Valuation of Bermudan-DB-Underpin Option

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Abstract

The study of embedded options has grown importance in pension design, with many novel forms of hybrid plan being proposed to meet the needs of employees and sponsors. In 2002, the State of Florida introduced (temporarily) a switching hybrid plan, under which employees were given the option to convert their defined contribution (DC) plans to defined benefit (DB) plans. The cost of such a switch is calculated in terms of the accumulated benefit obligation (ABO), which is the present value of the accrued benefit. If the ABO is greater than the DC account, the employee is assumed to fund the difference.

In this work we reconsider the switching hybrid plan, with additional downside protection for the employees. The new option is similar to the DB- Underpin hybrid design, also knowns as the floor-offset plan, but with a Bermudan-style exercise feature. We adopt an arbitrage-free pricing methodology to value the option, and specify the situation where early switch is not optimal.

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1 Two hybrid pendion plans: DB underpin and the Florida option

1.1 Introduction

Rising interest in risk sharing pension plans suggest the potential for new hybrid style pension designs. Hybrid pension plans combine aspects of both Defined Benefit (DB) and Defined Contribution (DC) plans, and may be designed to provide a more secure retirement benefit to employees compared with DC plans, while reducing the risks to employers compared with DB plans.

In this section, we will give a brief descriptions of two existing hybrid pension designs, the DB-underpin plan and the Florida's second election option. In both plans, as well as the new design explored in subsequent sections of this paper, we are considering true hybrid plans where employees have some combination of DC and DB benefits.

1.2 Notation and assumptions

We introduce some preliminary notation and assumptions.

c denotes the annual contribution rate (as proportion of salary) into the DC plan. We assume that contributions are paid annually. We also assume that all contributions are paid by the plan sponsor/employer, although this is easily relaxed to allow for employee contributions.

L_t denotes the salary from t to $t+1$ for $t = 0, 1, ..., T-1$, where t denotes years of service, and T denotes service at retirement. We assume that salaries increase deterministically at a rate μ_L per year, continuously convertible, so that

$$L_t = L_0 e^{\mu_L t}$$

r denotes the risk free rate of interest, convertible continuously.

b denotes the accrual rate in the DB plan.

$\ddot{a}(T)$ denotes the actuarial value at retirement of a pension of 1 per year.

S_t denotes the stochastic price index process of the funds in the DC account. We assume S_t follows a Geometric Brownian Motion, so that

$$\frac{dS(t)}{S(t)} = rdt + \sigma_S dZ_t^Q$$

1.3 DB-Underpin Plan

The DB Underpin pension plan, also known as the floor-offset plan in the North American, provides a guaranteed defined benefit (DB) minimum which underpins a DC plan. Plan sponsors make periodic contributions into the member's DC account, and separately contribute to the fund which covers the guarantee. Employees usually have limited investment options to make the guarantee value more predictable. At the retirement date, after T years of service, if the member's DC balance is higher than the guaranteed minimum defined benefit underpin, the plan sponsor will not incur any extra cost. However, if the DC benefit is smaller, the plan sponsor will cover the difference.

Using arbitrage-free pricing, we can calculate the present value of the cumulative cost of DB-underpin plan at time $t = 0$ as follows. We assume (more for clarity than necessity) that DB benefits are based on the final 1-year's salary, and we ignore all exits before retirement.

$$\underbrace{E^Q\left[\sum_{t=0}^{T-1} e^{-rt} cL_t\right]}_{\text{PV of Contribution to DC account}} \quad + \quad \underbrace{E^Q\left[e^{-rT}\left(bT\ddot{a}(T)L_{T-1} - \sum_{t=0}^{T-1}\frac{S_T}{S_t}cL_t\right)_+\right]}_{\text{Value of DB underpin option}}$$

Notice that the option value does not have an explicit solution, but can be determined using Monte Carlo simulation. See Chen and Hardy (2009) for details on the valuation and funding of the DB underpin option.

1.4 Florida Second Election Option

In 2002, public employees of the State of Florida were given an option to switch from their DC plan to DB plan anytime before their retirement date. The cost of participating in the DB plan is calculated by the accumulated benefit obligation (ABO), which is the present value of the accrued benefit, based on current service and pensionable salary.

We denote the ABO of the DB benefit for an employee with t years of service as K_t, so that

$$K_t = b\, L_{t-1}\, t\, \ddot{a}(T)\, e^{-r(T-t)}$$

Under the Florida Option hybrid plan, suppose the employee chooses to switch from DC to DB at time τ. If the ABO at τ is more than the DC account balance, the employee needs to fund the difference from her own resources. If the DC account is more than the ABO, at τ, then the employee retains the difference in a separate DC top-up account which can be withdrawn at retirement.

Mathematically, the present value of the total DC and DB benefit cost at inception is

$$\sup_{0\leq\tau\leq T} E^Q\left[\sum_{t=0}^{\tau-1} e^{-rt} c L_t + e^{-r\tau}\left(K_T e^{-r(T-\tau)} - K_\tau\right)\right]$$

where the first term, as in the previous section, is the present value of the DC contributions, and the second term is the additional funding required for the DB benefit, offset by the ABO at transition, which is funded from the DC contributions. The '$\sup$' indicates that the valuation assumes the switch from DC to DB is made at the optimal time to maximize the value of the benefits to the employee. Milevsky and Promislow (2004) studied the price and optimal switching time of the Florida option with deterministic assumptions.

2 Bermudan-style DB underpin plan

The Florida Option design has the advantage that it provides employees with the flexibility to choose their plan type based on their changing risk appetite. However, employees retain the investment risk through the DC period of membership, and also have the additional risk of a suboptimal choice of the switching time. Moreover, when the DC investment falls below the ABO of the DB plan, the employee may be unable to make up the difference.

Inspired by the idea of combining the DB-underpin plan with the Florida option, we investigate a new hybrid design, which adds a guarantee at the time of the switch from DC to DB. If the employee decides to switch at time τ, and her DC account is below the ABO at that time, then the plan sponsor will cover the difference.

2.1 Problem Formulation

We assume that contribution are made annually into the DC account until the employee switches to DB. Suppose we are valuing the benefits at a contribution time t. We let w_t denote the known account balance in the DC plan at t, immediately before the contribution at that date. Then at $t+\tau$, $\tau = 1, 2, ...$, we have stochastic process for future values of the DC account,

$$W_{t+\tau}|\mathcal{F}_t = w_t \frac{S_{t+\tau}}{S_t} + \sum_{u=t}^{\tau-1} \frac{S_{t+\tau}}{S_u} cL_u$$

where S_t is the price index process of the DC funds.

Assume that the employee has not switched from DC to DB before time t, and that the DC account is w_t at t. Then the present value at time t of the cost of future benefits (past and future service, DC and DB), which is denoted by the function $C(t, w_t)$, can be expressed in three parts:

1. The present value of the future contribution into the DC account before the member switches to DB plan.
2. The present value of the cost of the DB benefit, offset by the ABO at the time of the switch
3. The difference between the ABO and the DC balance, if positive.

We take the maximum value of the sum of these three parts, maximizing over all the possible switching dates, as follows.

$$\begin{aligned} C(t, w) = \sup_{0 \leq \tau \leq T-t} E_t^Q \Bigg[& \sum_{u=0}^{\tau-1} e^{-ru} cL_u + e^{-r\tau} \left(K_T e^{-r(T-t-\tau)} - K_{t+\tau} \right) \\ & + \quad e^{-\tau r} \left(K_{t+\tau} - W_{t+\tau} \right)_+ \Bigg| W_t = w \Bigg] \end{aligned}$$

Notice that the switching time τ is involved in all three parts, which makes the analysis more complex. However we can transform our problem into the simpler case given in equation

(1), using a put-call parity approach. Details are given in the appendix.

$$C(t,w) =$$
$$E_t^Q\left[K_T e^{-r(T-t)}\right] + \sup_{0\le\tau\le T-t} E^Q\left[e^{-\tau r}\left(W_{\tau+t} - K_{t+\tau}e^{-(T-t-\tau)r}\right)_+ \middle| W_t = w\right] - w \qquad (1)$$

The new formulation also consists of three terms:

- The PV of the DB plan benefits at time t
- A Bermudan type call option, with underlying W_t and strike K_t
- Offset by the available DC balance at t.

The first and third terms do not depending on the switching time, and the first part does not depend on the DC balance. To study the optimal exercising strategy, we omit the first and third part and define our value function as

$$\begin{aligned} v(t,w) &= \sup_{0\le\tau\le T-t} E_t^Q\left[e^{-\tau r}\left(W_{t+\tau} - K_{t+\tau}e^{-(T-t-\tau)r}\right)_+ \middle| W_t = w\right] \qquad (2)\\ &= \sup_{0\le\tau\le T-t} E_t^Q\left[e^{-\tau r}\left(\sum_{u=0}^{\tau-1}\frac{S_{t+\tau}}{S_{t+u}}cL_{t+u} + w\frac{S_{t+\tau}}{S_t} - b(t+\tau)\ddot{a}(T)L_{\tau+t-1}e^{-(T-\tau-t)r}\right)_+\right] \end{aligned}$$

We assume that the exercising dates are at the beginning of each year, before the contribution is made into the DC account, so the admissible exercising dates are $t = 0, 1, \cdots T$.

At time t, given that the DC account balance is w, we define the *exercising value* of the option, denoted $v^e(t,w)$, as the value if the member decided to switch at that date , and the *holding value*, denoted $v^h(t,w)$ as the value if the member decides not to switch. Then

$$\begin{aligned} v^e(t,w) &= (w - K_t)_+ \qquad (3)\\ &= \left(w - tbL_{t-1}\ddot{a}(T)e^{-(T-t)r}\right)_+ \qquad (4)\\ v^h(t,w) &= E_t^Q\left[e^{-r}v\left(t+1, (w+cL_t)\frac{S_{t+1}}{S_t}\right)\right] \qquad (5) \end{aligned}$$

The option is similar to a Bermudan call option, but with an underlying process following a generalized geometric Brownian motion. The option is not analytically tractable. In the next section will present some of the properties of the value function.

2.2 Characteristics of the value function

2.2.1 The value function at $T-1$, v_{T-1}

Since the option value at time $T-1$ depends only on the stock performance in the period $[T-1,T]$, it can be solved using the Black-Scholes Formula for the holding value

$$\begin{aligned}
v^h(T-1,w) &= E^Q\left[e^{-r}\left((w+cL_{T-1})\frac{S_T}{S_{T-1}} - bL_{T-1}T\ddot{a}(T)\right)_+\right] \\
&= N(d_1)(w+cL_{T-1}) - N(d_2)bL_{T-1}T\ddot{a}(T)e^{-r} \\
\text{where }\ d_1 &= \frac{1}{\sigma_S}\left[\ln\left(\frac{(w+cL_{T-1}}{bTL_{T-1}\ddot{a}(T)}\right) + \left(r+\frac{\sigma_S^2}{2}\right)\right] \\
d_2 &= d_1 - \sigma_S
\end{aligned}$$

thus

$$v(T-1,w) = \max\left(w - b(T-1)L_{T-2}\ddot{a}(T)e^{-r},\ v^h(T-1,w)\right)$$

2.2.2 The Exercise Frontier

The following proposition gives the general convexity and monotonicity of the value function

Proposition 1. *At each observation date $0 \le t \le T$, the value function $v(t,w)$ is a continuous, strictly positive, non-decreasing and convex function of w.*

The proof is given in Appendix.

Like other Bermudan- and American-type options, there exists a continuation region $\mathbf{C}$ and a stopping region (or exercising region) $\mathbf{D}$. When the time and DC account value pair, (t,w) is in the continuation region, it is optimal for the member to stay in the DC plan. Where (t,w) is in the stopping region, it is optimal for the member to switch to the DB plan. The option to switch is exercised when (t,w) moves into the stopping region. Mathematically, the continuation and stopping regions are defined as

- $v^h(t,w) > v^e(t,w) \Leftrightarrow (t,w) \in \mathbf{C}$
- $v^h(t,w) \le v^e(t,w) \Leftrightarrow (t,w) \in \mathbf{D}$

In order to value the option to switch we need to identify the continuation and stopping regions.

Consider first the case when $w < K_t$ at time t. We have $v^e(t,w) = (w - K_t)_+$, so

$$w < K_t \Rightarrow v^e(t,w) = 0$$

The value function, given in equation (2), is the expected value of a function bounded below by zero, and which has a positive probability of being greater than zero, which means that the expected value is strictly greater than zero. The holding function is the expected discounted value of the 1-year ahead value function (assuming the option is not exercised immediately), so it too must be strictly greater than 0.

So whenever $w < K_t$, we have $v^h(t,w) > v^e(t,w)$. Thus if $w \leq K_t$, it cannot be optimal to exercise. Therefore to explore the exercise frontier, we need only consider cases when $w > K_t$.

When $w > K_t$ we have $v^e(t,w) = w - K_t$

$$(t,w) \in \mathbf{D} \Rightarrow v(w,t) = v^e(t,w) \Rightarrow v(w,t) = w - K_t$$

and from Appendix B.0.3, $v(t,w) + h \geq v(t,w+h)$ for any $w \in \mathbf{R}$ and $h > 0$, which means that $v^h(t,w) + h \geq v^h(t,w+h)$ also.

First, assume that $w > K_t$ and that $(t,w) \in \mathbf{D}$,

$$\begin{aligned}
(t,w) \in D &\implies v^e(t,w) \geq v^h(t,w) \\
&\implies \forall h > 0 \quad v^e(t,w+h) = v^e(t,w) + h \geq v^h(t,w) + h \geq v^h(t,w+h) \\
&\implies (t,w+h) \in \mathbf{D}
\end{aligned}$$

Next, assume that $w > K_t$ and that $(t,w) \in \mathbf{C}$. Note that $v^h(t,w) + h \geq v^h(t,w+h)$ for all $h > 0$ implies that $v^h(t,w-h) \geq v^h(t,w) - h$

$$\begin{aligned}
(t,w) \in \mathbf{C} &\implies v^h(t,w) < v^e(t,w) \\
&\implies \forall h > 0 \quad v^h(t,w-h) \geq v^h(t,w) - h > v^e(t,w) - h = v^e(t,w-h) \\
&\implies (t,w-h) \in \mathbf{C}
\end{aligned}$$

These results show that it is optimal for the employee to switch to the DB plan only when his/her DC account balance is above a certain threshold at each possible switching time. Some numerical illustrations on the exercise boundary under a continuous setting will be presented in next section.

We let the function $\varphi(t)$ denote the boundary between the continuation and exercise regions. The results above can be summarized in the following proposition.

Proposition 2. *There exists a function $\varphi(t)$ such that*

$$v(t,w) = \begin{cases} v^e(t,w) & \textit{if } w \geq \varphi(t) \\ v^h(t,w) & \textit{if } w < \varphi(t) \end{cases}$$

Notice, it is possible, under certain parameters, that $\varphi(t) = \infty$ for some $t < T$, which means that it is not optimal to switch regardless of the DC account value. The next proposition will specify the situations where $\varphi(t) < \infty$.

Proposition 3. *The behavior of exercise boundary $\varphi(t)$ depending on the ratio $\frac{c}{b\ddot{a}(T)e^{-rT}}$*

i. If $\frac{c}{b\ddot{a}(T)e^{-rT}} < 1$, then $\varphi(t) < \infty, \forall t \in [0,T]$

ii. If $\frac{c}{b\ddot{a}(T)e^{-rT}} \geq 1$, there exists a $t_* \in [0,1,\cdots,T-1]$, such that $\varphi(t) = \infty, \forall t \leq t_*$, and $\varphi(t) < \infty, \forall t > t_*$.

(a) If $c > b\ddot{a}(T)\left((1-e^{-\mu_L})T + e^{-\mu_L}\right)e^{-r}$, $t_* = T-1$, which means $\varphi(t) = \infty$, $\forall t < T$

(b) If $\frac{c}{b\ddot{a}(T)e^{-rT}} = 1$, $t_* = 0$.

t_* is determined as $\lfloor t' \rfloor$, where t' satisfies

$$(t'+1)bL_{t'}\ddot{a}(T)e^{-r(T-t')} - t'bL_{t'-1}\ddot{a}(T)e^{-r(T-t')} - cL_{t'} = 0$$

The proof is given in the Appendix C. This proposition clearly demonstrate that the ratio between contribution rate to DC account and accrual rate of DB benefit will construct the shape of the exercise boundary. In extreme case, when contribution rate is much higher than the DB accrual rate, it is optimal for employee to wait until the retirement date, and the Bermudan-DB-underpin option simplifies to a DB-underpin plan.

3 Numerical Examples

Since the Bermudan-type DB switching option is not analytically tractable, numerical methods must be adopted. In this section, we evaluate the option using the least square method proposed by Longstaff and Schwartz (2001), and compare the cost of Bermudan-style DC/DB option with the floor offset from Chen and Hardy (2009), which is a European-style DC/DB option, and also with the Florida switching option.

The parameters are adopted from Milevsky and Promislow (2004) and Larrabee et al. (2016).

- $b = 0.016$, percentage value for people retire at age 65.
- $c = 0.125$, as adopted by Chen and Hardy (2009).
- $\mu_L = 0.0459$, assumed salary growth in Larrabee et al. (2016).
- $\sigma_S = 0.15$, $\ddot{a}(T) = 14.75$, $L_0 = 1$, $t = 0$, $W_0 = 0$ and $r = 0.04$.

3.1 Cost

Table 1 display the present value of each pension (option).

Time to Retirement	DB	DC	v(0,0)	Second Election	DB-Underpin
10yr	2.3911	1.2838	0.0089 (0.0001)	0	0.0031 (0.0012)
15yr	3.6941	1.9547	0.0409 (0.0003)	0	0.0186 (0.0021)
20yr	5.0729	2.6457	0.1078 (0.0006)	0.0287	0.0385 (0.0031)
30yr	8.0718	4.0903	0.3562 (0.0013)	0.2368	0.1062 (0.0055)
40yr	11.4165	5.6227	0.7460 (0.0024)	0.6095	0.2300 (0.0083)

Table 1: Cost of each pension plan

The values of three options, our Bermudan-DB-underpin, Florida's second election option and the DB-underpin plan, are all expressed as the value added to the DB plan. For example, the value of the Bermudan-DB underpin option is equal to $v(0, 0) = C(0, 0) - DB$. From the table, we can observe that

- For short term, all three options cost relatively insignificant compare to the DB plan, and Florida's second election has zero cost.
- For long term, Bermudan-DB-underpin and second election option cost much heavier than DB underpin, but at most 6.5% of the DB plan.
- The cost of Bermudan-DB-underpin can be greater than the sum of second-election option and DB underpin option.
- Increases in time to retirement greatly impact the value of the option. For example, the option prices for Bermudan-DB-underpin on a 40-year plan doubles a 30-year plan.

3.2 Sensitivity Tests

In this section, we present the sensitivity tests over five factors: c, μ_L, r, σ_S,b; and we set the time horizon to be 30 years. Details are displayed in Table 2 below.

Factor	Sensitivity Tests								
c	-0.04	-0.03	-0.02	-0.01	0.125	+0.01	+0.02	+0.03	+0.04
DC Value	2.7814	3.1086	3.4358	3.7630	4.0903	4.4175	4.7447	5.0719	5.3991
$v(0,0)$	0.0857	0.1355	0.1989	0.2703	0.3562	0.4503	0.5547	0.6687	0.7914
2^{nd} Election	0.0228	0.0571	0.1042	0.1641	0.2368	0.3206	0.4148	0.5190	0.6325
μ_L	-0.04	-0.03	-0.02	-0.01	0.459	+0.01	+0.02	+0.03	+0.04
DB Value	2.5304	3.3817	4.5194	6.0398	8.0718	10.7873	14.4165	19.2666	25.7484
$v(0,0)$	0.5296	0.4811	0.4339	0.3892	0.3587	0.3243	0.2977	0.2732	0.2543
2^{nd} Election	0.3433	0.3058	0.2770	0.2547	0.2368	0.2224	0.2081	0.1986	0.1885
DB underpin	0.4602	0.3522	0.2571	0.1769	0.1062	0.0771	0.0430	0.0319	0.0102
r	-0.04	-0.03	-0.02	-0.01	0.04	+0.01	+0.02	+0.03	+0.04
DB Value	26.7993	19.8534	14.7077	10.8958	8.0718	5.9797	4.4299	3.2817	2.4312
$v(0,0)$	0.0157	0.0443	0.1074	0.2112	0.3568	0.5106	0.6603	0.7935	0.8918
2^{nd} Election	0	0	0	0.0717	0.2368	0.4313	0.6156	0.7685	0.8859
σ_S	-0.04	-0.03	-0.02	-0.01	0.15	+0.01	+0.02	+0.03	+0.04
$v(0,0)$	0.2732	0.2895	0.3088	0.3303	0.3557	0.3833	0.4149	0.4440	0.4790
DB underpin	0.0225	0.0382	0.0635	0.0875	0.1062	0.1504	0.1709	0.2067	0.2645
b	-0.004	-0.003	-0.002	-0.001	0.016	+0.001	+0.002	+0.003	+0.004
DB Value	6.0538	6.5583	7.0628	7.5673	8.0718	8.5763	9.0808	9.5852	10.0897
$v(0,0)$	0.6091	0.5301	0.4641	0.4085	0.3564	0.3110	0.2720	0.2369	0.2064
2^{nd} Election	0.4899	0.4111	0.3439	0.2861	0.2368	0.1944	0.1576	0.1253	0.0986

Table 2: Sensitivity Test over (c, μ_L, r, σ_S and b)

The impact of each factor are summarized as below:

- Increase in contribution rate c would increase the value function as the employer is spending more money into DC account.
- Increase in salary growth would decrease the value function, but less significantly. The increases in ABO price is offset by the increase in DC contributions.
- Increase in market volatility would increase the price of the options.
- Increase in DB accrual rate b, would decrease the value function, since the fast accumulation of DB benefit would discourage employee from entering into DC account.

- Increase in risk-free rate r would increase the value function.

It is interesting to notice that the Bermudan-DB-underpin cost less than 10% of DB plan in most cases except for two. The first is when the salary growth is small, and the second is when the interest rate is high. However, in either case, the total cost that the employer will incur (sum of the DB benefit and the option value) is indeed smaller. The increases in the option value is offset by the decreases in the cost of DB benefit. One thing we want to point out is when the assumed interest rate is high, as adopted in Milevsky and Promislow (2004), the Bermudan-DB-underpin value is very close to the second-election option value. This is due to the fact that when the DC account is accumulating fast with higher interest rate, the guarantee will become costless comparing to the cost of the second election.

The variability in the option values shown in our sensitivity test, strongly suggest a need to adopt more sophisticated models (for example, interest rate model), and to consider more realistic assumptions (for example, different discounting rate for ABO calculations, employee contributions).

4 Conclusion and Future Work

In this paper, we discuss a new pension design, which combines the Florida's second election option and the DB-underpin option, to form a Bermudan-type DB-underpin plan. We summarize some key characteristics of the option, such as convexity and monotonicity. Also, we provide illustration on the behavior of early exercising region, and specifically include the situation that the Bermudan-DB-underpin will simplify into DB-underpin plan.

Our numerical illustration demonstrate that although the Bermudan-type DB underpin cost much more than the traditional DB-underpin plan or Florida's second election plan, it does not cost more than 10% of the DB plan in general. In cases when the cost of the option is high compare to DB benefit, we argue that the total costs incurred by the employer is indeed decreasing.

Here is a list of what we will study in the future

- Similar to DB-underpin in practice, the underlying DB plan is often smaller than a regular

DB benefit. Thus, we may set our guarantee applicable only to a certain proportion of the DB benefit (e.g. 90% of ABO is guaranteed, $0.9K_t$).

- Hybrid pension plan in continuous setting.

- Florida's Second Election option uses Accrued Benefit Obligation to calculate the price of switch, it maybe interesting to investigate using Projected Benefit Obligation, which we would believe to have large difference in both the shape of the exercising boundary as well as the option values.

- We may consider to adopt the actuarial valuation on Accrued Benefit Obligation, which often sets the discount rate different from market risk-free rate.

- Since the valuation is under arbitrage-free and complete market assumption, it may also interesting to assume salary and interest rate to be stochastic (where salary is assumed to be hedgable).

- Incorporate employee contribution c^* into wealth process maybe a more reasonable risk-sharing approach. It keeps the same contribution cost for the employer, but reduce the chance that employee will switch to DB plan with deficit, thus reduce the option value.

- The Bermudan-type DB-underpin maybe undesirable in the sense that the employers are bearing all the risk. It maybe interesting to study a similar option where employees are still guaranteed for the switch from DC to DB, but s/he must give up all his/her DC account regardless of its balance.

References and Notes

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A Cost Function

Here is the derivation of equation (1):

$$\begin{aligned}
C(t,w) &= \sup_{0\le\tau\le T-t} E^Q\Bigg[\sum_{u=0}^{\tau-1} e^{-ru}cL_{t+u} + e^{-r\tau}\left(K_T e^{-r(T-\tau-t)} - K_{t+\tau}e^{-(T-t-\tau)r}\right) \\
&\qquad + e^{-\tau r}\left(K_{t+\tau}e^{-(T-\tau-t)r} - W^{t,w}_{\tau+t}\right)_+ |\mathcal{F}_t\Bigg] \\
&= \sup_{0\le\tau\le T-t} E^Q\Bigg[\sum_{u=0}^{\tau-1} e^{-ru}cL_{t+u} + e^{-r\tau}\left(K_T e^{-r(T-\tau-t)} - K_{t+\tau}e^{-(T-t-\tau)r}\right) \\
&\qquad + K_{t+\tau}e^{-(T-t)r} - e^{-\tau r}W^{t,w}_{\tau+t} + e^{-\tau r}\left(W^{t,w}_{\tau+t} - K_{t+\tau}e^{-(T-t-\tau)r}\right)_+ |\mathcal{F}_t\Bigg] \\
&= \sup_{0\le\tau\le T-t} E^Q\Bigg[\sum_{u=0}^{\tau-1} e^{-ru}cL_{t+u} + K_T e^{-r(T-t)} - e^{-\tau r}W^{t,w}_{\tau+t} + e^{-\tau r}\left(W^{t,w}_{\tau} - K_{t+\tau}e^{-(T-t-\tau)r}\right)_+ |\mathcal{F}_t\Bigg]
\end{aligned}$$

To further reduce our equation, we need optional sampling theorem. First, observe that

$$\begin{aligned}
E^Q\left[e^{-(t-s)r}W^{s,w}_t|\mathcal{F}_s\right] &= E^Q\left[e^{-(t-s)r}\left(w\frac{S_t}{S_s} + \sum_{u=s}^{t-1}\frac{S_t}{S_u}cL_u\right)|\mathcal{F}_s\right] \\
&= w + e^{-(t-s)r}\sum_{u=s}^{t-1} e^{(t-u)r}cL_u = w + \sum_{u=s}^{t-1} e^{(s-u)r}cL_u
\end{aligned}$$

Define a new process X_t as

$$\begin{aligned}
X_t &= e^{-rt}W^{0,w}_t - \left(w + \sum_{u=0}^{t-1} e^{-ur}cL_u\right) \\
&= e^{-tr}w\frac{S_t}{S_0} - w + \sum_{u=0}^{t-1}\left(e^{-tr}\frac{S_t}{S_u} - e^{-ur}\right)cL_u
\end{aligned}$$

and it is easy to verify that X_t is a martingale:

$$\begin{aligned}
E^Q\left[X_t|\mathcal{F}_s\right] &= e^{-tr}w\frac{S_s}{S_0}e^{(t-s)r} - w + \sum_{u=0}^{s}\left(e^{-tr}\frac{S_s}{S_u}e^{(t-s)r} - e^{-ur}\right)cL_u + \sum_{u=s+1}^{t-1}\left(e^{-tr}e^{(t-u)r} - e^{-ur}\right)cL_u \\
&= e^{-sr}w\frac{S_s}{S_0} - w + \sum_{u=0}^{s-1}\left(e^{-sr}\frac{S_s}{S_u} - e^{-ur}\right)cL_u \\
&= X_s
\end{aligned}$$

Let $\tau \in [0, T-t]$ be any stopping time, by optional sampling theorem, we have

$$E^Q\left[X_{\tau+t}|\mathcal{F}_t\right] = X_t$$

$$\implies E^Q\left[\underbrace{e^{-(\tau+t)r}w\frac{S_{t+\tau}}{S_0} + \sum_{u=0}^{t+\tau-1}e^{-(t+\tau)r}\frac{S_{t+\tau}}{S_u}cL_u|\mathcal{F}_t}_{=e^{-(\tau+t)r}W_{t+\tau}^{t,W_t}}\right] = E^Q\left[w + \sum_{u=0}^{t+\tau-1}e^{-ur}cL_u|\mathcal{F}_t\right] + X_t$$

$$E^Q\left[e^{-\tau r}W_{t+\tau}^{t,W_t}|\mathcal{F}_t\right] = E^Q\left[\sum_{u=0}^{\tau-1}e^{-ur}cL_{u+t}|\mathcal{F}_t\right] + W_t$$

Substitute the last line into cost function, we have

$$\begin{aligned}
C(t,w) = \sup_{0\leq\tau\leq T-t}&\left\{E^Q\left[\sum_{u=0}^{\tau-1}e^{-ru}cL_{t+u}\right] - E^Q\left[e^{-\tau r}W_{\tau+t}^{t,w}|\mathcal{F}_t\right] + E^Q\left[K_Te^{-r(T-t)}\right]\right. \\
&\left.+E^Q\left[e^{-\tau r}\left(W_\tau - K_{t+\tau}e^{-(T-t-\tau)r}\right)_+\right]\right\} \\
&= \underbrace{E^Q\left[K_Te^{-r(T-t)}\right]}_{\text{The ABO of DB plan at t}} + \underbrace{\sup_{0\leq\tau\leq T-t}E^Q\left[e^{-\tau r}\left(W_{\tau+t}^{t,w} - K_{t+\tau}e^{-(T-t-\tau)r}\right)_+|\mathcal{F}_t\right]}_{\text{Price of the Option}} - w
\end{aligned}$$

B Characteristics of Value Function

Here is the proof of Proposition (1)

B.0.1 Value function is non-decreasing in w

For $h > 0$, we have $(x-k)_+ - (x+h-k)_+ \leq 0$, for all x, therefore,

$$
\begin{aligned}
&v(t,w) - v(t,w+h) \\
&\qquad \leq \sup_{0\leq\tau\leq T-t} E^Q\left[e^{-r\tau}\underbrace{\left\{\left(W^{t,w}_{t+\tau} - K_{t+\tau}e^{-(T-t-\tau)r}\right)_+ - \left(W^{t,w}_{t+\tau} + h\frac{S_{t+\tau}}{S_t} - K_{t+\tau}e^{-(T-t-\tau)r}\right)_+\right\}}_{\leq 0}\right] \\
&\qquad \leq 0
\end{aligned}
$$

since $h\frac{S_{t+\tau}}{S_t}$ is strictly positive a.s.. Notice, although value function is increasing in the initial DC balance, the cost function $C(t,w)$ is the opposite.

$$
\begin{aligned}
&C(t,w+h) - C(t,w) \\
&\leq \sup_{0\leq\tau\leq T} E^Q\left[e^{-r\tau}\left\{\left(W^{t,w}_{t+\tau} + h\frac{S_{t+\tau}}{S_t} - K_{t+\tau}e^{-(T-t-\tau)r}\right)_+ - \left(W^{t,w}_{t+\tau} - K_{t+\tau}e^{-(T-t-\tau)r}\right)_+\right\}\right] - h \\
&\leq \sup_{0\leq\tau\leq T} E^Q\left[e^{-r\tau}\left\{h\frac{S_{t+\tau}}{S_t} + \left(W^{t,w}_{t+\tau} - K_{t+\tau}e^{-(T-t-\tau)r}\right)_+ - \left(W^{t,w}_{t+\tau} - K_{t+\tau}e^{-(T-t-\tau)r}\right)_+\right\}\right] - h \\
&= 0
\end{aligned}
$$

B.0.2 Holding value function and exercise function is non-decreasing in w

For $x > y$,

$$
\begin{aligned}
v^h(t,x) - v^h(t,y) &= e^{-r}E^Q\left[\underbrace{v\left(t+1,(x+cL_t)\frac{S_{t+1}}{S_t}\right) - v\left(t+1,(y+cL_t)\frac{S_{t+1}}{S_t}\right)}_{\geq 0 \text{ from previous subsection}}\right] \\
&\geq 0
\end{aligned}
$$

Thus, v^h is increasing in DC account balance. It is also easy to see that

$$
\begin{aligned}
v^e(t,x) - v^e(t,y) &= (x - K_te^{-(T-t)r})_+ - (y - K_te^{-(T-t)r})_+ \\
&\geq 0
\end{aligned}
$$

B.0.3 Continuity of value function in w

For $x > y$, using the fact that $\sup[X]-\sup[Y] \leq \sup[X-Y]$ and $(x-k)_+-(y-k)_+ \leq (x-y)_+$, we have

$$\begin{aligned}
&|v(t,x) - v(t,y)| \\
&\leq \left| \sup_{0\leq\tau\leq T-t} E^Q \left[e^{-\tau r} \left(x\frac{S_{t+\tau}}{S_t} + \sum_{u=0}^{\tau} \frac{cS_{t+\tau}L_{u+t}}{S_{u+t}} - K_{t+\tau}e^{-(T-\tau-t)r} \right)_+ \right.\right. \\
&\qquad\qquad \left.\left. -e^{-\tau r} \left(y\frac{S_{t+\tau}}{S_t} + \sum_{u=0}^{\tau} \frac{cS_{t+\tau}L_{u+t}}{S_{u+t}} - K_{t+\tau}e^{-(T-\tau-t)r} \right)_+ \middle| \mathcal{F}_t \right] \right| \\
&\leq \sup_{0\leq\tau\leq T-t} E^Q \left[e^{-\tau r} \left((x-y)\frac{S_{t+\tau}}{S_t} \right) \right] \\
&\leq (x-y)
\end{aligned}$$

Thus, v is continuous in w and clearly

$$v(t, w+h) \leq v(t,w) + h \tag{6}$$

B.0.4 Convexity of value function

We follow similar to Hatem et al. (2002), and prove the convexity by induction. For any $w_1 > 0$ and $w_2 > 0$, and $0 \leq \lambda \leq 1$

$$\begin{aligned}
v^h(T-1, \lambda w_1 + (1-\lambda)w_2) &= E^Q \left[e^{-r} \left((\lambda w_1 + (1-\lambda)w_2 + cL_{T-1})\frac{S_T}{S_{T-1}} - bL_{T-1}T\ddot{a}(T) \right)_+ \right] \\
&\leq \lambda v^h(T-1, w_1) + (1-\lambda)v^h(T-1, w_2)
\end{aligned}$$

Thus, v^h satisfy the convexity at time T-1, and similarly v^e at time T-1. Since

$$v(T-1, w) = \max\left(v^e(T-1,w), v^h(T-1,w)\right)$$

v is also convex at time T-1.

We now assume that result hold for time $t+1$, where $0 \leq t \leq T-2$, then holding value at time t is

$$
\begin{aligned}
&v^h(t, \lambda w_1 + (1-\lambda) w_2) \\
&\quad = E^Q\left[e^{-r} v\left(t+1, (\lambda w_1 + (1-\lambda) w_2 + cL_t)\frac{S_{t+1}}{S_t}\right)\right] \\
&\quad \leq E^Q\left[e^{-r} \lambda v\left(t+1, (w_1 + cL_t)\frac{S_{t+1}}{S_t}\right)\right] + (1-\lambda) E^Q\left[e^{-r} \lambda v\left(t+1, (w_2 + cL_t)\frac{S_{t+1}}{S_t}\right)\right] \\
&\quad = \lambda v^h(t, w_1) + (1-\lambda) v^e(t, w_2)
\end{aligned}
$$

and since v^e holds the convexity for all t, thus, v^h is a convex function at t, which proves the convexity of v.

C Properties of $\varphi(t)$

This section provides proof for Proposition 3. First, notice that if $\varphi(t) < \infty$, then for sufficient large w

$$
\begin{aligned}
v(t, w) = v^e(t, w) \geq v^h(t, w) &= E\left[e^{-r} v\left(t+1, (w + cL_t)\,\frac{S_{t+1}}{S_t}\right)\right] \\
&\geq E\left[e^{-r} v^e\left(t+1, (w + cL_t)\,\frac{S_{t+1}}{S_t}\right)\right] \\
&= (w + cL_t)\, N\left(d_{1,t,w}\right) - (t+1) b L_t \ddot{a}(T) e^{-r(T-t-1)} e^{-r} N(d_{2,t,w})
\end{aligned}
$$

where

$$
\begin{aligned}
d_{1,t,w} &= \frac{1}{\sigma_S}\left[\ln\left(\frac{w + cL_t}{(t+1) L_t b \ddot{a}(T) e^{-(T-t-1)r}}\right) + \left(r + \frac{\sigma_S^2}{2}\right)\right] \\
d_{2,t,w} &= d_{1,t,w} - \sigma_S
\end{aligned}
$$

which is the Black-Scholes Formula, with initial stock price $w+cL_t$ and strike value $(t+1)bL_t\ddot{a}(T)e^{-r(T-t-1)}$. Here we define

$$\begin{aligned}
f(t) &= \lim_{w\to\infty} v^e(t,w) - E\left[e^{-r}v^e\left(t+1,(w+cL_t)\frac{S_{t+1}}{S_t}\right)\right] \\
&= \lim_{w\to\infty} v^e(t,w) - \big((w+cL_t)\,N\left(d_{1,t,w}\right) - (t+1)bL_t\ddot{a}(T)e^{-r(T-t-1)}e^{-r}N(d_{2,t,w})\big) \\
&= \lim_{w\to\infty} w - \big((w+cL_t)\,N\left(d_{1,t,w}\right) - (t+1)bL_t\ddot{a}(T)e^{-r(T-t)}N(d_{2,t,w})\big) - tbL_{t-1}\ddot{a}(T)e^{-r(T-t)} \\
&= \lim_{w\to\infty} -\big((t+1)bL_t\ddot{a}(T)e^{-r(T-t)}N(-d_{2,t,w}) - (w+cL_t)\,N\left(-d_{1,t,w}\right)\big) \\
&\qquad - cL_t + (t+1)bL_t\ddot{a}(T)e^{-r(T-t)} - tbL_{t-1}\ddot{a}(T)e^{-r(T-t)} \\
&= (t+1)bL_t\ddot{a}(T)e^{-r(T-t)} - tbL_{t-1}\ddot{a}(T)e^{-r(T-t)} - cL_t
\end{aligned}$$

Where second line to third is based on Put-Call Parity, and the third to fourth lines is based on the facts that $\lim_{w\to\infty} N(-d_{1,t,w}) = 0$, $\lim_{w\to\infty} N(-d_{2,t,w}) = 0$ and $\lim_{w\to\infty} wN(-d_{1,t,w}) = 0$. Clearly, whenever $\varphi(t) < \infty$, we have $f(t) \geq 0$.

I. If $\frac{c}{b\ddot{a}(T)e^{-rT}} < 1$, we prove $\varphi(t) < \infty, \forall t \in [0,T]$ by induction.
At time T, $\varphi(T) = TbL_{T-1}\ddot{a}(T) < \infty$.
At time t, assume $\varphi(t+1) < \infty$, first we observe that for sufficient large $w < \infty$, we have $v(t+1,w) = v^e(t+1,w)$. Next, we can show

$$\begin{aligned}
&\lim_{w\to\infty} v^h(t,w) - E\left[e^{-r}v^e\left(t+1,(w+cL_t)\frac{S_{t+1}}{S_t}\right)|\mathcal{F}_t\right] \\
=&\lim_{w\to\infty} E\left[e^{-r}\left(v\left(t+1,(w+cL_t)\frac{S_{t+1}}{S_t}\right) - v^e\left(t+1,(w+cL_t)\frac{S_{t+1}}{S_t}\right)\right)|\mathcal{F}_t\right] \\
=&0
\end{aligned}$$

The last line is due to the fact that if $w \geq \varphi(t+1)$

$$v\left(t+1,w\right) - v^e\left(t+1,w\right) = 0 < \varphi(t+1) < \infty$$

and for $w < \varphi(t+1)$

$$v\left(t+1,w\right) - v^e\left(t+1,w\right) \leq v^e\left(t+1,\varphi(t+1)\right) \leq \varphi(t+1) < \infty$$

since $v(t+1,w)$ is an increasing function of w (Appendix B.0.1). Thus, the difference is bounded by $\varphi(t+1) < \infty$ and we are able to apply the dominating convergence theorem.

Next,

$$\begin{aligned}\lim_{w\to\infty} v^e(t,w) - v^h(t,w) &= \lim_{w\to\infty} v^e(t,w) - E\left[e^{-r} v^e\left((w + cL_t)\,\frac{S_{t+1}}{S_t}\right)\right] \\ &= (t+1)bL_t \ddot{a} e_T^{-r(T-t)} - tb\ddot{a}(T)e^{-r(T-t)}L_{t-1} - cL_t \\ &> (t+1)bL_t \ddot{a} e_T^{-r(T-t)} - tb\ddot{a}(T)e^{-r(T-t)}L_{t-1} - L_t b\ddot{a}(T)e^{-rT} \\ &> 0\end{aligned}$$

Which implies $\varphi(t) < \infty$ (otherwise if $\varphi(t) = \infty$, $\lim_{w\to\infty} v^e(t,w) - v^h(t,w) \leq 0$). Thus $\varphi(t) < \infty, \forall t \in [0,T]$

II. For $\frac{c}{b\ddot{a}(T)e^{-rT}} \geq 1$, we split the proof into three parts: the first is when $\frac{c}{b\ddot{a}(T)e^{-rT}} > 1$, the second part and third part consider special cases when $c > b\ddot{a}(T)\left(\left(1 - e^{-\mu_L}\right)T + e^{-\mu_L}\right)e^{-r}$, and when $\frac{c}{b\ddot{a}(T)e^{-rT}} = 1$.

(a) If $\frac{c}{b\ddot{a}(T)e^{-rT}} > 1$, we first prove that there exists a t_* such that $\varphi(t) = \infty, \forall t \leq t_*$, then prove that $\varphi(t) < \infty, \forall t > t_*$ by induction from time T to $t_* + 1$ as in part (I).

i. Immediately we have $f(0) < 0$ that

$$\lim_{w\to\infty} v^e(0,w) - v^h(0,w) \leq f(0) < 0$$

and thus $\varphi(0) = \infty$. Also, notice we can write $f(t)$ in the form of $f(t) = e^{-\mu_L t}h(t)$ where $h(t)$ is a strict increasing function of time t. By the fact that $f(0) < 0$ and $f\left(\frac{\log\left(\frac{c}{b\ddot{a}(T)e^{-rT}}\right)}{r}\right) > 0$ where $\frac{\log\left(\frac{c}{b\ddot{a}(T)e^{-rT}}\right)}{r} > 0$ by assumption. Then, we can find t' such that

$$\begin{aligned}&f(t) < 0, t < t' \\ &f(t) = 0, t = t' \\ &f(t) > 0, t > t'\end{aligned}$$

Here we set $t_* = \lfloor t' \rfloor$, and we have

$$\lim_{w\to\infty} v^e(t,w) - v^h(t,w) \leq f(t) < 0, \forall t < t_*$$

and for $t = t_*$, first notice that

$$f(t_*) \leq 0 \implies c \geq b\ddot{a}(T)e^{-Tr}\left(t_* + 1 - t_* e^{-\mu_L}\right)e^{rt_*}$$

Next, for any finite $w > t_* L_{t_*-1} b\ddot{a}(T) e^{-r(T-t_*)}$, we have

$$\begin{aligned}
v^h(t_*, w) &= E^Q\left[e^{-r} v\left(t_*+1, \left(w + cL_{t_*}\right)\frac{S_{t_*+1}}{S_{t_*}}\right)\right] \\
&> E^Q\left[e^{-r}\left(\left(w+cL_{t_*}\right)\frac{S_{t_*+1}}{S_{t_*}} - (t_*+1)L_{t_*} b\ddot{a}(T) e^{-(T-t_*-1)r}\right)_+\right] \\
&\geq \max\left(0, w + cL_{t_*} - (t_*+1)L_{t_*} b\ddot{a}(T) e^{-(T-t_*)r}\right) \\
&\geq \max\left(0, w - t_* L_{t_*-1} b\ddot{a}(T) e^{-Tr} e^{t_* r}\right) \\
&= v^e(t_*, w)
\end{aligned}$$

The second to third line follows from Jensen's Inequality that

$$\begin{aligned}
&E^Q\left[e^{-r}\left(\left(w+cL_{t_*}\right)\frac{S_{t_*+1}}{S_{t_*}} - (t_*+1)L_t b\ddot{a}(T) e^{-(T-t_*-1)r}\right)_+\right] \\
&\geq \max\left(0, E^Q\left[e^{-r}\left(\left(w+cL_{t_*}\right)\frac{S_{t_*+1}}{S_{t_*}} - (t_*+1)L_t b\ddot{a}(T) e^{-(T-t_*-1)r}\right)\right]\right) \\
&= \max\left(0, w + cL_{t_*} - (t_*+1)L_t b\ddot{a}(T) e^{-(T-t_*)r}\right)
\end{aligned}$$

Thus, we have $v^h(t_*, w) > v^e(t_*, w), \forall w < \infty$, and $\lim_{t\to\infty} v^e(t_*, w) - v^h(t_*, w) \leq f(t_*) \leq 0$, which implies $\varphi(t_*) = \infty$.

ii. **At time T**, again we have $\varphi(T) < \infty$.
At time $t > t_*$, assume $\varphi(t+1) < \infty$. Exactly same as part (I), we have

$$\lim_{w\to\infty} v^e(t, w) - v^h(t, w) = f(t) > 0, t > t_*$$

Thus $\varphi(t) < \infty, \forall t > t_*$.

(b) If $c > b\ddot{a}(T)\left(\left(1 - e^{-\mu_L}\right)T + e^{-\mu_L}\right)e^{-r}$, immediately we notice that

$$\begin{aligned}
c &> b\ddot{a}(T)\left(\left(1 - e^{-\mu_L}\right)T + e^{-\mu_L}\right)e^{-r} \\
&= b\ddot{a}(T) e^{-r}\left(T - (T-1)e^{-\mu_L}\right) \\
&> b\ddot{a}(T) e^{-r} \geq b\ddot{a}(T) e^{-Tr}
\end{aligned}$$

Thus, we know there exist t_* as defined previously, and for time $T-1$,

$$\begin{aligned}
&\lim_{w\to\infty} v^e(T-1, w) - v^h(T-1, w) \\
&\quad = TbL_{T-1}\ddot{a}(T)e^{-r} - (T-1)bL_{T-2}\ddot{a}(T)e^{-r} - cL_{T-1} \\
&\quad < TbL_{T-1}\ddot{a}(T)e^{-r} - (T-1)bL_{T-2}\ddot{a}(T)e^{-r} - b\ddot{a}(T)\left(\left(1-e^{-\mu_L}\right)T + e^{-\mu_L}\right)e^{-r}L_{T-1} \quad = 0
\end{aligned}$$

Thus, $\varphi(t) = \infty, \forall t \leq T-1$, and the option simplifies to the DB-underpin option.

(c) When $\frac{c}{b\ddot{a}(T)e^{-rT}} = 1$, we have $f(0) = 0$. Thus $t_* = t' = 0$, immediately we have $\varphi(0) = \infty$.