STOCHASTIC MODELS FOR ACTUARIAL USE: THE EQUILIBRIUM MODELLING OF LOCAL MARKETS

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March 2007

Submission for the 17th International AFIR Colloquium to be held in Stockholm, Sweden, 12th-15th June 2007

ABSTRACT

In this article, a long-term equilibrium model of a local market is developed. Subject to minor qualifications, the model is arbitrage-free. The variables modelled are the returns on risk-free zero-coupon bonds—both index-linked and conventional—and on equities, as well as the inflation rate. The model is developed in discrete (nominally annual) time, but allowance is made for processes in continuous time subject to continuous rebalancing. It is based on a model of the market portfolio comprising all the above-mentioned asset categories. The risk-free asset is taken to be the one-year index-linked bond. It is assumed that, conditional upon information at the beginning of a year, market participants have homogeneous expectations with regard to the forthcoming year and make their decisions in mean–variance space. For the purposes of illustration, a descriptive version of the model is developed with reference to UK data. The parameters produced by that process may be used to inform the determination of those required for the use of the model as a predictive model. Illustrative results are given.

KEYWORDS

Stochastic investment models; actuarial models; equilibrium models; no-arbitrage models; United Kingdom

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1. INTRODUCTION

1.1 Numerous stochastic models have been developed in the actuarial literature. In these models, the issues of arbitrage and equilibrium are generally not addressed; the models tend to be based on ex-post estimates. This means that they are essentially developed as descriptive models. If such a model is used for predictive purposes, however, it may produce risk-adjusted expected returns that exceed those of the market for some asset categories and understate those of the market for others.

1.2 If a model is to be used to indicate under- or over-priced asset categories, then such a model is called for. For many applications, however, a model is required that will reflect market expectations. These applications include the estimation of fair-value prices of liabilities and the determination of benchmarks for the mandating of investment management and the measurement of investment performance. For the purposes of such applications, a model should be arbitrage-free. It should also arguably be an equilibrium model; that is, it should assume that, at any time, all market participants (including the financial institution concerned) are satisfied with their current exposures to the respective asset categories at current market prices after any adjustments at that time to their exposures and to those prices.

1.3 In this article, a long-term equilibrium model of a local market is developed. Subject to minor qualifications, the model is arbitrage-free. The variables modelled are the returns on risk-free zero-coupon bonds—both index-linked and conventional—and on equities, as well as the inflation rate. The model is developed in discrete (nominally annual) time, but allowance is made for processes in continuous time subject to continuous rebalancing. It is based on a model of the market portfolio comprising all the above-mentioned asset categories combined. That model is used as the basis of development of the arbitrage-free equilibrium model of its constituent asset categories. The risk-free asset is taken to be the one-year index-linked bond. It is assumed that, conditionally upon information at the beginning of a year, market participants have homogeneous expectations with regard to the forthcoming year and make their decisions in mean–variance space.

1.4 The distinction between a descriptive model and a predictive model is drawn in Thomson (2006). In this article, that distinction is used to distinguish between the development and parameterisation of the proposed model for descriptive purposes and its parameterisation for predictive purposes. For the purposes of illustration, a descriptive version of the model is developed with reference to United Kingdom data. The parameters produced by that process may be used to inform the determination of those required for the use of the model as a predictive model. Illustrative results are given.

1.5 Relevant literature is reviewed in section 2. The theory of the equilibrium model is developed in section 3. The theory of the market model—that is, the model of the market portfolio—is developed in section 4. In section 5, a descriptive version of the model is estimated with reference to UK data. Illustrative results of the predictive model are presented in section 6. Conclusions are drawn, and some suggestions for further research are given, in section 7.

2. LITERATURE REVIEW

2.1 There have been a number of publications on the topic of stochastic models of investment returns. The Wilkie (1986) model was the first published stochastic investment return model for actuarial use. That model uses auto-regressive integrated moving average (ARIMA) processes with transfer-function–noise equations (Box & Jenkins, 1976) to model all economic variables. Wilkie postulates that inflation is the independent driving variable and uses a cascade structure (i.e. the output of the model for one variable is used as input for the models for other variables) to model dividends, dividend yields, bank interest rates and yields on Consols. He uses a first-order autoregressive model for inflation. The output of the inflation model is then used as a predictor in the models for equity dividends, dividend yields and yields on Consols. The simulated Consols yield is used as a predictor in the model for the bank interest rates. In a subsequent paper (Wilkie, 1995) he extends the model to include a salary index, short-term interest rates, property rentals and rental yields, as well as yields on index-linked bonds; he also uses the autoregressive conditional heteroscedastic (ARCH) model (e.g. Engle, 1982) for inflation, allowing the volatility parameter to vary over time.

2.2 Huber (1997) and Hibbert, Mowbray & Turnbull (2001) show that the Wilkie model generates inconsistent relationships among inflation, bank interest rates and the yield on Consols. The autoregressive feature of equity returns in the model is shown to produce a distribution over a long period that displays lower volatility than empirical evidence suggests.

2.3 Thomson (1996) proposes a stochastic model of investment returns specifically for South Africa. He likewise proposes a vector ARIMA model. Variables modelled include inflation, money-market interest rates, long-term bond yields, equity dividend growth and dividend yields, direct-property rental growth and rental yields, and dividend growth and dividend yields on property unit trusts. The starting point of his model is the moving-average model of equity dividend growth. Inflation is then driven off that model via a transfer function. The models for long-term bond yields, money-market interest rates and direct-property rental yields are based on the carried-forward effect of past inflation. Property rental growth and dividend yields on property unit trusts are modelled via a transfer function using rental yield as the predictor. Property-trust dividend growth is likewise modelled using a transfer function but using as a predictor the short-term interest rate in excess of the carried-forward effect of inflation.

2.4 Whitten & Thomas (1999) review the stochastic asset model described in Wilkie (1995) and previous work on refining that model. The paper then considers the application of non-linear modelling to investment series, particularly ARCH techniques and threshold modelling. The paper suggests a threshold autoregressive system (Tong, 1990) as an improvement on the Wilkie (1995) model.

2.5 A model called the "TY model" was proposed by Yakoubov, Teeger & Duval (1999). Like Wilkie's, it is a cascade model with price inflation as the main driver variable. As in Wilkie (1995), the ARCH model was used for price inflation. The TY model also projects salary inflation. Other asset classes modelled are UK equities, UK fixed-coupon and index-linked gilts, cash and overseas equities. (The model is UK-based.) Forces of yields and returns are used throughout the model, many key relationships being additive rather than multiplicative. Unlike the Wilkie model, the TY equity model is earnings-based and not dividend-based. The force of the total return on equities is modelled as the sum of the forces of dividend yield (income component of return), earnings growth and change in earnings yield. For overseas equities however, only the total return in sterling terms is modelled.

2.6 All the above models include variously defined short-term and long-term interest rates. In effect, this implies a two-factor model for the term structure of interest rates. A model of the rest of the yield curve can be derived from the realisation of the short-term and long-term interest rates using the technique of principal-components analysis. (See Maitland (2002) for an application of this technique to the interpolation of the South African yield curve.)

2.7 Hibbert, Mowbray & Turnbull (unpublished) propose a model that generates consistent values for the term structure of real and nominal interest rates, inflation rates, equity capital returns and dividend yields. They use a two-factor Hull-White (1990) model for the real interest rates. Inflation is modelled in a similar manner, using a two-factor model. The equity return in excess of the risk-free rate is modelled using a two-state Markov regime-switching lognormal model. One regime has a higher expected return and lower volatility and the other regime a lower expected return with a much higher volatility. A matrix of transition probabilities governs the probability of the process staying in its current regime or switching regimes. A detailed description of the regime-switching lognormal model, as well as parameter estimation, is given by Hardy (2001).

2.8 Affine models are models in which the short-term interest rates can be expressed as (Dai & Singleton, 2000):

$$r(t) = \beta_0 + \sum_{i=1}^{M} \beta_i X_i(t);$$

where:

$X_i(t)$ are random state variables; and

M is the number of random factors driving the interest-rate model.

It has been shown (e.g. Duffie & Kahn, 1996) that affine models yield convenient closed-form expressions for the prices of zero-coupon bonds. For example, the price at time t of a zero coupon maturing at time T is expressed as:

$$P(t,T) = \exp\left(A(t,T) + \sum_{i=1}^{M} B_i(t,T) X_i(t) \right); \qquad (1)$$

where $A(t,T)$ and $B_i(t,T)$ are parameters expressed as functions of t and T whose forms need not concern us here. In other words, the exponent of the price formula is itself an affine function of the state variable. The model used by Hibbert, Mowbray & Turnbull (*op. cit.*) belongs to the affine class.

2.9 None of the above articles investigates the modelling of the market at equilibrium.

3. THE EQUILIBRIUM MODEL

In this section the theory of the arbitrage-free equilibrium model is developed. For the purposes of this section it is assumed that a model of the return during year t on the market portfolio has been developed, which may be expressed in the form:

$$\delta_{M;t} = \mu_{M;t} + \sigma_{M;t}\varepsilon_{M;t};$$

where:

$\mu_{M;t}$ is the expected return during that year, conditional on information at the start of that year;

$\sigma_{M;t}$ is the standard deviation of the return during that year, conditional on information at the start of that year; and
$\varepsilon_{M;t} \sim \mathrm{N}(0,1)$ is such that $\mathrm{cov}(\varepsilon_{M;t}, \varepsilon_{M;s}) = 0$ for $t \neq s$.

The development of this model is deferred to section 4.

3.1 ASSUMPTIONS

3.1.1 We assume that a local market comprises default-free index-linked and conventional zero-coupon bonds and equities. Here 'equity' is used with an extended meaning to include all undated risky capital assets (e.g. fixed property). It also includes foreign equity to the extent to which local investors (i.e. investors with liabilities in the local currency) invest in such equity. On the other hand the market is limited to capital assets in which equilibrium pricing may reasonably be supposed to be taking place. It therefore excludes unmarketable assets. Assets held by foreign investors in local capital are also excluded.

3.1.2 We further assume that market participants have homogeneous expectations and are able to borrow or lend unlimited amounts at the same risk-free return, and that the market is frictionless. At the end of a year, before decisions are made for the following year, the means and variances of factors affecting the average returns on each asset during the forthcoming year are known. (The choice of one-year intervals is arbitrary.) For this purpose, we define the return on an asset during a year as the average instantaneous real rate of return over the year. At the beginning of the year, portfolios are selected by optimisation in mean–variance space so that the market is in equilibrium. Real returns are used because, in the final analysis, equilibrium must relate to commodities, not to currencies. Here the mean and variance are those of the returns during the forthcoming year.

3.1.3 The returns on foreign equities are local real returns; i.e. returns in local currency net of local inflation. The prices of, and the returns on, such equities should be aggregated with local equities, weighted by market capitalisation in the local currency.

3.1.4 Arising from these assumptions, the capital-asset pricing model (CAPM) applies to the local market for a particular year, conditionally upon information and expectations at the end of the previous year.

3.1.5 In this article, a model of the form of equation (1) is used for zero-coupon bonds. Instead of adopting the usual approach of deriving the pricing formula from the process for the short-term interest rate, the reverse is done here. In this way one can allow greater generality in the modelling of the term structure, as well as using the current yield curve as the starting point for simulations.

3.1.6 In this article, a two-factor term-structure model is proposed**.** Although a two-factor model may adequately capture the volatility of the yield curve (e.g. Maitland, 2002), it suffers from the problem of not being able to mimic the correlation between the forward rates of different maturities. In particular, a two-factor model will over-estimate the correlation between forward rates for neighbouring maturities and under-estimate the correlation between forward rates with maturity dates far apart (Rebonato, 1998). This matter is further discussed in section 7.

3.2 INDEX-LINKED BONDS

3.2.1 Suppose that the price at time $t = 0,\ldots, T$ of an index-linked bond maturing at time $t + s$ is:

$$P_{I;t}(s) = \exp\left\{-Y_{I;t}(s)\right\}\ ; \tag{2}$$

where:

$$Y_{I;t}(s) = -\ln\left\{P_{I;t}(s)\right\} = f_{I;t}(s)\left\{1 + b_{I,1}(s)\eta_{1,t} + b_{I,2}(s)\eta_{2,t}\right\} \tag{3}$$

T is the time horizon to which projections will be required;

for $j = 1, 2$:

$$\eta_{j,t} = \sum_{i=1}^{6} a_{i,j}\varepsilon_{i,t}\ ; \text{ and} \tag{4}$$

$$\sum_{i=1}^{6} a_{i,j} = \sqrt{6}\ ; \tag{5}$$

$$\varepsilon_{i,t} \sim \mathrm{N}(0,1)\ ; \text{ and} \tag{6}$$

$$\mathrm{cov}\left(\varepsilon_{i,t}, \varepsilon_{k,t}\right) = 0 \text{ for } i \neq k\ . \tag{7}$$

Here, as elsewhere, b denotes a parameter of the model, which may be a function of s, but does not vary over time t; f denotes a parameter, which is a function of s, and varies over time t, but is known at time $t - 1$. The reason for the six dimensions referred to in equations (4) and (5) becomes apparent below. The various parameters are distinguished by their subscripts. The dependence of the parameter $f_{I;t}(s)$ on t is explained in ¶3.2.3

below. From (2) it follows that the return on that bond during year t —i.e. the interval $(t-1,t]$—is:

$$\delta_{I;t}(s) = \ln \frac{P_{I;t}(s)}{P_{I;t-1}(s+1)}$$
$$= Y_{I;t-1}(s+1) - Y_{I;t}(s). \tag{8}$$

The expected return is:

$$\mu_{I;t}(s) = Y_{I;t-1}(s+1) - f_{I;t}(s). \tag{9}$$

Thus, from (3):

$$\delta_{I;t}(s) = \mu_{I;t}(s) - f_{I;t}(s)\left\{b_{I,1}(s)\eta_{1,t} + b_{I,2}(s)\eta_{2,t}\right\}. \tag{10}$$

3.2.2 Since $P_{I;t}(0) \equiv 1$:

$$f_{I;t}(0) = 0 .$$

Also, without loss of generality:

$$b_{I;1}(0) = b_{I;2}(0) = 0 .$$

Therefore, from (8), the risk-free return for year t is:

$$\delta_{I;t}(0) = Y_{I;t-1}(1). \tag{11}$$

3.2.3 From (3) it may be seen that:

$$E\left\{Y_{I;t}(s)\right\} = f_{I;t}(s).$$

Also:

$$\frac{Y_{I;t}(s)}{s}$$

represents the yield curve at time t; and

$$\frac{f_{I;t}(s)}{s} \tag{12}$$

represents the expected yield curve at time t, conditional on information at time t – 1. Without loss of generality, $\eta_{1,t}$ and $\eta_{2,t}$ may be taken as the drivers of the short rate (s = 1) and a suitable long rate (say $s = \tau$) respectively so that:

$$\tfrac{1}{\tau} b_{I,1}(\tau) = 0 \text{ ; and}$$
$$b_{I,2}(1) = 0 .$$
(13)

3.2.4 It would be possible to define $Y_{I;t}(s)$ as:

$$Y_{I;t}(s) = -\ln\left\{P_{I;t}(s)\right\} = f_{I;t}(s) + b_{I,1}(s)\eta_{1,t} + b_{I,2}(s)\eta_{2,t} .$$

This would mean that the conditional volatility of real spot yields would be independent of the level of those yields. However, it would permit negative real yields. This in turn would allow arbitrage between index-linked bonds and commodities. If a real spot yield drops below zero, an investor can (in principle) earn a risk-free profit by shorting index-linked bonds and holding the goods (and rights to the services) comprising the index over the period to redemption. In order to ensure that the model does not allow such arbitrage, it is necessary to avoid negative yields on index-linked bonds. This is done by making the conditional volatility of real yields during each year proportionate to the conditional expected yields at the year-end (conditional, that is, on information at the beginning of the year). Because of the use of discrete time, this does not entirely avoid the possibility of negative yields, but it does reduce the probability of such yields.

3.3 INFLATION

The average instantaneous rate of inflation during year t is modelled as:

$$\gamma_t = \mu_{\gamma;t} + b_\gamma \eta_{3,t} ; \tag{14}$$

where:

$$\eta_{3,t} = \sum_{i=1}^{6} a_{i,3}\varepsilon_{i,t} \text{ ; and} \tag{15}$$

$$\sum_{i=1}^{6} a_{i,3} = \sqrt{6} . \tag{16}$$

The determination of $\mu_{\gamma;t}$ is explained in ¶3.4.2 below.

3.4 CONVENTIONAL BONDS

3.4.1 Suppose that the price at time t of a conventional bond maturing at time n is:

$$P_{C;t}(s) = \exp\left\{-Y_{C,0;t}(s)\right\} ; \tag{17}$$

where:

$$Y_{C;t}(s) = \ln\left\{P_{C;t}(s)\right\} = f_{C;t}(s)\left\{1 + b_{C,1}(s)\eta_{4,t} + b_{C,2}(s)\eta_{5,t}\right\}; \tag{18}$$

and, for j = 4, 5:

$$\eta_{j,t} = \sum_{i=1}^{6} a_{i,j}\varepsilon_{i,t} \text{ ; and} \tag{19}$$

$$\sum_{i=1}^{6} a_{i,j} = \sqrt{6}\,. \tag{20}$$

Then the return on that bond during year t is:

$$\delta_{C;t}(s) = Y_{C;t-1}(s+1) - Y_{C;t}(s) - \gamma_t\,; \tag{21}$$

and the expected return is:

$$\mu_{C;t}(s) = Y_{C;t-1}(s+1) - f_{C;t}(s) - \mu_{\gamma;t}\,. \tag{22}$$

Thus, from (14) and (18):

$$\delta_{C;t}(s) = \mu_{C;t}(s) - b_\gamma\eta_{3,t} - f_{C;t}(s)\left\{b_{C,1}(s)\eta_{4,t} - b_{C,2}(s)\eta_{5,t}\right\}. \tag{23}$$

As for index-linked bonds:

$$f_{C;t}(0) = b_{C,1}(0) = b_{C,2}(0) = b_{C,2}(1) = 0\,.$$

3.4.2 Suppose that the inflation risk premium

$$\phi_t = \mu_{C;t}(0) - \mu_{I;t}(0) \tag{24}$$

is constant, so that:

$$\phi_t = \phi \text{ for all } t.$$

From (11):

$$\begin{aligned}\mu_{I;t}(0) &= \delta_{I;t}(0)\\ &= Y_{I;t-1}(1);\end{aligned}$$

and from (22):

$$\mu_{C;t}(0) = Y_{C;t-1}(1) - \mu_{\gamma;t}\,.$$

Substituting these values into (24) gives:

$$\phi = Y_{C;t-1}(1) - \mu_{\gamma;t} - Y_{I;t-1}(1)\,;$$

i.e.:

$$\mu_{\gamma;t} = Y_{C;t-1}(1) - Y_{I;t-1}(1) - \phi \,. \tag{25}$$

3.4.3 As in the case of index-linked bonds (cf. ¶3.2.4), it would be possible to define the price of conventional bonds so that the conditional volatility of spot yields would be independent of the level of those yields. However, in this case, it would permit negative nominal yields. This in turn would allow arbitrage between index-linked bonds and commodity prices. If a nominal spot yield drops below zero, an investor can (in principle) earn a risk-free profit by shorting conventional bonds and holding cash over the period to redemption. Again, it is necessary to avoid negative yields, and similar observations apply.

3.5 EQUITIES

Suppose that the price of equities at time t is:

$$P_{E;t} = P_{E;t-1} \exp\left(\mu_{E;t} + b_{E,1}\eta_{6,t} \right);$$

where:

$\mu_{E;t}$ is the expected return;

$$\eta_{6,t} = \sum_{i=1}^{6} a_{i,6}\varepsilon_{i,t} \text{ ; and} \tag{26}$$

$$\sum_{i=1}^{6} a_{i,6} = \sqrt{6} \,. \tag{27}$$

Then the return on equities during year t is:

$$\delta_{E;t} = \mu_{E;t} + b_{E,1}\eta_{6,t} \,. \tag{28}$$

3.6 NOTIONAL RISKY ASSETS

3.6.1 If there are 6 risky assets in a market and an investor maintains constant exposure w_i (at market prices) to asset i during a year then, if all income is reinvested when paid, the total return is:

$$\sum_{i=1}^{6} w_i \delta_i \text{ ;}$$

where δ_i is the average return on asset i during that year.

3.6.2 Consider a set of 6 notional risky assets, whose return during year t is:

$$\delta_{i,t} = c + d\varepsilon_{i,t} \text{ for } i = 1,\ldots, 6; \tag{29}$$

where $\varepsilon_{i,t}$ is as defined in ¶3.2.1.

3.6.3 Now $\eta_{j,t}$ being a linear function of $\varepsilon_{i,t}$, and $\delta_{I;t}(s)$, $\delta_{C;t}(s)$ and $\delta_{E;t}$ being linear functions of $\eta_{j,t}$, it follows from ¶3.6.1 that any portfolio of index-linked bonds, conventional bonds and equities may be constructed from a portfolio of notional risky assets, and vice versa, with constant exposure to the constituents of the respective portfolios during year t. Furthermore, the decomposition of any portfolio of actual assets into the corresponding portfolio of notional assets constitutes a no-arbitrage hedging strategy, since the returns on the corresponding portfolio will be identical. This means that, as between the asset categories modelled, the model developed in this article is arbitrage-free.

3.6.4 In mean–variance space, as shown in Appendix A, the equilibrium market portfolio will reduce to equal exposure to each of these assets. The return on the market portfolio is thus:

$$\delta_{M;t} = \frac{1}{6}\sum_{i=1}^{6}\delta_{i,t}\ ; \tag{30}$$

and hence:

$$c = \mu_{M;t}\ ; \tag{31}$$

i.e. the expected return on the market portfolio. The variance of the return on that portfolio is:

$$\begin{aligned}\sigma_{M;t}^2 &= \mathrm{var}_{t-1}\left\{\frac{d}{6}\sum_{i=1}^{6}\varepsilon_{i,t}\right\}\\ &= \frac{d^2}{6^2}\sum_{i=1}^{6}\mathrm{var}_{t-1}\left\{\varepsilon_{i,t}\right\}\ \text{(since }\mathrm{cov}\left\{\varepsilon_{i,t},\varepsilon_{j,t}\right\}=0\text{ for } i\neq j\text{)}\\ &= \frac{d^2}{6};\end{aligned}$$

so that:

$$d = \sqrt{6}\sigma_{M;t}\,. \tag{32}$$

3.6.5 Substituting (31) and (32) into (29), we obtain:

$$\delta_{i,t} = \mu_{M;t} + \sqrt{6}\sigma_{M;t}\varepsilon_{i,t}\ ;$$

and hence, from (30):

$$\delta_{M;t} = \mu_{M;t} + \sigma_{M;t}\lambda_t\,; \tag{33}$$

where:

$$\lambda_t = \frac{1}{\sqrt{6}}\sum_{i=1}^{6}\varepsilon_{i,t}\,. \tag{34}$$

3.6.6 From (4), (15), (19) and (26):

$$\eta_{j,t} = \sum_{i=1}^{6} a_{i,j}\varepsilon_{i,t}\,; \tag{35}$$

where, from (5), (16), (20) and (27):

$$\sum_{i=1}^{6} a_{i,j} = \sqrt{6}\,. \tag{36}$$

3.6.7 Now let:

$$\sigma_{\eta_j,M;t} = \mathrm{cov}_{t-1}\left\{\eta_{j,t}, \delta_{M;t}\right\}.$$

Then, from (33) and (35):

$$\sigma_{\eta_j,M;t} = \mathrm{cov}_{t-1}\left\{\sum_{i=1}^{6} a_{i,j}\varepsilon_{i,t}, \sigma_{M;t}\lambda_t\right\};$$

and, from (34), (7), (6) and (36) respectively:

$$\begin{aligned}
\sigma_{\eta_j,M;t} &= \mathrm{cov}_{t-1}\left\{\sum_{i=1}^{6} a_{i,j}\varepsilon_{i,t}, \frac{\sigma_{M;t}}{\sqrt{6}}\sum_{i=1}^{6}\varepsilon_{i,t}\right\} \\
&= \frac{\sigma_{M;t}}{\sqrt{6}}\sum_{i=1}^{6} a_{i,j}\ \mathrm{var}_{t-1}\left\{\varepsilon_{i,t}\right\} \\
&= \frac{\sigma_{M;t}}{\sqrt{6}}\sum_{i=1}^{6} a_{i,j} \\
&= \sigma_{M;t}.
\end{aligned} \tag{37}$$

3.7 DEVELOPMENT OF THE EQUILIBRIUM MODEL

3.7.1 In order for an asset $X \in \{(I;t,s),(C;t,s),(E;t)\}$ to satisfy the CAPM during year t, we require that:

$$\mu_X = \delta_{I;t}(0) + k_t \sigma_{X,M}\,; \tag{38}$$

where:

$$k_t = \frac{\mu_{M;t} - \delta_{I;t}(0)}{\sigma^2_{M;t}}\,. \tag{39}$$

3.7.2 For a given model of the return on the market portfolio in year t, (39) may be used to determine k_t. For each asset category, given the covariance of its return with that of the market, (38) may then be used to determine its expected return.

3.7.3 In particular, for each index-linked bond:

$$\mu_{I;t}(s) = \delta_{I;t}(0) + k_t \sigma_{I,M;t}(s)\,; \tag{40}$$

where, from (10) and (37):

$$\begin{aligned} \sigma_{I,M;t}(s) &= \operatorname{cov}_{t-1}\left[\delta_{I;t}(s), \delta_{M;t}\right] \\ &= \operatorname{cov}_{t-1}\left[-f_{i;t}(s)\left\{b_{I,1}(s)\eta_{1,t} + b_{I,2}(s)\eta_{2,t}\right\}, \delta_{M;t}\right] \\ &= -\sigma_{M;t} f_{i;t}(s)\left\{b_{I,1}(s) + b_{I,2}(s)\right\}. \end{aligned} \tag{41}$$

3.7.4 Making $f_{I;t}(s)$ the subject of equation (9), we have, for s < τ:

$$f_{I;t}(s) = Y_{I;t-1}(s+1) - \mu_{I;t}(s)\,. \tag{42}$$

Substituting (41) into (40), we have:

$$\mu_{I;t}(s) = \delta_{I;t}(0) - k_t \sigma_{M;t} f_{i;t}(s)\left\{b_{I,1}(s) + b_{I,2}(s)\right\}.$$

Substituting this into (42) we then obtain:

$$f_{I;t}(s) = Y_{I;t-1}(s+1) - \delta_{I;t}(0) + k_t \sigma_{M;t} f_{i;t}(s)\left\{b_{I,1}(s) + b_{I,2}(s)\right\};$$

and solving for $f_{I;t}(s)$:

$$f_{I;t}(s) = \frac{Y_{I;t-1}(s+1) - \delta_{I;t}(0)}{1 - k_t \sigma_{M;t}\left\{b_{I,1}(s) + b_{I,2}(s)\right\}}\,. \tag{43}$$

3.7.5 In order to obtain the full yield curve, equation (43) will need to be evaluated for all values of s. A problem arises in the determination of $f_{I;t}(s)$ for the last point of the yield curve ($s = \tau$), where $b_{I,1}(\tau + 1)$ and $b_{I,2}(\tau + 1)$ are not defined. An assumption is required about the behaviour of the yield curve beyond τ. For the sake of simplicity it is assumed that, at any time t, the one-year forward rate for maturity at time $t+\tau$ is equal to the equivalent forward rate for maturity at time $t+\tau-1$; i.e.:

$$\frac{P_{I,t-1}(\tau)}{P_{I,t-1}(\tau+1)}=\frac{P_{I,t-1}(\tau-1)}{P_{I,t-1}(\tau)};$$

so that:

$$Y_{I;t-1}(\tau+1)=2Y_{I;t-1}(\tau)-Y_{I;t-1}(\tau-1)\,. \tag{44}$$

3.7.6 From (8) and (44) we have:

$$\begin{aligned}\delta_{I,t}(\tau)&=Y_{I;t-1}(\tau+1)-Y_{I;t}(\tau)\\&=2Y_{I,t-1}(\tau)-Y_{I,t}(\tau)-Y_{I,t-1}(\tau-1).\end{aligned}$$

Taking expectations on both sides and rearranging, we get:

$$f_{I;t}\left(\tau\right)=2Y_{I;t-1}(\tau)-Y_{I;t-1}(\tau-1)-\mu_{I,t}(\tau) \tag{45}$$

As above, since $b_{I,1}(\tau)=0$, we then obtain:

$$f_{I;t}(\tau)=\frac{2Y_{I;t-1}(\tau)-Y_{I;t-1}(\tau-1)-\delta_{I;t}(0)}{1-k_t\sigma_{M;t}b_{I,2}(\tau)}\,. \tag{46}$$

3.7.7 Similarly (i.e. as in ¶3.7.3), we require that, for each conventional bond:

$$\mu_{C;t}(s)=\delta_{I;t}(0)+k_t\sigma_{C,M;t}(s)\,; \tag{47}$$

where, from (15), (23) and (37):

$$\begin{aligned}\sigma_{C,M;t}(s)&=\text{cov}_{t-1}\left\{\delta_{C;t}(s),\delta_{M;t}\right\}\\&=\text{cov}_{t-1}\left\{-b_\gamma\eta_{3,t}-f_{C;t}(s)\left\{b_{C,1}(s)\eta_{4,t}+b_{C,2}(s)\eta_{5,t}\right\},\delta_{M;t}\right\}\\&=-\sigma_{M;t}\left[b_\gamma+f_{C;t}(s)\left\{b_{C,1}(s)+b_{C,2}(s)\right\}\right].\end{aligned} \tag{48}$$

3.7.8 Making $f_{C;t}(s)$ the subject of equation (22), we have, for s < τ:

$$f_{C;t}(s)=Y_{C;t-1}(s+1)-\mu_{\gamma,t}-\mu_{C;t}(s)\,; \tag{49}$$

Substituting (48) into (47), we have:

$$\mu_{C;t}(s) = \delta_{I;t}(0) - k_t \sigma_{M;t} \left[b_\gamma + f_{C;t}(s) \left\{ b_{C,1}(s) + b_{C,2}(s) \right\} \right].$$

Substituting this into (49) we then obtain:

$$f_{C;t}(s) = Y_{C;t-1}(s+1) - \mu_{\gamma;t} - \delta_{I;t}(0) + k_t \sigma_{M;t} \left[b_\gamma + f_{C;t}(s) \left\{ b_{C,1}(s) + b_{C,2}(s) \right\} \right];$$

and solving for $f_{C;t}(s)$:

$$f_{C;t}(s) = \frac{Y_{C;t-1}(s+1) - \mu_{\gamma,t} - \delta_{I;t}(0) + k_t \sigma_{M;t} b_\gamma}{1 - k_t \sigma_{M;t} \left\{ b_{C,1}(s) + b_{C,2}(s) \right\}}. \tag{50}$$

As for index-linked bonds:

$$f_{C;t}(\tau) = \frac{2Y_{C;t-1}(\tau) - Y_{C;t-1}(\tau - 1) - \mu_{\gamma;t} - \delta_{I;t}(0) + k_t \sigma_{M;t} b_\gamma}{1 - k_t \sigma_{M;t} b_{C,2}(\tau)} \tag{51}$$

3.7.9 For inflation, from (14) and (37):

$$\begin{aligned} \sigma_{\gamma,M;t} &= \operatorname{cov}_{t-1}\left\{ \gamma_t, \delta_{M;t} \right\} \\ &= \operatorname{cov}_{t-1}\left\{ b_\gamma \eta_{3,t}, \delta_{M;t} \right\} \\ &= b_\gamma \sigma_{M;t}. \end{aligned} \tag{52}$$

3.7.10 Finally, for equities:

$$\mu_{E;t} = \delta_{I;t}(0) + k_t \sigma_{E,M;t}; \tag{53}$$

where, from (28) and (37):

$$\begin{aligned} \sigma_{E,M;t} &= \operatorname{cov}_{t-1}\left\{ \delta_{E;t}, \delta_{M;t} \right\} \\ &= \operatorname{cov}_{t-1}\left\{ b_{E,1} \eta_{6,t}, \delta_{M;t} \right\} \\ &= b_{E,1} \sigma_{M;t}. \end{aligned} \tag{54}$$

3.8 SUMMARY OF THE EQUILIBRIUM MODEL

3.8.1 The parameters required are as follows:

- for $s = 1, \ldots, \tau$:

  $Y_{I;0}(s)$ and $Y_{C;0}(s)$; and

$b_{I,j}(s)$ and $b_{C,j}(s)$ for j = 1, 2; and

- b_γ;
- $b_{E,1}$; and
- for i, j = 1,…, 6:

  $a_{i,j}$.

3.8.2 For t = 1 we then determine the variables $\mu_{M;t}$ and $\sigma_{M;t}$, using the market model. Also:

$$\delta_{I;t}(0) = Y_{I;t-1}(1) \text{ (equation (11)).}$$

3.8.3 Using Monte Carlo methods we then simulate pseudorandom standard normal variables:

$$\varepsilon_{i,t} \text{ for } i = 1,\ldots, 6.$$

3.8.4 From the above values we calculate:

$$k_t = \frac{\mu_{M;t} - \delta_{I;t}(0)}{\sigma^2_{M;t}} \text{ (equation (39));}$$

for j = 1, …, 6:

$$\eta_{j,t} = \sum_{i=1}^{6} a_{i,j}\varepsilon_{i,t} \text{ (equation (35));}$$

$$\mu_{\gamma;t} = Y_{C;t-1}(1) - Y_{I;t-1}(1) - \phi \text{ (equation (25)}$$

$$\gamma_t = \mu_{\gamma;t} + b_\gamma \eta_{3,t} \text{ (equation (14));}$$

for s = 1, …, $\tau - 1$:

$$f_{I;t}(s) = \frac{Y_{I;t-1}(s+1) - \delta_{I;t}(0)}{1 - k_t\sigma_{M;t}\left\{b_{I,1}(s) + b_{I,2}(s)\right\}} \text{ (equation (43));}$$

$$f_{C;t}(s) = \frac{Y_{C;t-1}(s+1) - \mu_{\gamma,t} - \delta_{I;t}(0) + k_t\sigma_{M;t}b_\gamma}{1 - k_t\sigma_{M;t}\left\{b_{C,1}(s) + b_{C,2}(s)\right\}} \text{(equation (50));}$$

$$f_{I;t}(\tau) = \frac{2Y_{I;t-1}(\tau) - Y_{I;t-1}(\tau-1) - \delta_{I;t}(0)}{1 - k_t\sigma_{M;t}b_{I,2}(\tau)} \text{ (equation (46));}$$

$$f_{C;t}(\tau) = \frac{2Y_{C;t-1}(\tau) - Y_{C;t-1}(\tau-1) - \mu_{\gamma;t} - \delta_{I;t}(0) + k_t\sigma_{M;t}b_\gamma}{1 - k_t\sigma_{M;t}b_{C,2}(\tau)} \text{ (equation (51));}$$

for $s = 1, \ldots, \tau$:

$\sigma_{I,M;t}(s) = -\sigma_{M;t} f_{i;t}(s)\left\{b_{I,1}(s) + b_{I,2}(s)\right\}$ (equation (41));

$\sigma_{C,M;t}(s) = -\sigma_{M;t}\left[b_\gamma + f_{C;t}(s)\left\{b_{C,1}(s) + b_{C,2}(s)\right\}\right]$ (equation (48);

$\mu_{I;t}(s) = \delta_{I;t}(0) + k_t \sigma_{I,M;t}(s)$ (equation (40));

$\mu_{C;t}(s) = \delta_{I;t}(0) + k_t \sigma_{C,M;t}(s)$ (equation (47));

$\delta_{I;t}(s) = \mu_{I;t}(s) - f_{I;t}(s)\left\{b_{I,1}(s)\eta_{1,t} + b_{I,2}(s)\eta_{2,t}\right\}$ (equation (10));

$\delta_{C;t}(s) = \mu_{C;t}(s) - b_\gamma \eta_{3,t} - f_{C;t}(s)\left\{b_{C,1}(s)\eta_{4,t} + b_{C,2}(s)\eta_{5,t}\right\}$ (equation (23));

$\sigma_{E,M;t} = b_{E,1}\sigma_{M;t}$ (equation (54));

$\mu_{E;t} = \delta_{I;t}(0) + k_t \sigma_{E,M;t}$ (equation (53));

$\delta_{E;t} = \mu_{E;t} + b_{E,1}\eta_{6,t}$ (equation (28));

Finally, for $t < T$, we calculate:

for $s = 1, \ldots, \tau$:

$Y_{I;t}(s) = Y_{I;t-1}(s+1) - \delta_{I;t}(s)$ (from equation (8)); and

$Y_{C;t}(s) = Y_{C;t-1}(s+1) - \delta_{C;t}(s) - \gamma_t$ (from equation (21)).

3.8.5 The calculations in ¶¶3.8.2–4 are repeated for $t = 2, \ldots, T$.

3.9 EXTENSION TO OTHER ASSETS

3.9.1 Foreign bonds have not been included in the general model. However, their inclusion would be quite straightforward. They may be treated in a similar way to conventional bonds. But in addition to allowing for the erosion of value due to inflation, the model would have to allow, in a similar manner, for erosion of value due to exchange rates and (in the case of index-linked bonds) differences in inflation rates.

3.9.2 Credit risk on bonds may be treated as part of equity. While returns on credit risk are notoriously skew, they are offset by their effects on equity returns. If credit risk is included in equities, it may be expected that the skewness in equity returns will be reduced. Warrants should be similarly treated.

3.9.3 Derivative instruments and products issued by financial institutions should not be included. Only capital assets issued in the primary market to cover real investments in the economy should be included. For all other assets there are equal and opposite counterparties, whose holdings offset each other. The model may be used to price such instruments as described, for example, in Thomson (2005), but that is beyond the scope of this paper.

4. MARKET MODELS

4.1 In the equilibrium model, no consideration is given to the processes governing the variables $\mu_{M;t}$ and $\sigma_{M;t}$.

4.2 Depending on the local market, these variables can be treated as constants, or they can be modelled using either a regime-switching model or a vector autoregressive moving-average (VARMA) model. It would also be possible to treat some of these variables in one of these ways and the others in the other ways. The choice is largely a matter of optimal descriptive value. In order to optimise the trade-off between parsimony and fidelity, a criterion such as the Akaike information criterion (Aikake, 1974) may be used, both in the parameterisation of the models and in choosing between them. Because of the basis of such criteria, the choice may well be affected by the amount of data available.

4.3 Because regime-switching models may produce high kurtosis, they are not generally associated with decision-making in mean–variance space. The approach adopted in this paper would accommodate regime switches at year-ends. In effect, it would assume that there would be no price changes as a result of such a switch, though it would generally result in changes in the constitution of the market portfolio. As regards the following year's returns, homogeneous expectations at the start of that year include knowledge of the current regime. While these constraints constitute a degree of idealisation of the concept of regime-switching, they nevertheless allow for the effects of regime-switching to be accommodated. High kurtosis will not be observed cross-sectionally over an individual year, but it may appear longitudinally over a sequence of consecutive years.

4.4 On the other hand, because of their non-Markov characteristics, VARMA models are not generally associated with equilibrium. However, because equilibrium is conditional on information at the beginning of each year, and because the vector modelled (with components $\mu_{M;t}$ and $\sigma_{M;t}$) does not preclude such equilibrium, the general model described in this paper accommodates VARMA modelling where that is optimal.

4.5 The stochastic modelling of the volatility $\sigma_{M;t}$ accommodates models of autoregressive conditional heteroscedasticity. As in the case of regime switching, this allows for high kurtosis. For the sake of simplicity, it is assumed in this article that the ex-ante volatility is constant, denoted σ_M .

4.6 It is not possible to assume that $\mu_{M;t}$ is constant, otherwise whenever $\delta_{I,t}(0) > \mu_M$, we have $k_t < 0$, which means that the market price of risk is negative. In order to address this problem, the expected return on the market portfolio is expressed as a function of the risk-free return as follows:

$$\begin{aligned} \mu_{M,t} &= g\delta_{I,t}(0) + h \text{ for } \delta_{I,t}(0) > 0 \\ &= \delta_{I,t}(0) \text{ otherwise.} \end{aligned} \tag{55}$$

4.7 For $\delta_{I,t}(0) > 0$, equation (55) is justified on the grounds that the risk premium

$$\pi_t = \frac{\mu_{M,t} - \delta_{I,t}(0)}{\sigma_{M,t}} \tag{56}$$

is positive, though it may vary according to the level of $\delta_{I,t}(0)$. Substituting (55) into (56) and rearranging, we have:

$$(g-1)\delta_{I,t}(0) = \pi_t \sigma_{M,t} - h\,.$$

In general, since the sensitivity of the volatility of the return on the market to the risk-free return may be expected to be positive, it may be expected that $g > 1$. For $\delta_{I,t}(0) \le 0$, this does not apply; under such circumstances it is effectively being assumed that the risk premium is zero. While this is an arbitrary assumption, it is unlikely to apply frequently.

4.8 The exploration of alternative market models is left to further research.

5. DESCRIPTIVE ESTIMATION OF THE MODEL

5.1 The method of determination of the model parameters is explained in Appendix B for the purpose of the estimation of a descriptive version of the model. In this section the results of the descriptive estimation of the parameters are presented.
5.2 The historical data required include, for each year:

- the zero-coupon yield (conventional and index-linked) for each maturity modelled (in this case from 1 to 30 years at yearly intervals);
- the total return on equities;
- the inflation rate; and
- the composition of the market portfolio.

5.3 The composition of the market portfolio is represented by the split of the total investment market capitalisation into equity (as defined in this paper) and conventional and index-linked bonds. For this purpose, the market capitalisation of the bonds must be split by term to maturity. Since the bonds being modelled are zero-coupon bonds, each traded bond is decomposed into a series of zero-coupon bonds, which are aggregated by maturity date into annual buckets.

5.4 In ¶3.1.1, 'equity' is a broader asset class that includes all risky capital assets except bonds. However, for the purposes of the illustrative estimation of the descriptive model, the total market capitalisation of the FTSE All-Share Index was taken as a proxy for the market capitalisation of equities.

5.5 The yields on conventional bonds are obtained from the zero-coupon yield curves published by the UK Debt Management Office (DMO). These are denoted as *CONV*01,…,*CONV*30. The history of these yields is obtained from 31 December 1979 to 31 December 2006 at yearly intervals.

5.6 The yields on index-linked bonds were likewise obtained from zero-coupon yield curves published by DMO. The index-linked zero-coupon bond yields for maturities 1,…, 30 years are denoted as *ILB*01,…, *ILB*30. These were available at yearly intervals from 31 December 1985 to 31 December 2006.

5.7 Figure 1 shows the yield curves of conventional and index-linked bonds as at 31 December 2006.

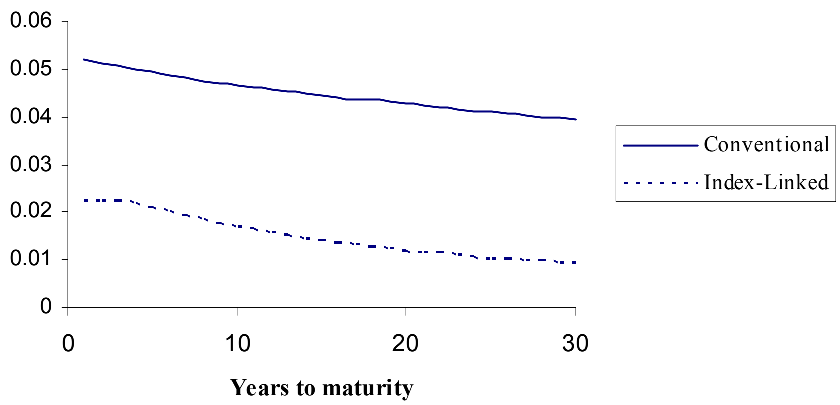
5.8 Historical inflation figures were derived as:

$$\overset{\circ}{\gamma}_t = \ln \frac{RPI_t}{RPI_{t-1}}$$

where RPI_t is the value of the UK retail prices index at the end of year t[1].

[1] Data supplied By Professor A D Wilkie, InQA Limited

Figure 1. Yield curves of conventional and index-linked bonds as at 31 December 2006



5.9 Historical equity returns are derived from the FTSE All-Share total-return index as follows:

$$\overset{\circ}{\delta}_{E,t} = \ln \frac{TRI_t}{TRI_{t-1}};$$

where TRI_t is the value of the relevant equity index at the end of year t[2].

5.10 Market capitalisations for bond markets are available only from 31 December 1998 onwards[3]. It was assumed that the split of total market capitalisation between equities and bonds prior to 1998 was the same as at 31 December 1998.

5.11 Since the yield curves for index-linked bonds were available only since 31 December 1985, it was assumed that the market capitalisation of those bonds was zero before that date. Since the market capitalisation of index-linked bonds is small compared with that of conventional bonds and equities, this is not expected to skew results significantly.

5.12 Table C.1 in Appendix C summarises the historical information used in the descriptive estimation of the model parameters. Table C.2 in that appendix shows the return on the market portfolio and the risk-free rate.

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[2] Source: Communication from info@ftse.com

[3] Source: www.dmo.gov.uk

5.13 For the purposes of estimating $\mu_{M;t}$, in terms of equation (55), a linear regression was carried out on the data shown in Table C.2; for this purpose, the risk-free return for the years prior to 1986 was calculated using the simplifying assumption that:

$$\delta_{I,t}\left(0\right)=\delta_{C,t}\left(0\right)-\phi\,.$$

It was found that the intercept constant h was not significant at the 95% level. Fixing the intercept at zero, we obtain an estimate of g = 1.833(p-value = 0.008).

5.14 The estimated parameters of the yields on zero-coupon conventional and index-linked bonds are shown in Table 1. The other model parameters were estimated as follows:
- $b_\gamma = -0.0083$;
- $b_{E,1} = 0.0685$ and
- $\sigma_M = 0.12026$.

The inflation risk premium (ϕ) was fixed at the arbitrary value of 0.3% per annum. Further research is required on the reliable estimation of the inflation risk premium.

5.15 The covariance matrix of η_{jt} and the coefficients a_{ij} (see Tables 2 and 3 respectively) were determined as described in Appendix B.

Table 1. Estimated parameters of the model for conventional and index-linked bonds

s	$Y_{I,0}$(s)	$b_{I,1}$(s)	$b_{I,2}$(s)	$Y_{C,0}$(s)	$b_{C,1}$(s)	$b_{C,2}$(s)
1	0.0220	0.1302	0.0000	0.0520	-0.1208	0.0000
2	0.0441	0.1300	-0.0001	0.1026	-0.1140	-0.0236
3	0.0661	0.1226	0.0192	0.1518	-0.1072	-0.0357
4	0.0868	0.0834	0.0253	0.2001	-0.0861	-0.0422
5	0.1047	0.0862	0.0502	0.2472	-0.0890	-0.0484
6	0.1205	0.0733	0.0571	0.2929	-0.0799	-0.0525
7	0.1346	0.0633	0.0609	0.3373	-0.0712	-0.0560
8	0.1472	0.0554	0.0628	0.3807	-0.0631	-0.0593
9	0.1587	0.0491	0.0636	0.4234	-0.0555	-0.0624
10	0.1691	0.0439	0.0637	0.4655	-0.0486	-0.0654
11	0.1785	0.0395	0.0635	0.5071	-0.0421	-0.0683
12	0.1872	0.0357	0.0632	0.5483	-0.0361	-0.0711
13	0.1951	0.0324	0.0629	0.5889	-0.0307	-0.0738
14	0.2024	0.0294	0.0627	0.6290	-0.0257	-0.0764
15	0.2092	0.0266	0.0627	0.6683	-0.0213	-0.0789
16	0.2154	0.0241	0.0627	0.7070	-0.0174	-0.0813
17	0.2211	0.0218	0.0630	0.7449	-0.0140	-0.0835
18	0.2265	0.0196	0.0634	0.7821	-0.0112	-0.0856
19	0.2315	0.0175	0.0639	0.8185	-0.0089	-0.0875
20	0.2362	0.0155	0.0646	0.8542	-0.0070	-0.0892
21	0.2407	0.0136	0.0655	0.8892	-0.0054	-0.0908
22	0.2450	0.0118	0.0665	0.9236	-0.0041	-0.0923
23	0.2491	0.0101	0.0677	0.9574	-0.0031	-0.0936
24	0.2530	0.0084	0.0690	0.9908	-0.0024	-0.0949
25	0.2569	0.0068	0.0702	1.0236	-0.0018	-0.0960
26	0.2607	0.0053	0.0714	1.0562	-0.0014	-0.0970
27	0.2645	0.0038	0.0725	1.0888	-0.0010	-0.0980
28	0.2684	0.0025	0.0736	1.1214	-0.0007	-0.0988
29	0.2722	0.0012	0.0746	1.1540	-0.0003	-0.0997
30	0.2760	0.0000	0.0756	1.1866	0.0000	-0.1005

Table 2. Covariances of η_{jt}

	$\eta_{1,t}$	$\eta_{2,t}$	$\eta_{3,t}$	$\eta_{4,t}$	$\eta_{5,t}$	$\eta_{6,t}$
$\eta_{1,t}$	6.00	0.11	2.04	3.49	0.84	0.02
$\eta_{2,t}$	0.11	5.78	-0.36	0.28	2.62	3.13
$\eta_{3,t}$	2.04	-0.36	3.90	1.03	0.02	1.00
$\eta_{4,t}$	3.49	0.28	1.03	3.09	1.03	-0.34
$\eta_{5,t}$	0.84	2.62	0.02	1.03	1.89	1.02
$\eta_{6,t}$	0.02	3.13	1.00	-0.34	1.02	3.75

Table 3. Coefficients a_{ij}

$j\backslash i$	1	2	3	4	5	6
1	2.449	0.000	0.000	0.000	0.000	0.000
2	0.046	2.403	0.000	0.000	0.000	0.000
3	0.833	-0.166	1.782	0.000	0.000	0.000
4	1.427	0.087	-0.083	1.018	0.000	0.000
5	0.344	1.084	-0.048	0.435	0.634	0.000
6	0.010	1.300	0.678	-0.403	-0.297	1.161

6. ILLUSTRATIVE RESULTS OF THE MODEL

6.1 In this section, illustrative results of the predictive model are presented.

6.2 Figures 2 to 7 show the results of the projections based on the parameters estimated from historical data and shown in the previous section. The simulated variables include short-term interest rates (one-year zero-coupon yields, both conventional and index-linked), long-term interest rates (20-year zero-coupon yields, both conventional and index-linked), inflation and equity returns. For each variable the mean and 95% confidence intervals are shown for each of the next 20 years based on 10,000 simulations of the model.

Figure 2. Mean and 95% confidence interval for equity returns

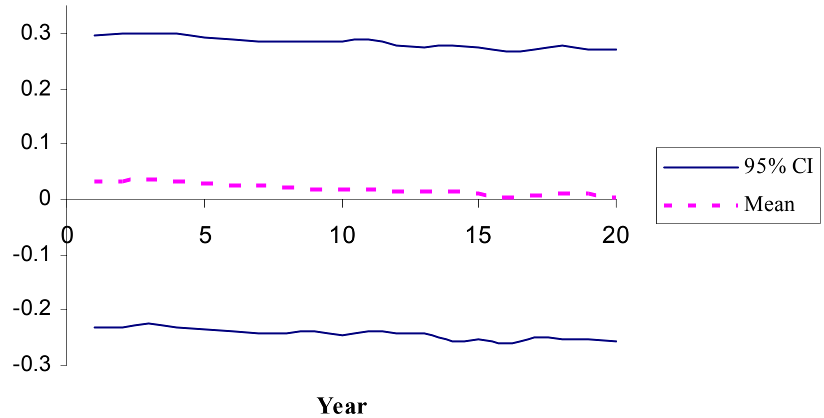


Figure 3. Mean and 95% confidence interval for inflation

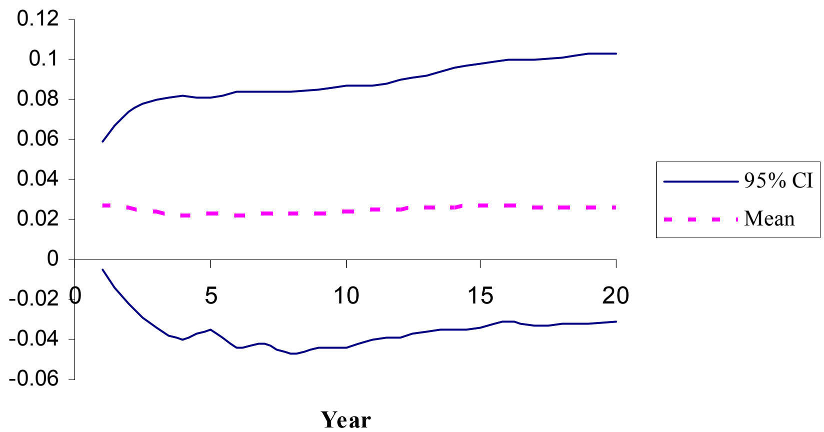


Figure 4. Mean and 95% confidence interval for yields on long-term conventional bonds

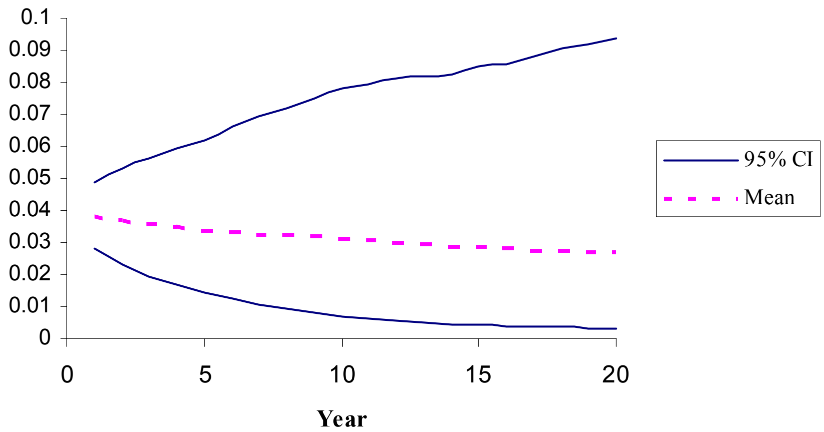


Figure 5. Mean and 95% confidence interval for yields on long-term index-linked bonds

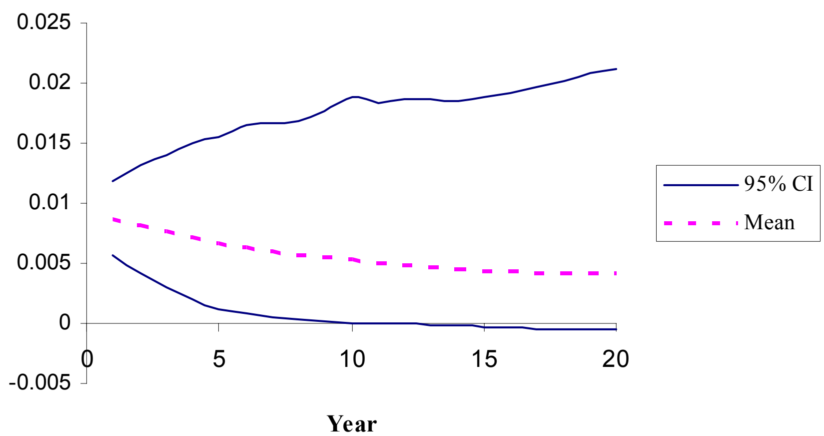


Figure 6. Mean and 95% confidence interval for yields on short-term conventional bonds

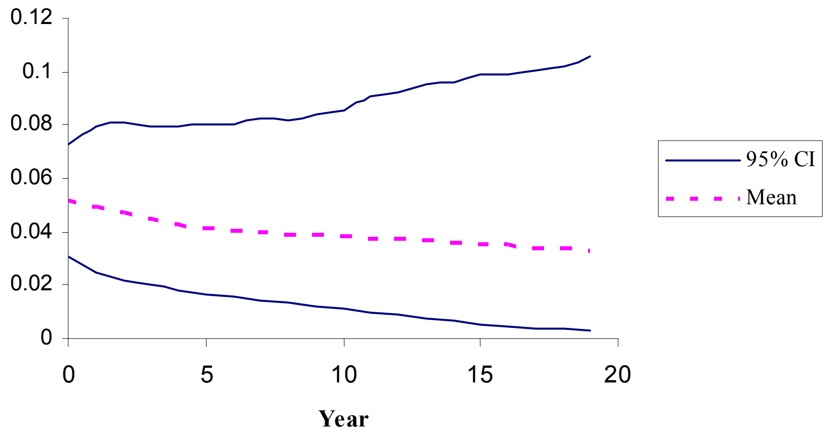
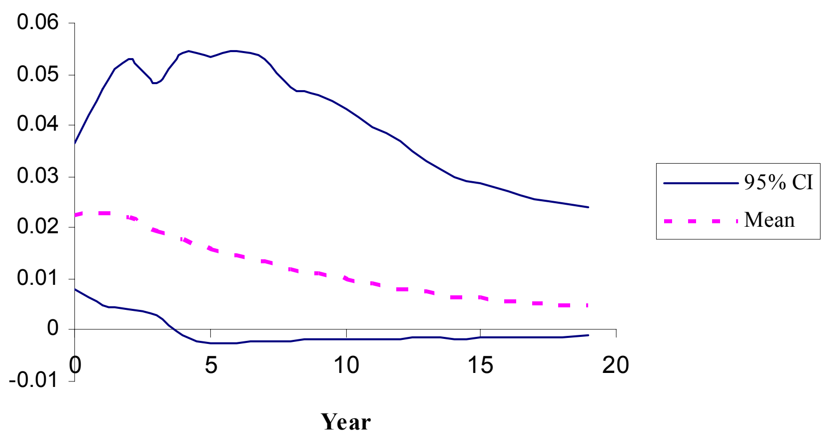


Figure 7. Mean and 95% confidence interval for yields on short-term index-linked bonds



7. CONCLUSIONS

7.1 The model presented in this paper is a long-term model of a local market. It comprises a model of the market portfolio, which, subject to certain constraints, may be specified by the user, as well as an equilibrium model of equity, bonds and inflation. While no arbitrage is present as between the asset classes modelled, there remains a small probability of negative yields (both nominal and real), which, in principle, allows a possibility of arbitrage. That may be avoided by eliminating projections that produce negative yields, or by applying a lower limit of zero, but this will have the effect of distorting the distribution of the returns on the asset classes so that arbitrage may be possible between asset classes, or so that the equilibrium equations do not apply, or so that the fidelity of the predictive model to the descriptive model is compromised.

7.2 In practice, negative yields should be monitored. If the effects are negligible in relation to the purpose to which the model is being applied, they may be ignored or avoided at the discretion of the user. Otherwise the model should not be used. It should be recognised, however, that, while negative yields allow arbitrage in principle, it may in practice be difficult to achieve. Particularly in the case of real yields, it is impossible to hold the basket of goods and services comprising a retail prices index without considerable cross-hedge risk or holding costs. Even in the case of nominal yields, there are costs in holding cash.

7.4 As mentioned in ¶3.1.6, a two-factor model of the term structure of interest rates overestimates the correlation between forward rates for neighbouring maturities and underestimates the correlation between forward rates with maturity dates far apart. Depending on the application for which the model is required, it may be necessary to consider a third factor for either or both of the models for conventional and index-linked bonds.

7.5 The following further research is required:

- the compilation of more historical data;
- the comparison of results for various markets;
- the estimation of the inflation risk premium and the modelling of inflation;
- the development of, and comparisons between, alternative models of the return on the market portfolio;
- the advantages and disadvantages of including a third factor in the bond pricing models; and
- an investigation of the problem of negative yields.

ACKNOWLEDGEMENTS

The authors acknowledge the assistance of Professor A.D. Wilkie and Mr M.R. Greenwood with regard to certain aspects of this paper. They also thank Professor A.D. Wilkie, Ms J. Jennett and Mr B. Davis for assistance with obtaining historical data. The financial assistance of the Actuarial Society of South Africa is acknowledged. Opinions expressed and conclusions drawn are those of the authors and are not to be attributed to the Society.

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APPENDIX A
THE EQUILIBRIUM MARKET PORTFOLIO OF NOTIONAL RISKY ASSETS

A.1 If the CAPM applies then, as shown by Elton & Gruber (1995: 100), the proportions invested in the respective assets of the market portfolio are:

$$x_i = \frac{z_i}{\sum_{i=1}^{6} z_i};$$

where:

$\mu_i - \delta = \sum_{j=1}^{6} z_j \sigma_{i,j}$ for $i = 1,\ldots, 6$;

δ is the risk-free return;

$\mu_i = E\{\delta_i\}$;

$\sigma_{i,j} = \text{cov}\{\delta_i, \delta_j\}$; and

δ_i is the return on the ith risky asset.

A.2 In the case under consideration the returns are instantaneous, but due to ¶3.6.1 the results are the same over integral periods if constant exposure is maintained. In this case:

$$\delta_i = c + d\varepsilon_i \text{ (equation (28))};$$

so that:

$\mu_i = c$; and

$\sigma_{i,j} = 0$ for $i \neq j$;

$\quad\quad d^2$ for $i = j$;

and hence:

$$c - \delta_i = d^2 z_i \text{ for } i = 1,\ldots, N.$$

Thus:

$$z_i = z_j = \frac{1}{N}.$$

APPENDIX B
DESCRIPTIVE ESTIMATION OF THE MODEL

B.1 INTRODUCTION

B.1.1 In this appendix, the method of determination of the model parameters is developed for the purpose of the estimation of a descriptive version of the model. The purpose of this process is to estimate values of the parameters required both for the equilibrium model and for a market model. While the mathematical specification of the equilibrium model does not require specification of the market model, the estimation of the former requires estimation of the latter. The latter is therefore dealt with first.

B.1.2 In determining the required parameters, we deliberately invoke the requirements of equilibrium, particularly through the use of the relationships between expected returns on the respective asset categories and the expected return on the market portfolio, as discussed in ¶3.7.1. This means that the estimates of these expected returns are not necessarily unbiased estimates ex post. Equilibrium is essentially established ex ante, and it is therefore important that, so far as it is possible, ex-ante expected values be estimated. Under the rational expectations hypothesis, which is normally invoked in the estimation of stochastic investment models, it is assumed that ex-post estimates are unbiased estimates of ex-ante expectations. Where that hypothesis conflicts with the requirements of equilibrium modelling, it is not invoked in this article.

B.2 ESTIMATION OF THE MARKET MODEL

B.2.1 Consider a sample historical period $t = t_0 + 1, t_0 + 2, \ldots, t_0 + T$. Let $\mathring{\gamma}_t$ denote the continuously compound rate of inflation during year *t*. Let $\mathring{y}_{I;t}(s)$ and $\mathring{y}_{C;t}(s)$ denote the effective continuously compound spot yields at time *t* on zero-coupon bonds—index-linked and conventional respectively—maturing at time $t + s$. We assume that, for each *t*, these have been graduated (either parametrically or non-parametrically) using standard techniques for the fitting of yield curves (see for example Van Deventer *et. al*., (2004)). The corresponding one-year returns are:

$$\mathring{\delta}_{I;t}(s) = \mathring{Y}_{I;t-1}(s+1) - \mathring{Y}_{I;t}(s) \text{ ; and} \tag{B.1}$$

$$\mathring{\delta}_{C;t}(s) = \mathring{Y}_{C;t-1}(s+1) - \mathring{Y}_{C;t}(s) - \mathring{\gamma}_t \text{ ;} \tag{B.2}$$

where:

$\mathring{Y}_{I;t}(s) = s\,\mathring{y}_{I;t}(s)$; and

$\mathring{Y}_{C;t}(s) = s\,\mathring{y}_{C;t}(s)$.

Let $\mathring{\delta}_{E;t}$ be the continuously compound (real) return on equities during year t.

B.2.2 Let:

- $\mathring{w}_{E;t}$ be the observed proportion of the market portfolio in equities, by market capitalisation, in year t; and
- $\mathring{w}_{I;t}(s)$ and $\mathring{w}_{C;t}(s)$ be the corresponding proportions in index-linked and conventional bonds respectively, with payment dates at time $t + s$.

For the purposes of calculation of the above proportions, coupon-paying bonds need to be notionally stripped into zero-coupon bonds. Bond payments need to be notionally apportioned between integral payment dates. Allowance needs to be made for lags in index-linking. Double-counting of corporate equity holdings needs to be avoided. Approximations may need to be made in order to avoid excessive data collection or to accommodate missing data, especially where the inclusion of such data would merely produce spurious accuracy.

B.2.3 From the above values, the return on the market portfolio during year t may be calculated as:

$$\mathring{\delta}_{M;t} = \sum_{s=1}^{\infty}\left\{\mathring{w}_{I;t}(s)\,\mathring{\delta}_{I;t}(s) + \mathring{w}_{C;t}(s)\,\mathring{\delta}_{C;t}(s)\right\} + \mathring{w}_{E;t}\,\mathring{\delta}_{E;t}\,. \tag{B.3}$$

From these values we may estimate σ_M as follows:

$$\hat{\sigma}_M = \sqrt{\frac{1}{T-1}\sum_{t=1}^{T}\left(\mathring{\delta}_{M;t} - \hat{\mu}_M\right)^2}\,. \tag{B.4}$$

B.2.4 As explained in ¶4.6, we may determine the ex-ante estimate of $\mu_{M;t}$ as:

$$\begin{aligned}\hat{\mu}_{M,t} &= g\delta_{I,t}(0) + h, \text{ for } \delta_{I,t}(0) > 0 \\ &= \delta_{I,t}(0) \text{ otherwise.}\end{aligned} \tag{B.5}$$

B.3 ESTIMATION OF THE EQUILIBRIUM MODEL

As stated in ¶3.8.1, the parameters required are as follows:

- for all required values of s :

  $Y_{I;0}(s)$ and $Y_{C;0}(s)$; and

  $b_{I,j}(s)$ and $b_{C,j}(s)$ for j = 1, 2; and
- b_γ ;
- $b_{E,1}$; and
- for i, j = 1,…, 6:

  $a_{i,j}$.

B.3.1 ESTIMATION OF $Y_{I;0}(s)$ AND $Y_{C;0}(s)$

From equation (3), $Y_{I;0}(s)$ may be estimated as:

$$\overset{\circ}{Y}_{I;0}(s) = -\ln\left(\overset{\circ}{P}_{I;0}(s)\right);$$

where $\overset{\circ}{P}_{I;0}(s)$ is the observed value of $P_{I;0}(s)$. Similarly:

$$\overset{\circ}{Y}_{C;0}(s) = -\ln\left(\overset{\circ}{P}_{C;0}(s)\right).$$

B.3.2 ESTIMATION OF $b_{I,j}(s)$

B.3.2.1 The values of $b_{I,j}(s)$ may be estimated as follows. From (41) it is clear that, since $\sigma_{M;t}$ is constant (say σ_M) for all t, the value of:

$$\chi_I(s) = \frac{\sigma_{I,M;t}(s)}{f_{I;t}(s)} \qquad \text{(B.6)}$$

will also be constant. From this definition:

$$\chi_I(s) = \frac{1}{f_{I;t}(s)} \operatorname{cov}_{t-1}\left\{\delta_{I;t}(s), \delta_{M;t}\right\}.$$

Since $f_{I;t}(s)$ is known ex ante at time t – 1 (though it is unobservable ex post), and since $\chi_I(s)$ is defined ex ante, we may write:

$$\chi_I(s) = \operatorname{cov}_{t-1}\left\{\frac{\delta_{I;t}(s)}{f_{I;t}(s)}, \delta_{M;t}\right\}.$$

Since this value is constant, we may write:

$$\chi_I(s) = \operatorname{cov}\left\{\frac{\delta_{I;t}(s)}{f_{I;t}(s)}, \delta_{M;t}\right\}.$$

Substituting equation (42), we obtain:

$$\chi_I(s) = \operatorname{cov}\left\{\frac{\delta_{I;t}(s)}{Y_{I;t-1}(s+1) - \mu_{I;t}(s)}, \delta_{M;t}\right\}.$$

The ex-post estimate of the ex-ante value of $\chi_I(s)$ is then given by:

$$\hat{\chi}_I(s) = \frac{1}{T-1}\sum_{t=1}^{T} \frac{\mathring{\delta}_{I;t}(s) - \hat{\mu}_{I;t}(s)}{\mathring{Y}_{I;t-1}(s+1) - \hat{\mu}_{I;t}(s)}\left(\mathring{\delta}_{M;t} - \hat{\mu}_{M;t}\right). \qquad \text{(B.7)}$$

From (39) we have:

$$\hat{k}_t = \frac{\hat{\mu}_{M;t} - \mathring{\delta}_{I;t}(0)}{\hat{\sigma}_M^2};$$

and from (40):

$$\begin{aligned}\hat{\mu}_{I;t}(s) &= \mathring{\delta}_{I;t}(0) + \hat{k}_t \hat{\sigma}_{I,M;t}(s) \\ &= \mathring{\delta}_{I;t}(0) + \hat{k}_t \hat{\chi}_I(s) \hat{f}_{I;t}(s).\end{aligned} \qquad \text{(B.8)}$$

This linear constraint explains the division by $T-1$ in equation (B.7).

B.3.2.1 Substituting (42) into (B.8) we obtain:

$$\hat{\mu}_{I;t}(s) = \mathring{\delta}_{I;t}(0) + \hat{k}_t \hat{\chi}_I(s)\left\{\mathring{Y}_{I;t-1}(s+1) - \hat{\mu}_{I;t}(s)\right\};$$

and thus:

$$\hat{\mu}_{I;t}(s) = \frac{\mathring{\delta}_{I;t}(0) + \hat{k}_t \hat{\chi}_I(s) \mathring{Y}_{I;t-1}(s+1)}{1 + \hat{k}_t \hat{\chi}_I(s)}. \qquad \text{(B.9)}$$

Substituting this into (B.7) we obtain, after some algebra:

$$\hat{\chi}_I(s) = \frac{\frac{1}{T-1}\sum_{t=1}^{T}\left(1-\kappa_{I;t}(s)\right)\left(\overset{\circ}{\delta}_{M;t}-\hat{\mu}_{M;t}\right)}{1+\frac{1}{T-1}\sum_{t=1}^{T}\hat{k}_t\kappa_{I;t}(s)\left(\overset{\circ}{\delta}_{M;t}-\hat{\mu}_{M;t}\right)}; \qquad \text{(B.10)}$$

where:

$$\kappa_{I;t}(s) = \frac{\overset{\circ}{Y}_{I;t\text{-}1}(s+1)-\overset{\circ}{\delta}_{I;t}(s)}{\overset{\circ}{Y}_{I;t\text{-}1}(s+1)-\overset{\circ}{\delta}_{I;t}(0)}.$$

On substituting the value of (B.10) into (B.9) we obtain ex-post estimates $\hat{\mu}_{I;t}(s)$ of the ex-ante expected returns. Ex post, these are clearly biased estimates of $\mu_{I;t}(s)$; that is a consequence of the need to estimate ex-ante expected values, which may be ex-post biased. As mentioned in ¶B.1.2, we are deliberately avoiding the rational-expectations hypothesis to the extent that it conflicts with the requirements of equilibrium modelling.

B.3.2.2 From (10) we may write:

$$\boldsymbol{x}_t = -\left(\hat{\boldsymbol{b}}_{I,1}\eta_{1,t} + \hat{\boldsymbol{b}}_{I,2}\eta_{2,t}\right); \qquad \text{(B.11)}$$

where:

$$\boldsymbol{x}_t = \begin{pmatrix} x_{t,1} \\ \vdots \\ x_{t,\tau*} \end{pmatrix};$$

$$x_{t,s} = \frac{\overset{\circ}{\delta}_{I;t}(s)-\overset{\circ}{\mu}_{I;t}(s)}{\overset{\circ}{f}_{I;t}(s)}$$

$$\hat{\boldsymbol{b}}_{I;j} = \begin{pmatrix} \hat{b}_{I;j}(1) \\ \vdots \\ \hat{b}_{I;j}(\tau) \end{pmatrix};$$

and $\hat{b}_{I;j}(s)$ is the estimate of $b_{I;j}(s)$ to be determined. This may be done by finding the first two principal components (e.g. Jackson, 1991), as follows.

B.3.2.3 First we estimate the covariance matrix of $\boldsymbol{x}_t$ as:

$$\hat{\boldsymbol{\Sigma}} = \begin{pmatrix} \hat{\sigma}_{11} & \cdots & \hat{\sigma}_{1\tau^*} \\ \vdots & & \vdots \\ \hat{\sigma}_{\tau^*1} & \cdots & \hat{\sigma}_{\tau^*\tau^*} \end{pmatrix};$$

where:

$$\hat{\sigma}_{ij} = \frac{1}{T-1}\sum_{t=1}^{T} x_{ti}x_{tj} \ .$$

Once again, we are working with an (ex-post biased) estimate of ex-ante expectations.

B.3.2.4 Next we determine the first two eigenvalues l_1 and l_2 and the eigenvectors $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ of $\hat{\boldsymbol{\Sigma}}$, so that:

$$\boldsymbol{U}'\hat{\boldsymbol{\Sigma}}\boldsymbol{U} = \boldsymbol{L};$$

where:

$$\boldsymbol{U} = \left(\boldsymbol{u}_1 \quad | \quad \boldsymbol{u}_2\right);$$

$$\boldsymbol{u}_1 = \begin{pmatrix} u_{1,1} \\ \vdots \\ u_{1,\tau^*} \end{pmatrix};$$

$$\boldsymbol{u}_2 = \begin{pmatrix} u_{2,1} \\ \vdots \\ u_{2,\tau^*} \end{pmatrix}; \text{ and}$$

$$\boldsymbol{L} = \begin{pmatrix} l_1 & 0 \\ 0 & l_2 \end{pmatrix}.$$

The eigenvalues and eigenvectors may be determined either by means of the power method (Jackson, *op. cit.*: 451–3) or by means of more efficient techniques available in numerous computer packages (*ibid.*, 453–5). The matrix $\boldsymbol{L}$ is the variance matrix of the principal components (their covariances being zero as they are uncorrelated). The principal-component scores corresponding to the observed values $\boldsymbol{x}_t$ are:

$$z_{1,t} = \boldsymbol{u}_1'\boldsymbol{x}_t \text{ and } z_{2,t} = \boldsymbol{u}_2'\boldsymbol{x}_t \ .$$

B.3.2.5 Now we need to determine $\hat{\boldsymbol{b}}_{I,1}$ and $\hat{\boldsymbol{b}}_{I,2}$. Assuming that the third and higher-order principal components may be ignored, we have (Jackson, *op. cit.*: 15):

$$x_{t,s} = u_{1,s} z_{1,t} + u_{2,s} z_{2,t} . \tag{B.12}$$

Let:

$$\eta_{1,t} = c_{11} z_{1,t} + c_{12} z_{2,t} \text{ ; and} \tag{B.13}$$

$$\eta_{2,t} = c_{21} z_{1,t} + c_{22} z_{2,t} . \tag{B.14}$$

Then, from (B.11) and (B.12), for all values of t:

$$-\left\{ \hat{b}_{I,1}(s) \left(c_{11} z_{1,t} + c_{12} z_{2,t} \right) + \hat{b}_{I,2}(s) \left(c_{21} z_{1,t} + c_{22} z_{2,t} \right) \right\} = u_{1,s} z_{1t} + u_{2,s} z_{2t} .$$

Equating the coefficients of $z_{1,t}$ and those of $z_{2,t}$, we have, respectively:

$$-\left\{ \hat{b}_{I,1}(s) c_{11} + \hat{b}_{I,2}(s) c_{21} \right\} = u_{1,s} \text{ ; and} \tag{B.15}$$

$$-\left\{ \hat{b}_{I,1}(s) c_{12} + \hat{b}_{I,2}(s) c_{22} \right\} = u_{2,s} . \tag{B.16}$$

In particular, for $s = 1$ and τ, we have, from (13):

$$\begin{aligned} c_{11} \hat{b}_{I,1}(1) &= -u_{11}; \\ c_{21} \hat{b}_{I,2}(\tau) &= -u_{1\tau}; \\ c_{12} \hat{b}_{I,1}(1) &= -u_{21}; \text{ and} \\ c_{22} \hat{b}_{I,2}(\tau) &= -u_{2\tau}. \end{aligned} \tag{B.17}$$

B.3.2.6 Now from (41) and (B.6):

$$\hat{b}_{I,1}(1) = -\frac{\hat{\chi}_I(1)}{\hat{\sigma}_M} \text{ ; and}$$

$$\hat{b}_{I,2}(\tau) = -\frac{\hat{\chi}_I(\tau)}{\hat{\sigma}_M} .$$

Thus, from (B.17), we obtain:

$$c_{11} = -\frac{u_{11}}{\hat{b}_{I,1}(1)} ;$$

$$c_{21} = -\frac{u_{1\tau}}{\hat{b}_{I,2}(\tau)} ;$$

$$c_{12} = -\frac{u_{21}}{\hat{b}_{I,1}(1)} \text{; and}$$

$$c_{22} = -\frac{u_{2\tau}}{\hat{b}_{I,2}(\tau)}.$$

B.3.2.7 Equations (B.15) and (B.16) may be represented as:

$$\hat{\boldsymbol{B}}_I \boldsymbol{C} = \boldsymbol{U};$$

where:

$$\hat{\boldsymbol{B}}_I = \left(\hat{\boldsymbol{b}}_{I,1} \quad | \quad \hat{\boldsymbol{b}}_{I,2} \right) \text{; and}$$

$$\boldsymbol{C} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}.$$

Thus:

$$\hat{\boldsymbol{B}}_I = \boldsymbol{U}\boldsymbol{C}^{-1}.$$

B.3.3 ESTIMATION OF b_γ

From equation (25) we may estimate $\mu_{\gamma;t}$ as:

$$\hat{\mu}_{\gamma;t} = \overset{\circ}{Y}_{C;t-1}(1) - \overset{\circ}{Y}_{I;t-1}(1) - \phi.$$

An ex-post estimate of the ex-ante value of $\sigma_{\gamma,M}$ may be determined as:

$$\hat{\sigma}_{\gamma,M} = \frac{1}{T-1} \sum_{t=1}^{T} \left(\overset{\circ}{\gamma}_t - \hat{\mu}_{\gamma;t} \right) \left(\overset{\circ}{\delta}_{M;t} - \hat{\mu}_{M;t} \right).$$

From equation (52), b_γ may then be estimated as:

$$\hat{b}_\gamma = \frac{\hat{\sigma}_{\gamma,M}}{\hat{\sigma}_M}.$$

From equation (14) we may also derive:

$$\hat{\eta}_{3,t} = \frac{\overset{\circ}{\gamma}_t - \hat{\mu}_{\gamma;t}}{\hat{b}_\gamma}. \tag{B.18}$$

B.3.4 ESTIMATION OF $b_{C,j}(s)$

B.3.4.1 For conventional bonds, as for index-linked bonds, we have:

$$\hat{\mu}_{C;t}(s) = \overset{\circ}{\delta}_{I;t}(0) + \hat{k}_t \hat{\chi}_C(s) \hat{f}_{C;t}(s)\,; \tag{B.19}$$

where $\hat{\chi}_C(s)$ is the ex-post estimate of the ex-ante value of:

$$\chi_C(s) = \frac{\sigma_{C,M;t}(s)}{f_{C;t}(s)}.$$

Hence, from (49), after rearranging:

$$\hat{\mu}_{C;t}(s) = \frac{\overset{\circ}{\delta}_{I;t}(0) + \hat{k}_t \hat{\chi}_C(s)\left\{\overset{\circ}{Y}_{C;t-1}(s+1) - \hat{\mu}_{\gamma;t}\right\}}{1 + \hat{k}_t \hat{\chi}_C(s)}. \tag{B.20}$$

From equation (49), we have:

$$\hat{f}_{C;t}(s) = \overset{\circ}{Y}_{C;t-1}(s+1) - \hat{\mu}_{\gamma,t} - \hat{\mu}_{C;t}(s)\,.$$

As for index-linked bonds, we obtain:

$$\hat{\chi}_C(s) = \frac{\dfrac{1}{T-1}\displaystyle\sum_{t=1}^{T}\left(1 - \kappa_{C;t}(s)\right)\left(\overset{\circ}{\delta}_{M;t} - \hat{\mu}_{M;t}\right)}{1 + \dfrac{1}{T-1}\displaystyle\sum_{t=1}^{T}\hat{k}_t \kappa_{C;t}(s)\left(\overset{\circ}{\delta}_{M;t} - \hat{\mu}_{M;t}\right)}\,; \tag{B.21}$$

where:

$$\kappa_{C;t}(s) = \frac{\overset{\circ}{Y}_{I;t\text{-}1}(s+1) - \hat{\mu}_{\gamma;t} - \overset{\circ}{\delta}_{C;t}(s)}{\overset{\circ}{Y}_{I;t\text{-}1}(s+1) - \hat{\mu}_{\gamma;t} - \overset{\circ}{\delta}_{I;t}(0)}\,.$$

B.3.4.2 From (24) we may write:

$$\boldsymbol{x}_t = -\left(\hat{\boldsymbol{b}}_{C,1}\eta_{4,t} + \hat{\boldsymbol{b}}_{C,2}\eta_{5,t}\right). \tag{B.22}$$

where:

$$\boldsymbol{x}_t = \begin{pmatrix} x_{t,1} \\ \vdots \\ x_{t,\tau*} \end{pmatrix};$$

$$x_{t,s} = \frac{\overset{\circ}{\delta}_{C;t}(s) - \overset{\circ}{\mu}_{C;t}(s) + \hat{b}_\gamma \hat{\eta}_{3,t}}{\overset{\circ}{f}_{C;t}(s)};$$

$$\hat{\boldsymbol{b}}_{C;j} = \begin{pmatrix} \hat{b}_{C;j}(1) \\ \vdots \\ \hat{b}_{C;j}(\tau*) \end{pmatrix};$$

and $\hat{b}_{C;j}(s)$ is the estimate of $b_{C;j}(s)$ to be determined. This may be done by finding the first two principal components as in section B.3.2.

B.3.4.3 The matrix U is similarly calculated, and we then have:

$$c_{11} = -\frac{u_{11}}{b_{C,1}(1)};$$

$$c_{21} = -\frac{u_{1\tau}}{b_{C,2}(\tau)};$$

$$c_{12} = -\frac{u_{21}}{b_{C,1}(1)}; \text{ and}$$

$$c_{22} = -\frac{u_{2\tau}}{b_{C,2}(\tau)}.$$

Here we have:

$$\hat{\eta}_{4,t} = c_{11} z_{1,t} + c_{12} z_{2,t}; \text{ and} \qquad \text{(B.23)}$$

$$\hat{\eta}_{5,t} = c_{21} z_{1,t} + c_{22} z_{2,t}. \qquad \text{(B.24)}$$

The matrix

$$\hat{\boldsymbol{B}}_I = \left(\hat{\boldsymbol{b}}_{I,1} \quad | \quad \hat{\boldsymbol{b}}_{I,2} \right)$$

may then be similarly derived as

$$\hat{\boldsymbol{B}}_I = \boldsymbol{UC}^{-1}.$$

B.3.5 ESTIMATION OF $b_{E,1}$

From (53):

$$\hat{\mu}_{E;t} = \overset{\circ}{\delta}_{I;t}(0) + \hat{k}_t \hat{\sigma}_{E,M;t}.$$

From (54) it is clear that, since $\sigma_{M;t} = \sigma_M$ is constant, $\sigma_{E,M;t}$ will also be constant (say $\sigma_{E,M}$). Let:

$$\hat{\sigma}_{E,M} = \frac{1}{T-1}\sum_{t=1}^{T}\left(\overset{\circ}{\delta}_{E;t} - \hat{\mu}_{E;t}\right)\left(\overset{\circ}{\delta}_{M;t} - \hat{\mu}_{M;t}\right). \qquad \text{(B.25)}$$

From (53) we have:

$$\hat{\mu}_{E;t} = \overset{\circ}{\delta}_{I;t}(0) + \hat{k}_t \hat{\sigma}_{E,M;t}. \qquad \text{(B.26)}$$

Substituting (B.26) into (B.25), we have, after some rearrangement:

$$\hat{\sigma}_{E,M} = \frac{\dfrac{1}{T-1}\displaystyle\sum_{t=1}^{T}\left(\overset{\circ}{\delta}_{E;t} - \overset{\circ}{\delta}_{I;t}(0)\right)\left(\overset{\circ}{\delta}_{M;t} - \hat{\mu}_M\right)}{1 + \dfrac{1}{T-1}\displaystyle\sum_{t=1}^{T}\hat{k}_t\left(\overset{\circ}{\delta}_{M;t} - \hat{\mu}_M\right)}.$$

From (54):

$$b_{E,1} = \frac{\hat{\sigma}_{E,M}}{\hat{\sigma}_M}.$$

Also, from (28):

$$\hat{\eta}_{6,t} = \frac{\delta_{E;t} - \mu_{E;t}}{b_{E,1}} \qquad \text{(B.27)}$$

B.3.6 ESTIMATION OF $a_{i,j}$

B.3.6.1 The estimation of $a_{i,j}$ proceeds by Cholesky decomposition of the sample covariance matrix:

$$\hat{\boldsymbol{\Sigma}} = \begin{pmatrix} \hat{\sigma}_{11} & \cdots & \hat{\sigma}_{16} \\ \vdots & & \vdots \\ \hat{\sigma}_{61} & \cdots & \hat{\sigma}_{66} \end{pmatrix}.$$

where:

$$\hat{\sigma}_{ij} = \frac{1}{T-1}\sum_{t=1}^{T}\eta_{i,t}\eta_{j,t} .$$

First, using the values of $\eta_{j,t}$, the residuals of the descriptive model, as determined in (B.13), (B.14), (B.18), (B23), (B.24), and (B.27), we define:

$$\boldsymbol{\eta}_t = \begin{pmatrix} \eta_{1,t} \\ \vdots \\ \eta_{6,t} \end{pmatrix}.$$

Now we calculate the sample covariance matrix $\hat{\boldsymbol{\Sigma}}$.

B.3.6.2 From equation (35):

$$\begin{aligned} \hat{\sigma}_{ij} &= \operatorname{cov}\left(\eta_i, \eta_j\right) \\ &= \sum_{k=1}^{6}\sum_{l=1}^{6} a_{ki}a_{lj} \operatorname{cov}\left(\varepsilon_k, \varepsilon_l\right) \\ &= \sum_{k=1}^{6} a_{ki}a_{kj}. \end{aligned}$$

We now require the matrix:

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & \cdots & a_{16} \\ \vdots & & \vdots \\ a_{61} & \cdots & a_{66} \end{pmatrix}$$

such that

$$\boldsymbol{A}'\boldsymbol{A} = \boldsymbol{\Sigma} .$$

This may be found by Cholesky decomposition.

APPENDIX C
DATA

Table C.1. Summary of historical data

Year	Equity	Conv. Bonds	Index-Linked Bonds	Equity Index	CPI	CONV01	CONV30	ILB01	ILB30
1979	82.7%	17.32%	0.00%	227.36	60.68	14.72%	12.61%	0.64%	
1980	82.7%	17.32%	0.00%	293.19	69.86	13.34%	12.18%	1.97%	
1981	82.7%	17.32%	0.00%	310.10	78.28	14.32%	13.91%	9.05%	
1982	82.7%	17.32%	0.00%	383.25	82.51	10.09%	9.28%	4.92%	
1983	82.7%	17.32%	0.00%	470.03	86.89	9.19%	8.28%	4.71%	
1984	82.7%	17.32%	0.00%	588.57	90.87	10.13%	8.35%	4.60%	
1985	81.0%	16.96%	2.08%	682.94	96.05	11.49%	8.49%	4.35%	3.86%
1986	81.0%	16.96%	2.08%	835.48	99.62	10.75%	8.33%	3.13%	3.69%
1987	81.0%	16.96%	2.08%	870.22	103.30	8.73%	8.21%	2.10%	4.21%
1988	81.0%	16.96%	2.08%	926.59	110.30	12.22%	7.15%	3.58%	4.05%
1989	81.0%	16.96%	2.08%	1,204.70	118.80	12.46%	7.80%	3.56%	3.96%
1990	81.0%	16.96%	2.08%	1,032.25	129.90	11.43%	8.75%	3.62%	4.57%
1991	81.0%	16.96%	2.08%	1,187.70	135.70	10.37%	8.76%	3.84%	4.51%
1992	81.0%	16.96%	2.08%	1,363.79	139.20	6.40%	9.20%	2.75%	3.92%
1993	81.0%	16.96%	2.08%	1,682.17	141.90	4.93%	6.39%	2.05%	3.10%
1994	81.0%	16.96%	2.08%	1,521.44	146.00	7.08%	7.99%	3.78%	3.88%
1995	81.0%	16.96%	2.08%	1,803.09	150.70	5.94%	7.79%	2.96%	3.65%
1996	81.0%	16.96%	2.08%	2,013.66	154.40	6.47%	7.45%	3.27%	3.77%
1997	81.0%	16.96%	2.08%	2,411.00	160.00	7.15%	6.04%	3.29%	3.06%
1998	81.0%	16.96%	2.08%	2,673.92	164.40	5.33%	4.12%	2.76%	2.02%
1999	85.5%	12.65%	1.82%	3,242.06	167.30	6.22%	4.20%	3.44%	1.51%
2000	86.0%	12.15%	1.89%	2,983.81	172.20	5.40%	3.99%	3.12%	1.54%
2001	84.7%	13.16%	2.10%	2,523.88	173.40	3.95%	4.50%	2.53%	2.07%
2002	79.9%	17.08%	2.99%	1,893.73	178.50	3.71%	4.37%	1.78%	2.11%
2003	81.6%	15.73%	2.69%	2,207.38	183.50	3.89%	4.55%	1.24%	1.92%
2004	81.1%	16.10%	2.77%	2,410.75	189.90	4.55%	4.28%	1.73%	1.42%
2005	81.4%	15.60%	2.96%	2,847.02	194.10	4.36%	3.86%	1.53%	0.93%
2006	81.9%	14.90%	3.22%	3,221.42	202.70	5.20%	3.96%	2.20%	0.90%

Table C.2. The return on the market portfolio and the risk-free rate

Year	$\delta_{M,t}$	Risk-free
1980	0.1023	0.0034
1981	-0.0655	0.0167
1982	0.1921	0.0875
1983	0.1397	0.0462
1984	0.1550	0.0441
1985	0.0876	0.0430
1986	0.1482	0.0435
1987	0.0249	0.0313
1988	-0.0025	0.0210
1989	0.1528	0.0358
1990	-0.2012	0.0356
1991	0.0988	0.0362
1992	0.1197	0.0384
1993	0.1879	0.0275
1994	-0.1250	0.0205
1995	0.1350	0.0378
1996	0.0773	0.0296
1997	0.1354	0.0327
1998	0.0889	0.0329
1999	0.1453	0.0276
2000	-0.0890	0.0344
2001	-0.1440	0.0312
2002	-0.2412	0.0253
2003	0.1023	0.0178
2004	0.0485	0.0124
2005	0.1257	0.0173
2006	0.0596	0.0153