

COPULA: A NEW VISION FOR ECONOMIC CAPITAL AND APPLICATION TO A FOUR LINE OF BUSINESS COMPANY

ASTIN Topic 3: Risk control

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Abstract/Resume:

The problem of risk accumulation is an important one, as shown by events such as the September 11th 2001 terrorist attacks. However, most Economic Capital assessment models encounter difficulties when trying to incorporate the dependence of claim costs between different Lines of Business (LOBs).

These models indeed often use multivariate distributions that are well studied in the literature (e.g. the multivariate Normal distribution). With such multivariate distributions, each marginal (i.e. distribution of claim costs for a specific LOB) belongs to the same distribution family (e.g. Normal). This property is however not observed in practice. Therefore, if current models use a well-studied multivariate probability distribution function (pdf), they can incorporate the dependency between LOBs, but do not take into account the characteristics of each of them. On the contrary, current models can also use distributions of claim costs specific to each LOB, but suppose them to be independent from one another.

In this article, we suggest the use of copula theory as a solution to this problem. A copula is a function that embodies all the information about the dependence between the components of a random vector. When a copula is applied to marginal distributions that are subject to some minor technical requirements, but do not necessarily belong to the same distribution family it results in a proper multivariate distribution.

We use this property to model the overall distribution of claim costs of a four-LOB company. By using different copulas we show that the dependence structure has a substantial impact on the Economic Capital of that firm. In particular, we pay attention to situations in which attritional losses from different LOBs compensate for each other to produce stable total results, while at the same time extreme losses tend to occur simultaneously across different LOBs. This type of dependence could not be modelled by the multivariate normal distribution, which is at the root of many current Economic Capital assessment models.

Keywords:

Copula, Economic Capital, Value at Risk, tail dependence, probability of ruin, DFA, solvency

1 Introduction

The September 11th 2001 terrorist attacks were a turning point for the non-life insurance industry for at least two reasons: because of the amount of indemnity involved, but also because that single event impacted several LOBs whose claim costs were often assumed to be independent from one another. It thus provided evidence that claim costs are not independent.

The question is then “how can one incorporate such a dependence in the Economic Capital assessment process”? Until recently there were only two options: either to model the dependence between LOBs, or to model the characteristics of each LOB. Indeed, in order to incorporate the dependence between LOBs, most Economic Capital assessment models used well-studied multivariate probability distribution functions (pdf) (Normal, Student, Pareto...). For any of these multivariate distributions, the marginals (i.e. the distributions of claim costs for each LOB) belong to the same distribution family (e.g. Normal). This characteristic is however not experienced in practice. This is the reason most models assume the different LOBs to be independent, but take into account their effective annual loss distributions.

In this article, we suggest the use of copula theory as a solution to this problem. A copula is a function that embodies all the information about the dependence structure between the components of a random vector. When it is applied to marginal distributions (that are subject to some minor technical requirements, but do not necessarily belong to the same distribution family), it results in a proper multivariate distribution. As a consequence, this theory enables us to incorporate an *exhaustive* modelling of the dependence structure between annual claim costs for each LOB, while allowing them to be modelled by different distributions.

With this model, we show the effects of risk management that would be based on the independence of claim costs from one LOB to another when they are actually dependent. In particular, we emphasise dependence structures that simultaneously allow for the technical results induced by attritional losses to compensate each other, while extreme events are highly dependent between different LOBs. Such situations are unfavourable to the insurer who may face a capital drain without warning.

This article is articulated as follows: we first give a brief introduction to copula theory, with its definition as well as main results and properties. Proof of the statements will not be provided but one can refer to the bibliography. We then apply these results to model a virtual 4-LOB insurance company and derive conclusions for the assessment of Economic Capital.

2 Copula theory

Copula theory is related to the study of multivariate distributions with given marginals. This is of special interest for the non-life insurance industry, where the distribution of claim costs for each LOB has long been studied. It appears that each LOB has a specific annual claim costs distribution, sometimes very skewed and fat-tailed. So that, at the level of the entire company, no widespread existing multivariate distribution can model such a variety of marginals. We will see that copula theory helps handle such situations.

2.1 What is a copula?

A copula (or n-copula) C is the joint cumulative distribution function (cdf) of a random vector $U = (U_1, \dots, U_n)$ of \mathbb{R}^n whose marginals U_1, \dots, U_n are uniform $U[0;1]$.

$$C(u_1, \dots, u_n) = \Pr(U_1 \leq u_1, \dots, U_n \leq u_n) \quad u_1, \dots, u_n \in [0;1] \quad (1.)$$

One of the main areas of interest of copula theory is given by the following theorem:

□ Theorem (Sklar 1959):

Let F be an n -dimensional cdf with continuous marginals F_1, \dots, F_n . Then F has a unique copula representation:

$$F(x_1, \dots, x_n) = C(F_1(x_1), \dots, F_n(x_n)) \quad (2.)$$

A copula is thus a function that, when applied to univariate marginals, results in a proper multivariate pdf. That pdf, because it embodies all the information about the random vector, contains the information about the dependence structure of its components. Using copulas in this way splits the distribution of a random vector into individual components (marginals) with a dependence structure (the copula) between them without losing any information. Note that this theorem does not require F_1, \dots, F_n to be identical or even to belong to the same distribution family.

With a view to assessing the Economic Capital of a company with different LOBs, the theorem implies that the study of the distribution of claim costs for the entire company can be split into two parts: first, studying the distribution of claim costs relative to a given LOB, and then taking into account the dependence between those LOBs by adopting a copula.

The theorem also provides a way to generalise previous models using well-studied multivariate pdfs. Since all these distributions have continuous marginals, there exists a unique, implicit, copula that links them. Once this implicit copula is isolated, it can then be reused with different marginals to better reflect the characteristics of the modelled company.

2.2 Fréchet bounds and partial ordering for the set of copulas

Among the whole set of copulas, three are of special nature: C^- , C^\perp and C^+ , which are respectively the copulas of *Counter-comonotony*, *Independence*, and *Comonotony*. Counter-comonotony is the extreme negative dependence, independence is the absence of dependence, and comonotony is the extreme positive dependence structure:

$$C^-(u_1, \dots, u_n) = \max\left(\sum_{i=1}^n u_i - n + 1; 0\right) \quad (3.)$$

$$C^\perp(u_1, \dots, u_n) = \prod_{i=1}^n u_i \quad (4.)$$

$$C^+(u_1, \dots, u_n) = \min(u_1; \dots; u_n) \quad (5.)$$

where $u_1, \dots, u_n \in [0;1]$.

Remark: C^\perp and C^+ are copulas whatever the dimension n , but C^- is only a copula in the bivariate case. Nevertheless C^- is for any dimension n (n greater than two) the pointwise best possible bound, i.e. for any u in $Dom C^-$ there exists a copula C such that $C(u) = C^-(u)$.

As an illustration, when companies consider that the losses they are facing are independent from one LOB to another, they consider that the loss cdf for the entire company is:

$F(x_1, \dots, x_i, \dots, x_n) = \prod_{i=1}^n F_i(x_i)$, where n stands for the number of LOBs for the company, F_i is the loss cdf for LOB i , and x_i the loss amount on LOB i .

This is equivalent to writing $F(x_1, \dots, x_i, \dots, x_n) = C^\perp(F_1(x_1), \dots, F_i(x_i), \dots, F_n(x_n))$.

With a view to introducing a first comparison between copulas, we define a partial ordering over the set of copulas:

□ Definition (Nelsen [1998]):

A copula C_1 is said to be smaller than a copula C_2 if

$$\forall (u_1, \dots, u_n) \in [0,1]^n, \quad C_1(u_1, \dots, u_n) \leq C_2(u_1, \dots, u_n)$$

One writes $C_1 \prec C_2$

\prec is called the concordance ordering, and is related to the first order stochastic dominance over cdfs.

We can remark that the ordering $C^- \prec C \prec C^+$ holds for any copula C .

From this definition, we will say that:

A copula is a positive dependence structure if $C^\perp \prec C \prec C^+$

A copula is a negative dependence structure if $C^- \prec C \prec C^\perp$

Note that \prec is a partial ordering. There exist copulas that are neither positive nor negative dependence structures.

This definition provides us with a first classification among copulas, but it remains theoretical. That is the reason we introduce measures of the dependence induced by the copula.

2.3 Concordance measure and Tail dependence

As mentioned in 2.1, a copula embodies all the dependence structure between the components of the random vector it models. In particular, we introduce dependence measures that can be expressed in terms of the selected copula. These measures belong to two subcategories depending on whether they quantify a global trend (concordance measures), or an asymptotical dependence between extreme events(tail dependence).

The concordance measures must be related to the linear correlation coefficient for elliptical distributions. The linear correlation coefficient is of common use in actuarial sciences, yet as stressed by Embrechts (1999) [c] using it for distributions other than elliptical can prove to be misleading. Therefore in presence of skewed heavy tailed distributions, concordance measures are a good substitute for the linear correlation coefficient: they belong to $[-1;1]$, equal zero when the random variables are independent and the concordance ordering over copulas induce the ordering of the measure. Two widespread concordance measures are the already known: Kendall's tau and Spearman's rho.

□ Definition: Kendall's Tau of random vector (X, Y) is defined by

$$\tau(X, Y) = \Pr\{(X - X')(Y - Y') > 0\} - \Pr\{(X - X')(Y - Y') < 0\} \quad (6.)$$

where (X', Y') is an independent copy of (X, Y) .

The Kendall's correlation coefficient is thus the probability of concordance minus the probability of discordance.

□ Definition: Spearman's Rho of a random vector (X, Y) is

$$\rho_S(X, Y) = \rho_P(F_X(X), F_Y(Y)) \quad (7.)$$

where F_X and F_Y stand for the cdf of X and Y , and ρ_P is Pearson's linear correlation coefficient.

Tail dependence is a major concept of risk management. Its purpose is to quantify the dependence that may exist for a bivariate distribution in the upper or lower quadrant. Contrary to Kendall's Tau and Spearman's Rho, tail dependence is a local measure of dependence.

The upper tail dependence of a random vector (X, Y) is defined by

$$\lambda_U(X, Y) = \lim_{\alpha \rightarrow 1} \Pr\left(X > F_X^{-1}(\alpha) \mid Y > F_Y^{-1}(\alpha)\right) \quad (8.)$$

The lower tail dependence of a random vector (X, Y) is defined by

$$\lambda_L(X, Y) = \lim_{\alpha \rightarrow 0} \Pr\left(X < F_X^{-1}(\alpha) \mid Y < F_Y^{-1}(\alpha)\right) \quad (9.)$$

So that $\lambda_U, \lambda_L \in [0, 1]$

Hence the upper tail dependence answers the question: "assuming that an extreme loss impacts LOB_i , what is the probability that an extreme loss also impacts LOB_j ".

As we see in the following section, dependence structures are of a great variety from one copula family to another.

2.4 Comparative study of three copula families: the Normal, Student and Gumbel copulas.

This brief introduction of copula theory is sufficient to proceed to a first comparison of three copula families: the Normal, Student and Gumbel copulas. Each of them expresses a special form of dependence that will be used in the third part of this document.

Both the Normal and Student copulas belong to a special category of copulas: the elliptical copulas. As stated in 2.1, starting from an already known multivariate pdf may prove to be a convenient way to create copulas. Under the assumptions of Sklar's theorem, a corollary exists that states that for any $u_1, \dots, u_n \in [0; 1]$, the following expression holds:

$$C(u_1, \dots, u_n) = F\left(F_1^{-1}(u_1), \dots, F_n^{-1}(u_n)\right) \quad (10.)$$

□ **Normal copula:** it owes its name to one of its properties: when applied to univariate $N(0, 1)$ marginals, the resulting distribution boils down to a multivariate normal distribution. This copula is thus (explicitly or implicitly) at the root of much of the Economic Capital assessment models we referred to in the introduction, so a special interest is given to its properties.

The Normal copula can be expressed as:

$$C_{\rho}^{Ga}(u) = \Phi_{\rho}^n\left(\Phi^{-1}(u_1), \dots, \Phi^{-1}(u_n)\right), \quad \rho \text{ the correlation matrix, } u_1 \dots u_n \in [0; 1] \quad (11.)$$

where Φ_p^n and Φ are respectively the multivariate and univariate normal cdfs.

One of its main interests is that in the bivariate case, it can reach the whole spectrum of dependence: from C^- to C^+ , including C^\perp ($\tau_{i,j} = \frac{2}{\pi} \arcsin(\rho_{i,j})$, $\rho_{S_{i,j}} = \frac{6}{\pi} \arcsin\left(\frac{\rho_{i,j}}{2}\right)$).

Nonetheless it does not allow for extreme events to be dependent unless $\rho_{i,j} = \pm 1$.

- **Student Copula:** Although it is another elliptical copula, the dependence structure it models is slightly different from that of the Normal copula. It can reach the whole range of dependence from C^- and C^+ in the bivariate case, except independence. It indeed always presents a double trend, positive and negative that only vanishes for $R_{i,j} = \pm 1$, where $C_{\nu,R}^n(u_1, \dots, u_n) = t_{\nu,R}^n(t_\nu^{-1}(u_1), \dots, t_\nu^{-1}(u_n))$ (12.) with $t_{\nu,R}^n$ the n-variate student distribution with ν degrees of freedom, and R the correlation matrix.

Another important difference from the Normal copula is that it is able to express a dependence between extreme events:

$$\lambda_{U_{i,j}} = \lambda_{L_{i,j}} = 2 \bar{t}_{\nu+1} \left(\sqrt{\nu+1} \sqrt{\frac{1-R_{i,j}}{1+R_{i,j}}} \right), \quad i, j \in \{1, \dots, n\} \quad (13.)$$

where $\bar{t}_{\nu+1}$ is the survival function of a student random variable with $\nu+1$ degrees of freedom.

No analytical formula links the parameters of the Student copula with the Kendall's Tau or Spearman's Rho.

- **Gumbel copula:** contrary to the Normal and Student copulas, it is not derived from a known multivariate distribution, but is part of the Archimedian copulas. The Gumbel copula is very different from the previously described copulas. It can only model independence or positive dependence structures and, as it depends on a single parameter (to be compared to the $\frac{n(n-1)}{2}$ pairs of components of an n-variate random vector), it has no allowance for special treatment of some components.

$$C(u_1, \dots, u_n) = \exp \left[- \left(\sum_{i=1}^n (-\ln u_i)^\theta \right)^{1/\theta} \right], \quad \theta \geq 1 \quad (14.)$$

The positive dependence structure is expressed by the relation between θ and τ :

$$\tau = 1 - \frac{1}{\theta} \quad (15.)$$

Moreover the upper tail dependence is strong while the lower tail dependence always equals 0:

$$\lambda_U = 2 - 2^{1/\theta} \quad (16.)$$

$$\lambda_L = 0 \quad (17.)$$

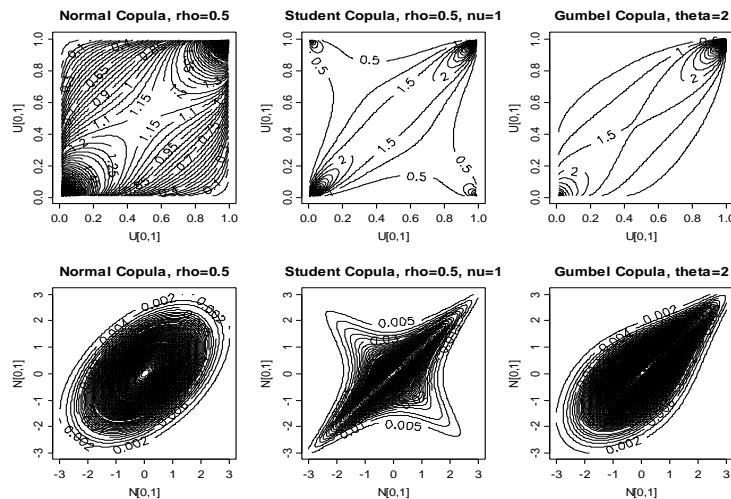
With a view to assessing Economic Capital, the main interest for using a Gumbel copula is that it confronts the solvency of a company to unfavourable scenarios (stress scenarios),

i.e. where major events tend to be linked, while the most common claims remain independent.

The following graphs illustrate the differences we mentioned for these three families of copulas in the bivariate case. They represent the density of the copulas

$$(c(u_1, \dots, u_n) = \frac{\partial^n C}{\partial u_1 \dots \partial u_n}(u_1, \dots, u_n)),$$

and the density of the distributions built on the selected copula, with marginals both equal to $N(0,1)$. Note that we chose to use the same marginals for both of the components to show some properties of the copulas, but using different marginals is of course possible.



With this brief overview of the nature of copulas, we will see how they may be incorporated into the Economic Capital assessment model of a virtual 4-LOB insurance company and explore the advantages of using them.

3 Dependence and Economic Capital

We have seen that copulas are a convenient tool to model dependence between the components of a random vector. Sklar's theorem (2.) shows that copula theory fits well into an insurance context, where claims occurring in different LOBs have different distributional characteristics.

The interactions of losses between LOBs then have to be collected in order to choose the copula that models the company best.. Inferential statistics for copulas are not developed herein, but the reader may refer to the bibliography nonetheless.

We now present a model for assessing the Economic Capital of a company. The aim of this model is to fully consider the dependence between claim costs, and to investigate the effect of assuming them to be independent when they are not.

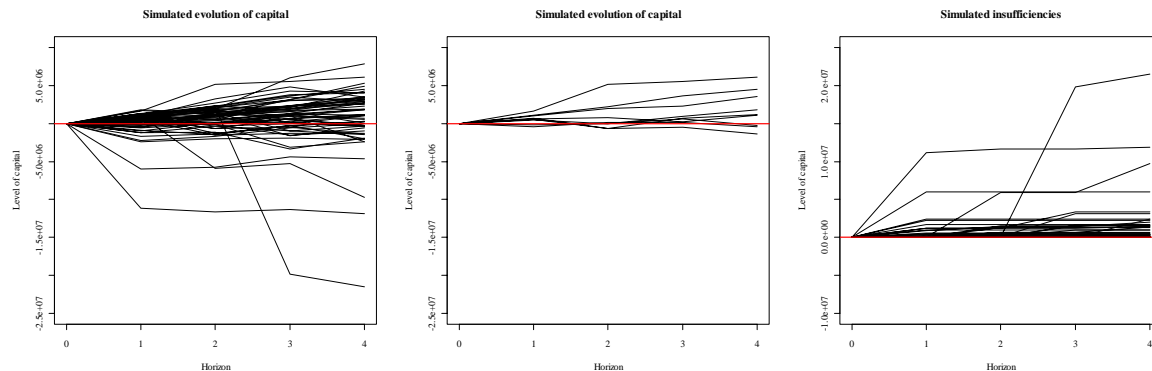
3.1 Selected ruin criterion

We now study the potential ruin of the company and derive a measurement that quantifies the capital required. The definition of ruin is thus a crucial element of the following study.

In order to fulfil the common requirements of a business plan, a company must project its position in the future to anticipate the consequences of its strategic options. In the context of our study we decided to project the results of the company over a four-year horizon.

We consider that the Economic Capital of the company is the level of capital that the company should hold to maintain its probability of ruin below a given threshold. α From a mathematical point of view, the Economic Capital is thus defined as the Value at Risk (VaR) of level α of the distribution of the ruin intensities.

Consider the following projection scenarios of the level of capital:



A closer examination of the simulated scenarios indicates three kinds of paths: those for which the level of capital always remains positive, those for which it will remain negative and those for which it will temporarily become negative.

The ruin of the company is thus not simply the presence of a negative capital at the end of the projection. For any projected year, we consider that the company is ruined either if its capital is not sufficient to absorb the loss of that projected year, or if it was already ruined during a previous projected year.

As a consequence we introduce the *insufficiency* as the maximum ruin intensity generated since the beginning of the projection (the insufficiency is defined as zero if the capital remains positive).

The insufficiency is hence always increasing from one projected year to another, so that only the values at the end of the time horizon need to be stored.

The Economic Capital of the company is therefore estimated by the minimum level of capital sufficient to absorb the final insufficiency of a proportion α of the scenarios.

This definition is more restricting than absorbing a proportion α of the simulated losses over all of the scenarios and all of the projected years.

After having defined a measurement, we can now discuss the structure of the company we have chosen to study.

3.2 General description of the company

In order to emphasize the importance of the dependence between claim costs, we will neglect or simplify some sources of uncertainty. In particular, we consider that:

- There is no misestimating in the reserving process (the ultimate loss is known)

- The management has a thorough understanding of its company: no specification nor estimation error in the determination of the loss distribution for each LOB
- There is no uncertainty concerning the investment income: all the invested assets are risk-free assets (flat yield structure).
- There are no acquisition or administration expenses
- There are no taxes

Moreover some modelled factors have a simplified behaviour:

- Cash-flows occur in the middle of the year
- Policies cover a one-year guaranty.

As a result, the balance sheet and profit and loss account of our company are as follows:

A	L
Assets	<div style="border: 1px solid black; padding: 5px; margin-bottom: 5px;">Capital</div> <div style="border: 1px solid black; padding: 5px; margin-bottom: 5px;">Profit Before Tax</div> <div style="border: 1px solid black; padding: 5px; margin-bottom: 5px;">Claims Reserve</div> <div style="border: 1px solid black; padding: 5px;">Unearned Premium Reserve</div>
L	P
<div style="border: 1px solid black; padding: 5px; margin-bottom: 5px;">Paid Claims</div> <div style="border: 1px solid black; padding: 5px;">Technical Result</div>	<div style="border: 1px solid black; padding: 5px; margin-bottom: 5px;">Written Premiums</div> <div style="border: 1px solid black; padding: 5px;">Investment Income</div>

The evolution of the company is therefore ruled by a few equations.

- Unearned Premium Reserve ('UPR')

$$UPR(t) = UPR(t-1) + WP(t) - EP(t) \quad (14.)$$

where $WP(t)$ is the amount of written premium at year t , and $EP(t)$ the amount of earned premium at year t . So that if $v_{Pr\ emium}$ is the vector of the premium earning pattern, we can write:

$$EP(t) = \sum_{i=0}^t v_{Pr\ emium}[t-i] WP(t-i) \quad (15.)$$

- Claim reserve

Let v_{Claims} be the n -dimensional vector of the payment pattern, and $S(t)$ the ultimate claims for accident year t , then the following relationship holds:

$$Paid(t) = \sum_{i=0}^{n-1} v_{Claims}[t-i] S(t-i) \quad (16.)$$

$$Claim\ Re\ s(t) = Claim\ Re\ s(t-1) + S(t) - Paid(t) \quad (17.)$$

□ **Technical cash-flow**

Since it is assumed to be uniformly distributed over the year, it can be approximated by a linear tendency:

$$Technical_CashFlow(t) = \frac{1}{2} \{ [WP(t-1) - Paid(t-1)] + [WP(t) - Paid(t)] \} \quad (18.)$$

□ **Investment Income**

Let r_{Asset} be the return on invested assets, then the investment income $InvInc$ is:

$$InvInc(t) = r_{Asset} \cdot Asset(t-1) + Technical_CashFlow(t) (\sqrt{1 + r_{Asset}} - 1) \quad (19.)$$

□ **Profit before Tax**

Since there are no expenses:

$$PbT(t) = WP(t) - Paid(t) - [UPR(t) - UPR(t-1)] - [Claim\ Re\ s(t) - Claim\ Re\ s(t-1)] + InvInc(t) \quad (20.)$$

These simplified dynamics do not allow for any source of uncertainty other than the claim cost, which is the heart of our model.

3.3 Modelling and simulation process of the aggregated loss distribution

To describe the dependence of annual claim costs across LOBs, we apply the fundamental reasoning of copula theory by splitting the individual behaviours (distribution of claim costs for each LOB) from their potential dependence (choice of the copula).

For S_i , the annual loss distribution for LOB i , we consider the collective model:

$$S_i = \sum_{j=1}^{N_i} X_j \quad (21.)$$

where N_i stands for the annual claim frequency, and X_j the individual average claim cost.

Generally this expression does not boil down to a known cdf, so we replace the real distribution by the empirical cdf built on one million simulations of S_i . The selection of one million is purely subjective; it is the result of a compromise between computation time and precision in the description of the tail of the distribution.

The dependence is then incorporated via the copula, which provides, by simulation, the percentiles of the marginal distributions (see for example equation 10).

In order to compare the results for different copulas, the software we created is designed as follows:

1. An initialising processor, which handles the one million simulations of the annual claim costs for each LOB, as well as the creation of the dedicated empirical cdfs.
2. A second processor, which projects the financial position of the company, including interactions with the different items of the balance sheet and profit and loss account.

The consistency of the results is guaranteed by the use of the same seed to initialise the generator of pseudo random series.

3.3.a) Claim Costs over each LOB

The company we will study has four lines of business. The main characteristics are described in the table below. It must be noted that they may not reflect a realistic situation, yet they allow for incorporating the impact of the volatility of the distribution of annual claim costs, of the payment patterns, of earnings patterns etc....

		LOB 1	LOB 2	LOB 3	LOB 4	
Premium	Pricing principle	Expected Value				
	Security loading	0%	20%	2%	-10%	
	Earning pattern	Year 1	100%	100%	50%	50%
		Year 2		50%	50%	
Payment	Payment pattern	Year 1	100%	100%	50%	40%
		Year 2			30%	30%
		Year 3			20%	20%
		Year 4				10%
Technical losses	Frequency distribution	Poisson				
	Parameter	10				
	Claim costs distribution	European Pareto	European Pareto	LogNormal	European Pareto	
	Parameters	a = 100 000 $\alpha = 2.1$	a = 100 000 $\alpha = 2.1$	$\mu = 2$ $\sigma = 3$	a = 100 000 $\alpha = 1.8$	
	Allocated Capital	0.25 €				

Where

Distribution	Pdf	mean	Variance
European Pareto	$\begin{cases} \frac{\alpha a^\alpha}{x^{\alpha+1}} & \text{if } x > a \\ 0 & \text{otherwise} \end{cases}$	$\frac{\alpha a}{\alpha - 1} \quad \text{if } \alpha > 1$	$\frac{\alpha a^2}{(\alpha - 1)^2 (\alpha - 2)} \quad \text{if } \alpha > 2$
LogNormal	$\begin{cases} \frac{1}{\sigma x \sqrt{2\pi}} e^{-\frac{1}{2} \left(\frac{\ln(x) - \mu}{\sigma} \right)^2} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$	$e^{\mu + \frac{\sigma^2}{2}}$	$e^{2\mu + \sigma^2} (e^{\sigma^2} - 1)$

Moreover the invested assets have a flat 5% risk-free return.

Note that the overall capital of the company is deliberately set to €1. The aim of this parameterisation is to have an almost zero, yet positive amount of capital, so that by computing the $Var_{\alpha}[Insufficiency]$ we directly obtain the level of capital the company should hold to produce a probability of ruin over four years of $1-\alpha$.

In the table above LOB 2 is a copy of LOB1 except for an additional security loading of 20% of the pure premium. The policies related to any of these two LOBs cover a one year period and are assumed to be written the first of January of each year. The written premiums are therefore entirely earned within each year, so that no UPR needs to be held. Neither does a Claim reserve need to be held, since each claim occurring during an accounting year is assumed to be entirely paid by the end of that year. The results of these two LOBs are thus highly dependent on the adequacy of the premium with regards to the insured risk.

LOB 3 has both a UPR and claim reserve. A security loading exists but is low (2%). Its claim cost distribution is the least volatile of the four.

Finally LOB 4 has the most fat-tailed distribution of claim costs of the four LOBs (in particular its variance is not defined), while premiums are insufficient.

Intuitively, if one had to rank these LOBs by probable importance in order of losses of the company, the result would be:

1. LOB 4
2. LOB 1
3. LOB 2
4. LOB 3

3.3.b) Selected copulas

While all the previous assumptions were supposed to be provided by the management of the company, we consider that management did not inquire into the dependence structure of their company. We assume that the Economic Capital of the company is assessed by its management by assuming the annual claim costs to be independent one from another. In reality, there exists a dependence structure which we assume to be:

- A Normal copula, with correlation matrix

$$\begin{vmatrix} 1 & 0.5 & 0 & -0.9 \\ 0.5 & 1 & 0 & -0.4 \\ 0 & 0 & 1 & 0 \\ -0.9 & -0.4 & 0 & 1 \end{vmatrix}$$

Under this assumption, LOBs 1 and 4 (which are the most volatile) tend to produce technical results that compensate each other. LOBs 2 and 4 are somewhat compensating. LOBs 1 and 2 are positively dependent, while LOB 3 remains independent from the other LOBs.

- A Student copula with the same correlation matrix as the Normal copula, and with coefficient ν equal to one. That value of ν induces a strong tail dependence (see equation 13)
- A Gumbel copula with coefficient θ equal to two. This copula expresses that all the four LOBs are equally positively dependent with each other (Kendall's tau equals to 50% (equation 15)), while a tight tail dependence also exists.

3.4 Results of the simulations

For each assumption 500 000 simulations of the evolution of the company over a four-year horizon were calculated. The results are summarised in the following table:

Amounts in Euros (000's)

Copula		Student	Normal	Independence	Gumbel
Initial probability of ruin		41%	46%	54%	54%
		(13%)*	(8%)		+ 0%
VaR [Insufficiency]	90%	(33%)	(36%)	4 402	+ 62%
	95%	(22%)	(29%)	6 546	+ 64%
	99%	(1%)	(13%)	14 213	+ 64%
	99.5%	+ 7%	(8%)	19 899	+ 65%

*Percentages indicate relative difference compared to the Independence scenario.

In this table, the initial probability of ruin refers to the probability of ruin of the company with an almost-zero capital. For instance, when the company has almost no capital and its LOBs are supposed to be independent, its probability of ruin is 54%. This probability would be reduced 13% if the real dependence structure between LOBs is a Student copula with the set of parameter mentioned in the previous section.

The above VaRs give the Economic Capital of the company to maintain its probability of ruin below 10%, 5%, 1% and 0.5% respectively, over a four-year horizon. For instance, the Economic Capital of the company for a 90% protection is €4 402 k if the LOBs are independent. This amount is reduced by 31% if the copula of the company is Student with the aforementioned set of parameters.

The study of this table brings about two conclusions:

- Similar initial probabilities of ruin can occur despite significantly different ruin distributions.

This phenomenon is due to both the almost-zero initial amount of capital of the company, and some properties of the aggregated distribution of claim costs. In particular, LOBs 1, 2 and 4 have fat-tailed loss distributions. These distributions can thus generate losses, rather

frequently, that exceed the level of written premiums and investment income. In the absence of capital, any of them can ruin the company.

Moreover the Gumbel copula has a zero lower tail dependence, which means that low percentile losses (hence more frequent but of lesser relative intensity) are independent. This explains the similarity of the initial probability of ruin between the independence assumption and the Gumbel copula assumption.

From that point on, the distributions of capital insufficiencies are very different. The positive dependence as well as large upper tail dependence induced by the Gumbel copula make important and even extreme losses more likely to occur during the same accident year on different LOBs. This results in an increase of the required Economic Capital at given protection level compared to the independence assumption.

- Even for a lower initial probability of ruin, generated extreme losses may be more intense than anticipated by the independence assumption.

To illustrate this assertion, let have a closer look at the capital insufficiencies generated by the Student copula. The first remark is that this is the assumption set that provides the greatest reduction of the initial probability of ruin compared to the independence assumption. This relative reduction keeps decreasing as the level of protection sought increases, until it finally produces a capital insufficiency for high percentiles.

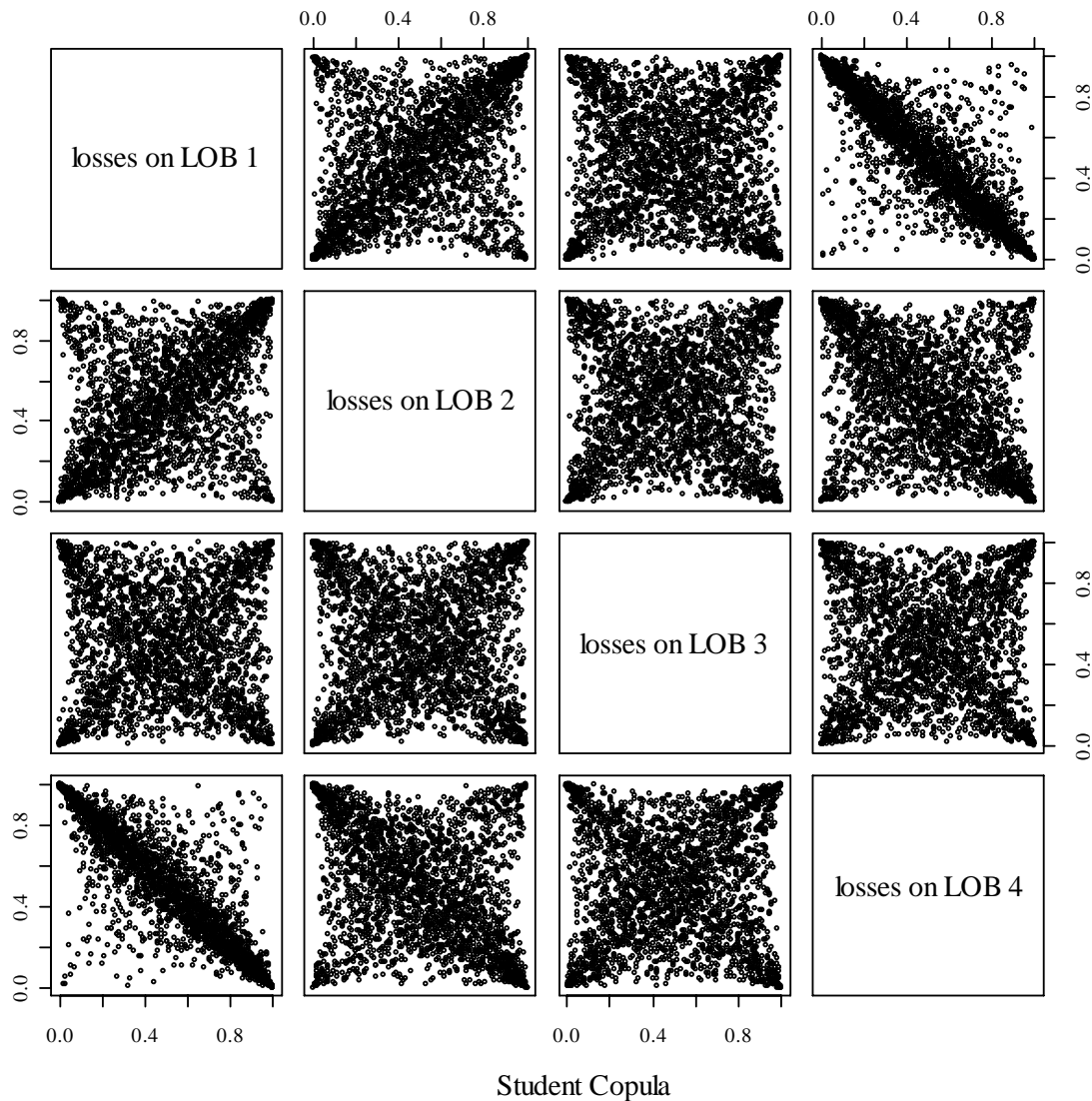
How is such a situation possible? To answer this, we must relook at some properties of the Student copula.

The dependence structure induced by this copula is very particular. For instance, it does not allow claim costs to be independent across LOBs. A comparison of the dependograms of claim costs for both the Normal and Student copula provides insight.

A dependogram is a graph in which each dot stands for a joint observation, let say the losses occurred for LOBs i and j during a same accident year. Each coordinate represents thus the rank of the considered loss for, let say LOB i , among all the losses that were ever experienced for that LOB. We then divide each coordinate by the number of observations we have for the considered LOB. It thus provides with a way of representing empirical percentiles. As the number of observations increases these empirical percentiles converge to the real percentile of the underlying distribution, which are the items the copula controls.

In the following graphs, we only focus on the 0.5% most severe claim costs for each LOB.

Dependogram of Technical Losses



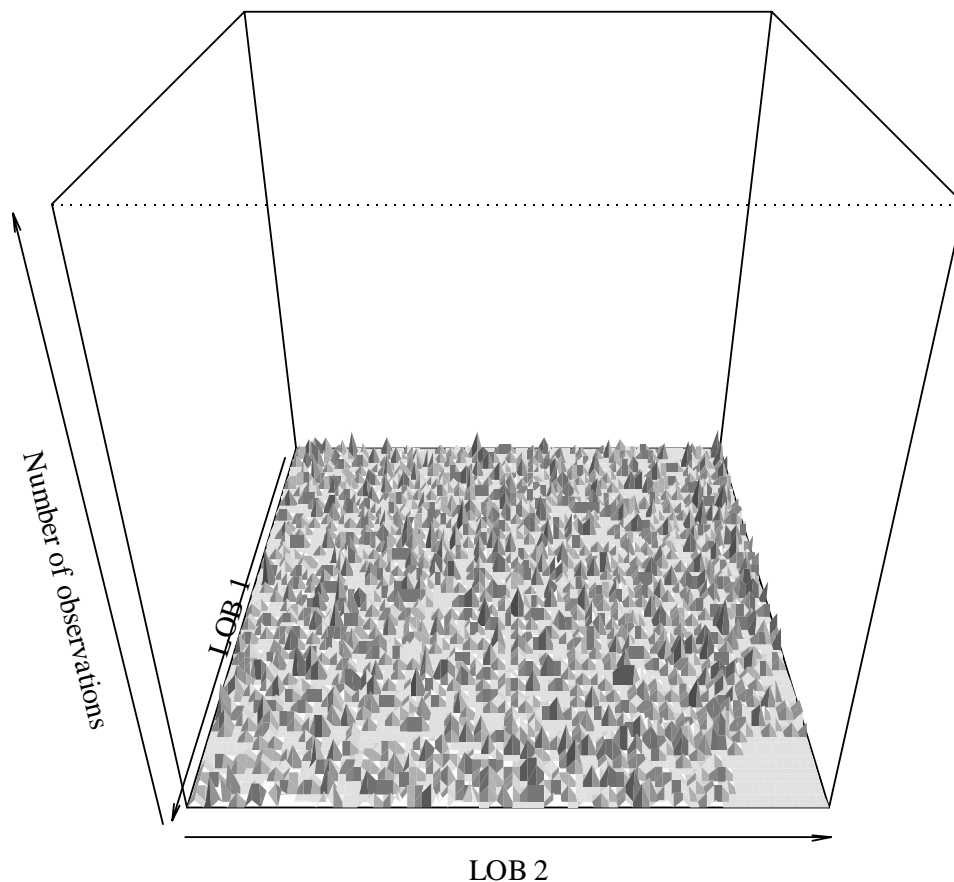
On the contrary the Student copula always expresses (except for a coefficient of the matrix equal to 1 or -1) a double trend, both positive and negative (cross form). So that even for LOBs 1 and 4 severe losses are likely to occur simultaneously.

Furthermore, with this parameterisation, the Student copula generates a relation of dependence that is more pronounced (dots are more aggregated over the main trend), which explains a tendency for induced technical results to compensate each other and so a substantial reduction of the initial probability of ruin.

Nevertheless, even if the initial situation is more favourable to the company than under the independence assumption, using a Student copula enables the generation of severe capital insufficiencies for high percentiles. This is the combined result of two properties of the copula: the aforementioned presence of a positive trend, and the existence of an upper tail dependence.

In order to get a better understanding of the situation for high percentiles, let consider LOBs 1 and 2 as well as the following series of graphs. These graphs are "modified dependograms": the 0.5% most intense ruins are represented and grouped by sections ($80 \times 80 = 6\,400$ squares). Considering these two LOBs is not as innocent a choice as it may seem: the ruin process is highly dependent on the choice of the marginals. By selecting two LOBs with similar loss distributions, we avoid any imbalance that may interfere with the properties of the copula.

Dependogram of Ruin Intensities



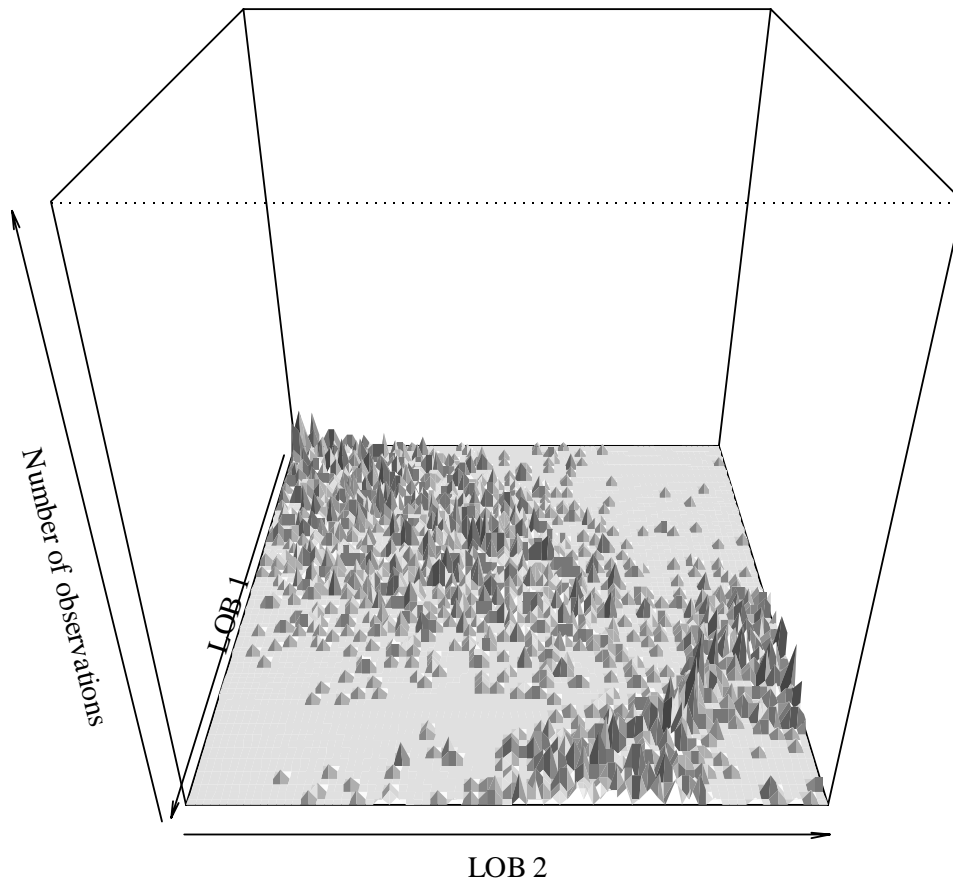
Independent Copula

For the independence assumption, the ruin intensities are uniformly distributed, except for an area that corresponds to simultaneous extreme events on both LOBs 1 and 2. We mentioned in 2.2 that the independence could also be expressed as a copula. This expression enables us to compute the upper tail dependence for that copula, which equals zero. This means that the probability for extreme losses to occur simultaneously on two LOBs equals zero. This however is an asymptotic property so that for any finite threshold α , the probability for a loss of percentile greater than α to occur is positive, but low. This implies that for a simulation process, the finite number of simulations will produce scarce, if any, stress scenarios.

For the assessment of the Economic Capital of a company, it means that such stress scenarios are not anticipated in the predicted ruin intensities.

The Normal copula also does not allow for these stress scenarios (except for linear correlation coefficients 1 and -1) as shown in the following graph:

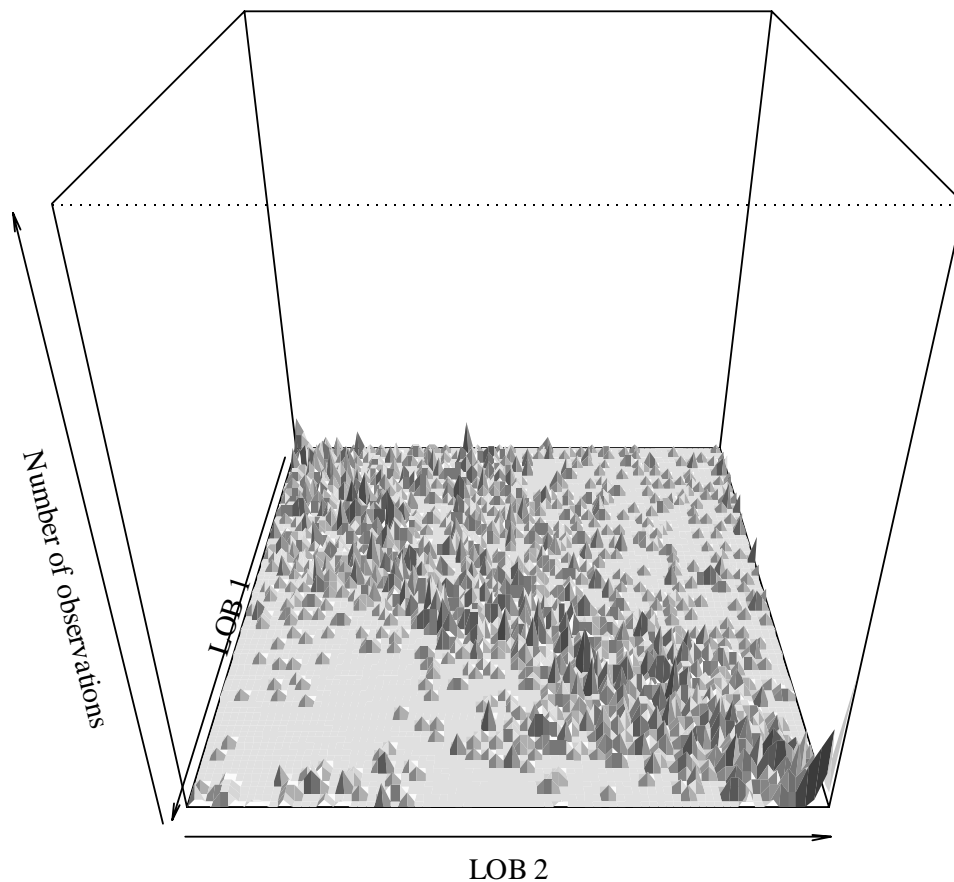
Dependogram of Ruin Intensities



Normal Copula

The Student copula on the contrary has a 56% upper tail dependence, which means that if an extreme loss impacts, let say, LOB 1 there is a 56% probability that an extreme loss also impacts LOB 2:

Dependogram of Ruin Intensities

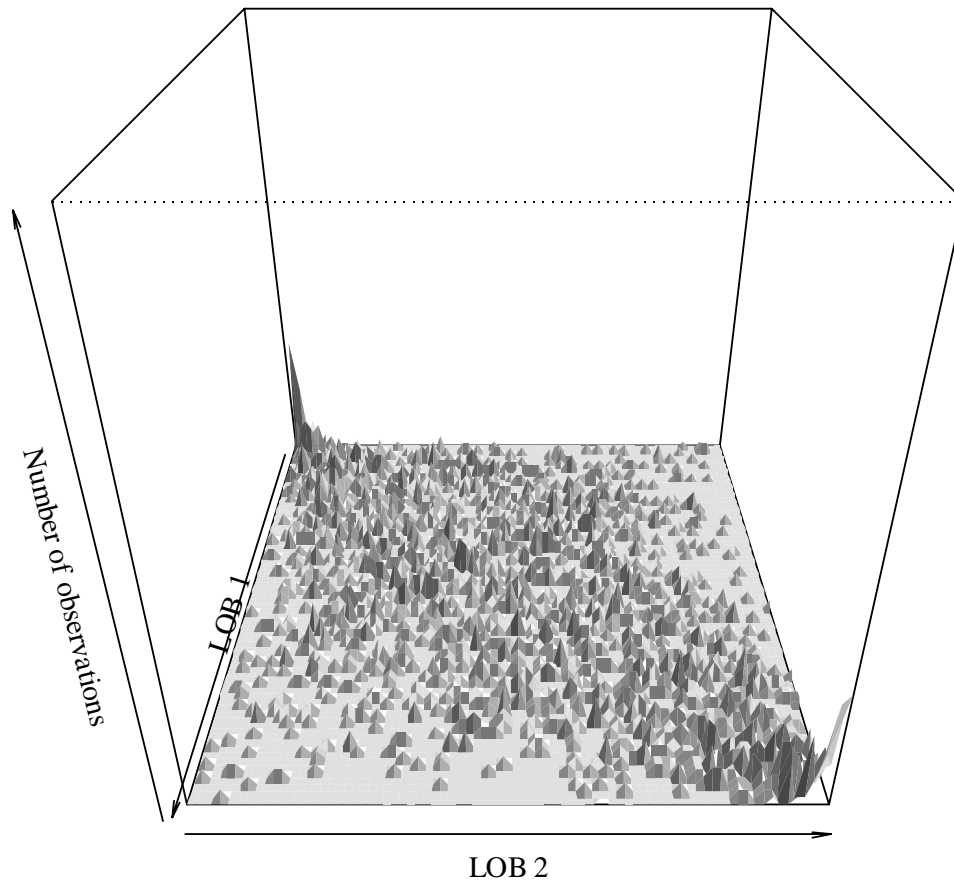


Student Copula

The absence of losses simultaneously severe on LOB 1 and relatively low on LOB 2 is related to their respective link with LOB 4, which has the heaviest distribution of claim costs. LOB 4 contributes significantly to reaching the 0.5% most severe ruins. For the "lowest" ruins, the pronounced negative dependence between LOBs 1 and 4 prevents medium to heavy losses for LOB 1 from occurring at the same time as medium to heavy losses on LOB 2.

For the Gumbel copula, the dependogram emphasises both its symmetric structure and the presence of a 58% (equation 16) upper tail dependence:

Dependogram of Ruin Intensities



Gumbel Copula

4 Conclusion

In this article we studied an Economic Capital assessment model for a four-LOB insurance company. We used copula theory to fully incorporate the impact of a potential dependence of claim costs between different LOBs. We stressed that using copulas also allowed each LOB to have a specific distribution of losses. These distributions could be different from one another, and did not necessarily belong to the same distribution family.

With this model we showed that the assumption that claim costs are independent from one LOB to another, when they are not, could result in an underestimation of the required Economic Capital for high percentiles.

In particular, among all the scenarios that could lead to a capital underestimation we pointed out that by using a Student copula, we can model a situation where attritional losses produce technical results that tend to compensate each other, while extreme losses tend to occur during the same accident year across different LOBs.

We also mentioned that even if the Gumbel copula may not reflect a realistic situation, it can produce stress scenarios useful to test the solvency of a company.

Potential developments of the present work may be a focus on inferential statistics for copulas, as well as improvements of the model including asset risk modelling, or a study of reinsurance programs.

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