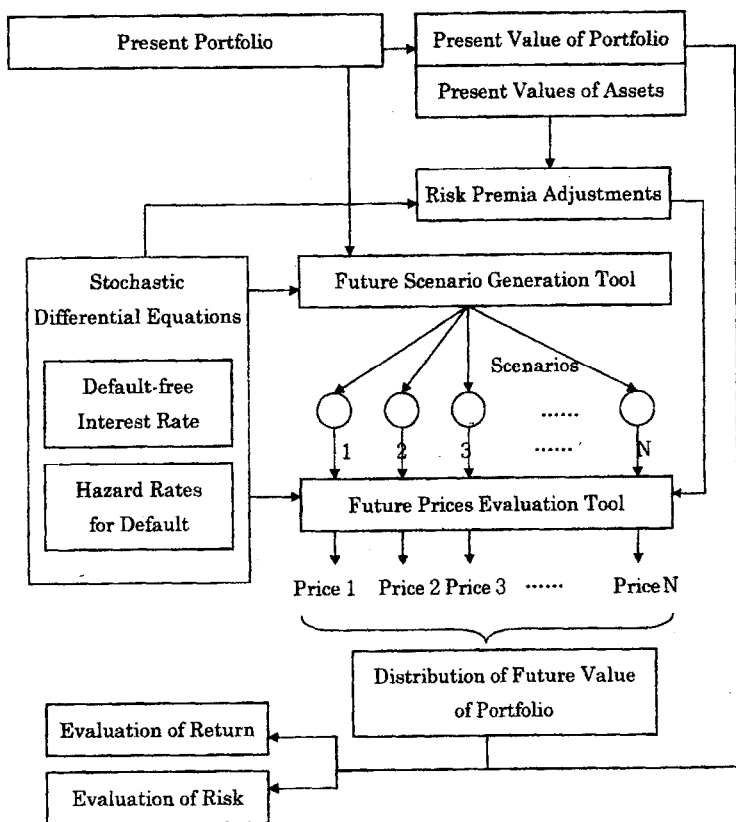


Figure 1. Image of the model structure