Building and Testing Economic Scenario Generators for PC ERM

Gary Venter[1]

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Topics

- How to test economic scenario generators
- Advances in affine models of interest rates
- Equity and foreign exchange modeling
- Special issues for ERM

Specialized ESG Needs for PC ERM

- As part of economic capital modeling, PC companies simulate thousands of scenarios of their income statements and balance sheets, and the scenarios should be realistic and have a realistic distribution
- Most asset modeling is trading focused but PC ERM estimates probability distribution of price changes for a fixed portfolio over various time horizons; so real world vs. risk neutral
- ESGs model economic factor drivers such as interest rates, credit spreads, equity prices, inflation, foreign exchange
- Realism e.g. – arbitrage-free curves important: if scenarios allow arbitrage not realistic and can distort analysis; similar problem with forcing yield curve to follow smooth curves
- Want similar realism for economic factors that drive liability values, but might need both real world and risk neutral models for liability drivers like inflation if economic value of liabilities needed

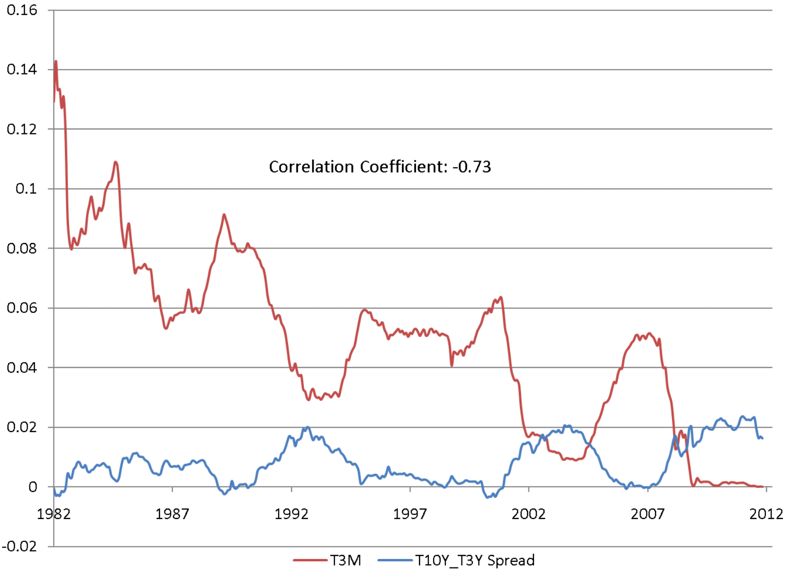
How To Evaluate ESGs

- Basically measure how well they capture statistical properties of historical processes
- Time-series properties
  - Autocorrelation
  - Moments of changes
  - Moments of volatility
  - Risks to excess profit potential of longer investments
- Cross-sectional properties
  - Distribution of yield curve shapes
  - Volatilities by maturity
- Measure both by simulations from model
- Even cross-sectional targets based on time-series history

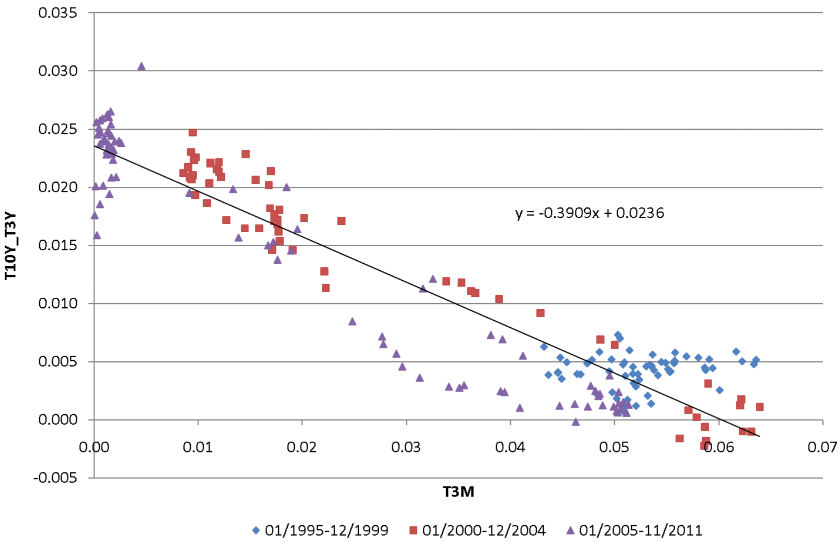
Some Historical Properties of Treasury Rates

- Rates highly autocorrelated
- Positive skewness and excess kurtosis, but both fairly modest
- Fluctuation around temporary levels that only slowly revert to long-term mean
- Rates tend to increase with maturity while volatilities decrease
- Longer rate spreads lower when short-rate is higher
- Volatility slowly reverts to long-term volatility but with occasional spikes
- As zero-coupon bonds mature, values tend to increase while rates decrease: risk issue for excess returns of longer bonds
  - This tends to be more so for longer bonds and when the yield curve is steeper

Yield curve Shapes Indicated by 10 - 3 Year Spreads



10 - 3 Year Spreads Conditional on 3 Month Rate



Modeling Choices

- Interest rates can be inferred from model of short-rate plus market price of risk (affine models) or modeled with correlated procss for all forward rates simultaneously (like Libor Market Model – LMM)
- Credit spreads often use a separate model but can be integrated into rate model
- Stochastic volatility can make time series more realistic but not always needed in cross-section
- Several approaches have been attempted for association of interest rates and inflation but none totally satisfactory
- FX correlations with each other complex and need more advanced copulas, but they are more volatile than most series so correlations with other series may be low enough to ignore in short time horizons

Affine Short Rate Models

- Diffusion, possibly jump diffusion, possibly multi-factor models for short-term rate process
- Appeal is that entire yield curve can be produced using a virtually closed form formula
- Needs risk-neutral short-rate process to price bonds to get real-world yield curve
- Risk-neutral process is real-world process plus a market-price of risk trend shift for each factor
- Some more recent generalizations (essentially affine, semi-affine, extended affine) build in more complicated price of risk formulas to get more realistic risk-neutral processes and yield curves

Affine Example - the CIR Model

$$dr(t) = \kappa[\theta - r(t)]\,dt + \eta\sqrt{r(t)}dB_r(t).$$

- The short rate at t is $r(t)$, reverting to mean θ at speed κ with volatility $\eta\sqrt{r(t)}$
- Square-root process cannot go negative because once r(t) gets to zero there is no stochastic effect and drift becomes $\kappa\theta$
- Yield curve derives from discounting along risk-neutral process that has drift increased by a market price of risk
- Same yields can be computed directly by closed-form formula
- Unfortunately this process is too simple to capture short-rate dynamics and yield-curve shapes are too limited
- Modeling short-rate as a sum of three unobserved CIR processes (but not two) gives adequate variety of yield curves
- That may produce temporary mean reversion as well but not stochastic volatility

Stochastic Volatility Example - the Chen Model

$$
\begin{aligned}
dv(t) &= \mu\left[\overline{v} - v(t)\right]dt + \eta\sqrt{v(t)}dB_v(t) \\
d\theta(t) &= \nu\left[\overline{\theta} - \theta(t)\right]dt + \zeta\sqrt{\theta(t)}dB_\theta(t) \\
dr(t) &= \kappa\left[\theta(t) - r(t)\right]dt + \sqrt{v(t)}dB_r(t).
\end{aligned}
$$

- The short rate at t is $r(t)$, reverting to temporary mean $\theta(t)$ at rate κ with volatility $\sqrt{v(t)}$
- $\theta(t)$ reverts to its mean $\overline{\theta}$ as a square-root process
- $v(t)$, volatility of r(t), also square-root process, reverting to $\overline{v}$
- Called an $A_1(3)$ process
  - Like actuarial notation, A doesn't mean anything, just a place to hang subscripts
  - 3 means 3 factor model, 1 indicates that 1 of the 3 factors impacts the volatility of r
  - $A_0(3)$ has no stochastic volatility and $A_3(3)$ does not allow some needed correlation among factors, but $A_2(3)$ used too

$A_2(3)$ Model

- Based on two unobserved driver processes Y_1 and Y_2, which are correlated square-root processes both > 0
- Volatility and reverting mean of the short rate are linear combinations of the two drivers, with non-negative coefficients
- Here θ and v are the reverting mean and volatility of r
- The notation refers to a 3 factor model where the volatility of rates is a function of 2 of them

$$\begin{aligned}\theta(t) &= a_\theta + b_\theta Y_1(t) + c_\theta Y_2(t) \\ v(t) &= a_v + b_v Y_1(t) + c_v Y_2(t)\end{aligned}$$

$A_2(3)$ Formulas

- The diffusions for Y_1 , Y_2 and r are:

$$
\begin{aligned}
dY_1(t) &= \kappa_{11}\left[\theta_1 - Y_1(t)\right]dt + \kappa_{12}\left[\theta_2 - Y_2(t)\right]dt + \sqrt{Y_1(t)}dB_1(t) \\
dY_2(t) &= \kappa_{21}\left[\theta_1 - Y_1(t)\right]dt + \kappa_{22}\left[\theta_2 - Y_2(t)\right]dt + \sqrt{\beta_{22}Y_2(t)}dB_2(t) \\
dr(t) &= \kappa_{33}\left[\theta(t) - r(t)\right]dt + \sqrt{v(t)}B_r(t) + \\
&\qquad p\sqrt{Y_1(t)}B_1(t) + q\sqrt{Y_2(t)}B_2(t)
\end{aligned}
$$

- Y_1 and Y_2 are square-root diffusions with interactions in their drifts but independent innovations
- r is the same as in the Chen model but it has interaction in its innovations that are related to Y_1 and Y_2

Bond Prices

- To get real-world bond prices you discount the payoff back to the valuation date using a risk-neutral short-rate future history
- We still want the real-world yield curve to be arbitrage-free
- Bond price is mean discounted value over all future paths
- The risk-neutral process is the real process plus a risk adjustment to the drift
- In the affine model this adjustment is a scalar but in the extended and semi affine models it is a linear function of the factors
- To get one simulation of the 30-year bond price at $t =$ one year from now you could simulate one simulation to t then many extensions of that out to 31 years and discount back
- The advantage of affine models and their generalizations is there is an almost closed form formula for the whole yield curve at the end of the first simulation to t

Semi-Affine Model

- Increase in drift of each factor to get risk-neutral process is a function of that factor and possibly other factors
- For $A_2(3)$ there are 11 parameters that determine the 3 adjusted drifts as functions of the current states of the factors
- Impossible to fit risky and risk-free curves with just 3 risk factors
- Market prices of risk evolve over time and are related to volatility of rates, level, slope and curvature of yield curve, all captured by semi-affine
- Relationship of volatility to level and across maturities also better captured
- Better captures variety of shapes of real-world and risk-neutral curves and relationship of shape to volatility
- Also more realistic moments of distributions of yields over time

Yield Curve Calculation

- Let $P(t, \tau)$ denote the price at time t of a zero-coupon bond paying 1 at time $t + \tau$
- The yield is the log of the inverse of the bond price
- For admissible affine models the bond prices are:

$$P(t, \tau) = exp\left[A(\tau) - B(\tau)' Y(t)\right] \tag{1}$$

where A and B are solutions of differential equations based on the parameters of the risk-neutral factors Y:

$$\begin{aligned} \frac{dA(\tau)}{d\tau} &= -\tilde{\theta}'\tilde{\kappa}' B(\tau) + \frac{1}{2}\sum_{i=1}^{N}\left[\Sigma' B(\tau)\right]_i^2 \alpha_i - \delta_0, \\ \frac{dB(\tau)}{d\tau} &= -\tilde{\kappa}' B(\tau) - \frac{1}{2}\sum_{i=1}^{N}\left[\Sigma' B(\tau)\right]_i^2 \beta_i + \delta_y. \end{aligned}$$

Key Recent Papers

- Lund, Andersen, Benzoni, 2004. "Stochastic Volatility, Mean Drift, and Jumps in the Short Rate Diffusion: Sources of Steepness, Level and Curvature," Econometric Society 2004 North America Winter Meetings 432, Econometric Society.
  - Models short-rate process as three-factor diffusion like Chen but with Poisson process jumps of mean zero normal size
  - More accurately represents stochastic volatility, which is highly persistent process but with occasional shocks
  - Also matches autocorrelation of rates and other features of the short-rate process
- Feldhtter, Peter, Can Affine Models Match the Moments in Bond Yields? (April 30, 2008)
  - Finds that semi-affine models capture stochastic and time-series properties of yield curves and simpler models do not
  - Risk to excess returns of longer investments, higher moments of rates, and volatility of rates of various maturities captured
  - Semi-affine gives control of relationship of real-world and risk-neutral rates to achieve this

Adding Credit Spreads to $A_2(3)$ Model

- Yield spreads are rates for defaultable corporate bonds minus Treasury rates, at different levels of default risk
- Now yield spreads as well as volatility and reverting mean of the short rate are linear combinations of the two drivers, with non-negative coefficients
- Here θ and v are the reverting mean and volatility of r, s is the risk-free to AA spread and u is the AA to BBB spread
- Add spreads to short-rate to get short risky rates and use same market prices of risk for the three factors to get the risky and risk-free yield curves

$$\begin{aligned}
\theta(t) &= a_\theta + b_\theta Y_1(t) + c_\theta Y_2(t) \\
v(t) &= a_v + b_v Y_1(t) + c_v Y_2(t) \\
s(t) &= a_s + b_s Y_1(t) + c_s Y_2(t) \\
u(t) &= a_u + b_u Y_1(t) + c_u Y_2(t)
\end{aligned}$$

Properties of Risky Bond Yields

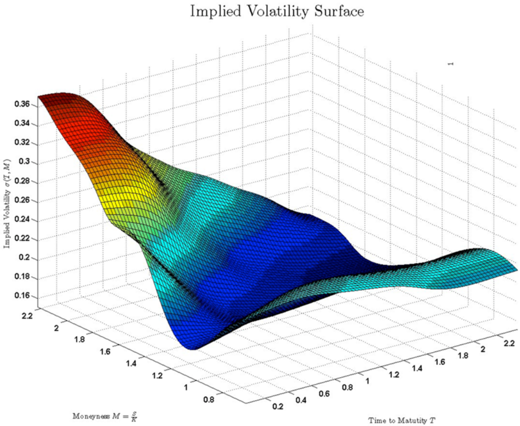
- Similar to those for Treasuries
- Longer yields are higher with lower variance
- Wide variety of yield-curve shapes
- Spread to Treasuries increases by maturity
- Higher short rate linked to a compression of spreads among longer maturities
- High correlation of Treasury and risky rates, but decreasing by maturity
- Correlation of lags of Treasury and risky rates still high but decreases by lag
- Spreads can move with or against Treasury rates, in part associated with inflation and economic activity

Adding Inflation

- Could add as a fourth factor Y_4 with mean and volatility linear functions of Y_1 and Y_2.
- Would need market price of risk for Y_4 to price instruments subject to inflation risk
- Market value of liabilities for instance
- Would simulate risk neutral versions of Y_1 and Y_2 using their market prices of risk use market price of risk of Y_4 for resulting risk-neutral inflation process.
- Could calibrate risk price to inflation-adjusted bonds
- Correlation of inflation and interest rates low over short periods but high in the long run

Equities - The Volatility Surface

- Problem with geometric Brownian motion - volatility provides all option prices but options give different implied volatilities
- Near-term options have higher implied volatilities as do out-of-the-money options, this for 8/6/10 from sctcm blog



Can Be Solved by Jump Diffusion

- Use separate Poisson frequencies for up and down jumps
- Also power-tail jump sizes for both up and down jumps (exponentials for logs)
- Steve Kou found options all consistently priced by one set of parameters
- For SP 500 estimated by MLE by Ramezani and Zeng, Maximum likelihood estimation of the double exponential jump-diffusion process, Annals of Finance (2007) 3:487
- Also can add stochastic volatility

Exchange Rate Modeling

- Look at numerous currencies' dollar exchange rates
- Models in literature include interest rate parity and purchasing power parity
- These assume that exchange rate expected movements maintain value of bond investment or pricing of available commodities
- Empirically do not work well
- Correlated AR1 processes seem to capture a lot of the movements of exchange rates
- Exchange rate volatility tends to be high so modeling error structure more important than modeling expected values
- Correlations and volatilities tend to change over time so perhaps looking at last ten years may be reasonable

Exchange Rate Correlation

- Rates across countries correlated with tail dependence, but could have high tail dependence with low overall correlation
- t-copula can't handle that but individuated t can: like t but each variate gets its own dof ν_n; also called grouped t copula as often several variates get the same dof.
- For two variates with the same dof ν, the tail dependence is $2t_{\nu+1}(-\sqrt{(\nu+1)(1-\rho)/(1+\rho)}$ as in the t-copula. If the dofs differ, it is between the values from the individual dofs, but closer to that of the higher. Denote:
- $h_n(y)$ = inverse chi-squared distribution with ν_n dof at y
- w_n = inverse of t-distribution with ν_n dof at u_n, over $\sqrt{\nu_n}$
- $J_{nm} = n, m$ element of inverse of correlation matrix ρ
- Then IT copula density at $(u_1, \ldots u_n)$ is:

$$\int_0^1 \frac{\prod_{n=1}^{N} \sqrt{(1+w_n^2)^{1+\nu_n} h_n(y)}\,\Gamma(\nu_n/2)\Gamma([1+\nu_n]/2)}{exp\left(\frac{1}{2}\sum_{n,m} J_{nm} w_n w_m \sqrt{h_n(y)h_m(y)}\right)\sqrt{det(\rho)2^N}}dy$$

FX Estimation

- Can compute density numerically for MLE
- Penalizing MLE for extra parameters led to only two dofs, one near 7 and one large, indicating normal-copula behavior for those
- Do not need dof to be integers, using beta distribution calculation of t-distribution
- Larger currencies like yen, Euro, pound were in group with 7 dof
- Perhaps large movements in these currencies tends to come from events that affect the dollar primarily, so occur together, whereas smaller currencies are more likely to have large idiosyncratic movements

Parameterization of Affine by SMM

- Parameters of 3 short-rate processes fit to short-rate history
- Current values of Y_1, Y_2, the 3 short-rates and market prices of risk calibrated to 3 current yield curves

$$
\begin{aligned}
\theta(t) &= a_\theta + b_\theta Y_1(t) + c_\theta Y_2(t) \\
v(t) &= a_v + b_v Y_1(t) + c_v Y_2(t) \\
u(t) &= a_u + b_u Y_1(t) + c_u Y_2(t) \\
s(t) &= a_s + b_s Y_1(t) + c_s Y_2(t).
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&\qquad p\sqrt{Y_1(t)}B_1(t) + q\sqrt{Y_2(t)}B_2(t)
\end{aligned}
$$

Simulated Method of Moments (SMM)

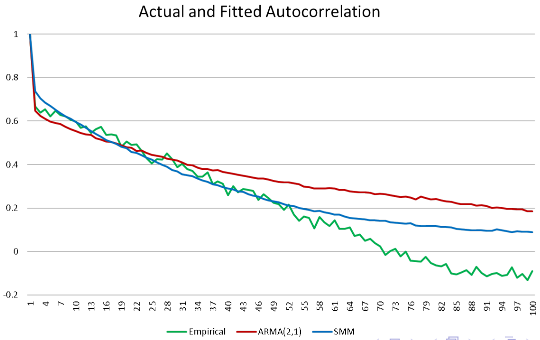
- Looks for parameters so that when they are used to simulate a long series, that series has statistical properties close to those of the historical series
- Any statistical property can be called a generalized moment
- Disadvantage is you make an ad hoc choice of properties to match so there is no real statistical inference
- Advantage is you fit better to historical properties than you do in more formal methods
- Also not too hard if you have a non-linear optimizer handy
- We fit moments and autocorrelation spectra of changes in rates and their absolute values, as well as cross-correlation spectra among the 3 series

Pros and Cons of SMM

- Not efficient in statistical sense – other estimators may have lower variance
- May be more robust – less sensitive to unsual history
  - "Efficient (estimation) may pay close attention to economically uninteresting but statistically well-measured moments." A Cross-Sectional Test of an Investment-Based Asset Pricing Model, John H. Cochrane, Journal of Political Economy, 1996
  - Comparing Multifactor Models of the Term Structure, Michael W. Brandt, David A. Chapman: "... the successes and failures of alternative models are much more transparent using economic moments. For example, it is easy to see that a particular model can match the observed cross- and auto-correlations but not the conditional volatility structure. In contrast, when models are estimated (by efficient methods), it is much more difficult to trace a model rejection to a particular feature of the data. In fact, the feature of the data responsible for the rejection may be in some obscure higher-order dimension that is of little interest to an economic researcher."

Medical CPI Modeled as Sum of Two AR1s Fit Both Ways

- Match moments of monthly movements and of absolute value of movements plus auto-correlation spectrum
- Fit slightly more general time series model by MLE, very similar to two AR1s by MCMC
- Data displays a slight long-term cycle model does not capture, so data not from this model, efficiency not necessarily key



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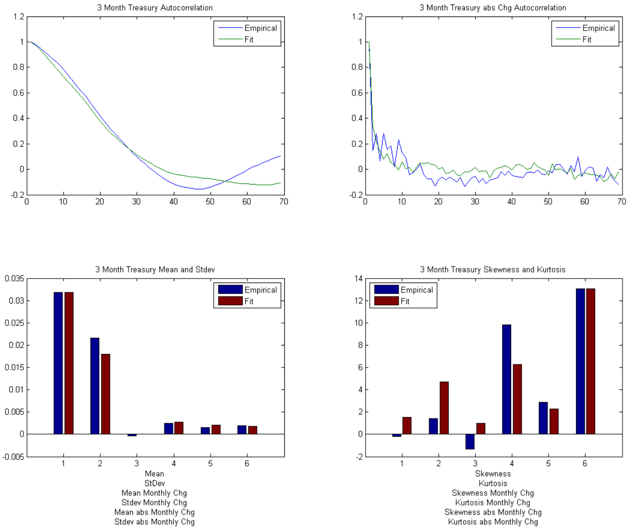
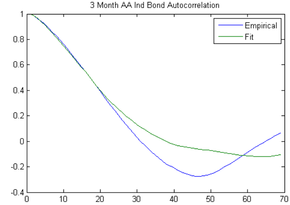
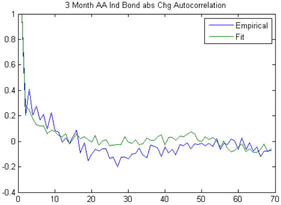
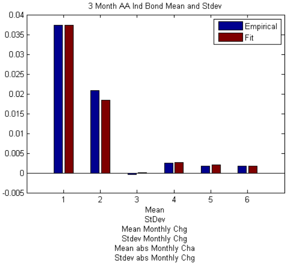
Three Month Treasury Fitting vs Empirical 1995-2011

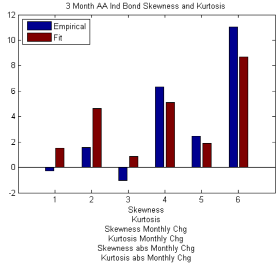


Three Month AA Ind Bond Fitting vs Empirical 1995-2011

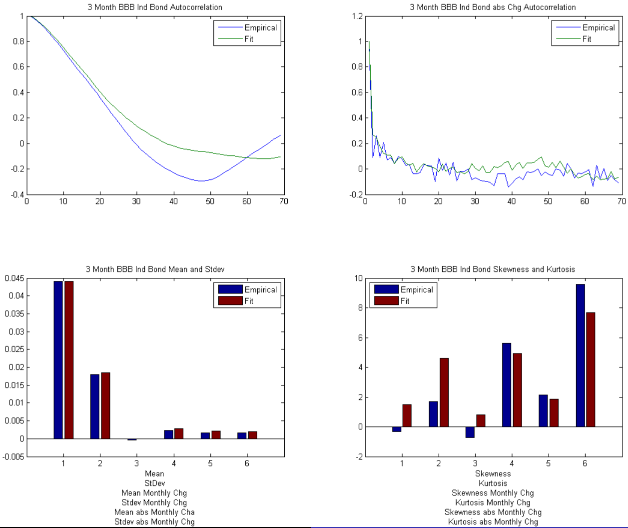




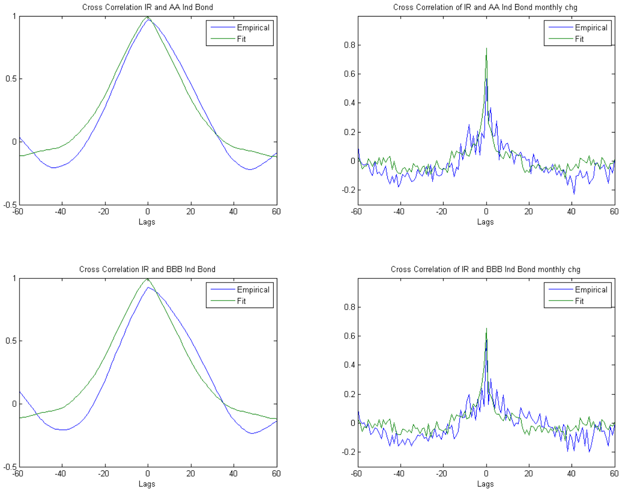




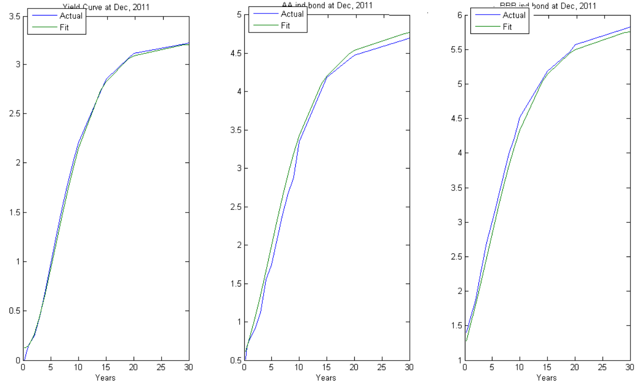
Three Month BBB Ind Bond Fitting vs Empirical 1995-2011



Three Month Treasury, AA and BBB Ind Bond Cross Correlation vs Empirical 1995-2011



Treasury, AA and BBB Ind Bond Yield Curve Fitting at Dec 2011



- Similar results when model extended to include Y_4 which is inflation, using Y_1 and Y_2 for reverting mean and volatility

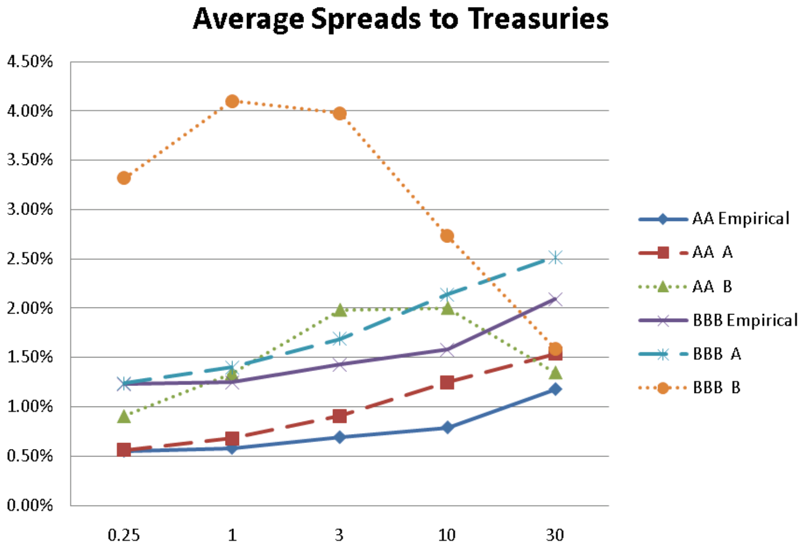
Fitting by MCMC

- Fits short rates, yield curves, and market price of risk parameters simultaneously for entire history
- Reasonable with semi-affine as model allows for changes in market prices of risk
- Starts with fairly wide prior distributions for all of the parameters and simulates Bayesian posterior of parameters conditional on the data
- After a while should converge to a set of parameters

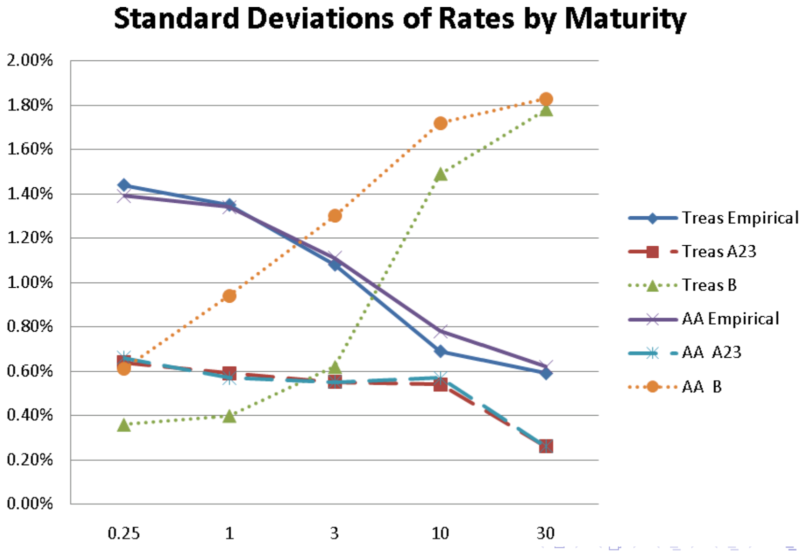
Testing Simulations

- Can test fit by comparisons with time series history
- Sometimes given only simulated scenarios from black-box models and want to test reasonableness of distribution of simulated results one year ahead
- Here did comparison of A23 fit by SMM, called model A, with a black-box output, call it model B
- This approach could be part of model validation

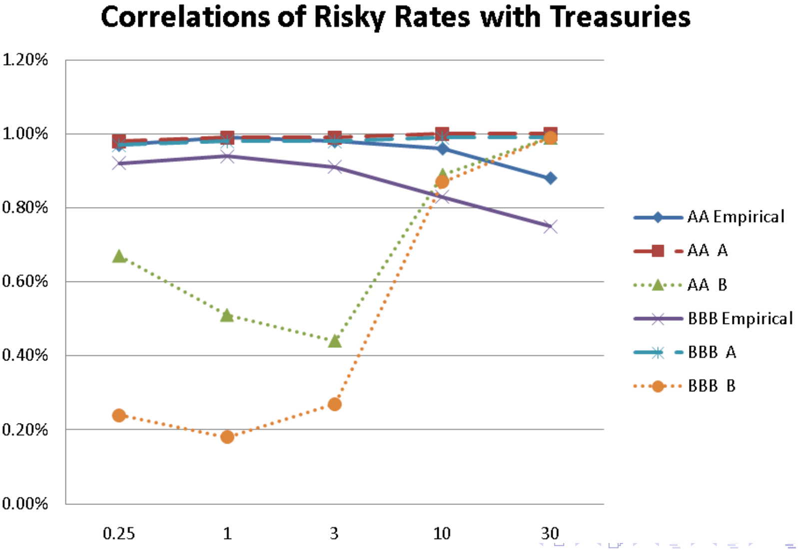
Year Ahead Forecast Compared to History and B Average Spreads



Year Ahead Forecast Compared to History and B Rate Volatility



Year Ahead Forecast Compared to History and B Rate Correlations



Shape Metrics

Table: 10-3 Spread vs. 3 month Slope and Spread

	Empirical	A	B
Slope T	-0.39	-0.39	0.94
Slope AA	-0.42	-0.41	-0.11
Slope BBB	-0.39	-0.41	0.40
Spread T	0.003	0.002	0.010
Spread AA	0.003	0.002	0.011
Spread BBB	0.003	0.001	0.012