Selecting the Best Pricing Model to Conform to a Country's Available Data

Uğur Karabey*, Şule Şahin* and Ayşe Arık*

*Department of Actuarial Sciences
Hacettepe University
Ankara, TURKEY

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Outline

Introduction

- Mortality modelling of Turkish mortality rates
- Studying on longevity risk
- **Aim:** to model the Turkish mortality rates using different mortality models, compare the model forecasts, price longevity bonds and measure the longevity risks of the bonds.

Data

- **Turkish Statistical Institute (TURKSTAT)**
- **Turkish census data** (1938-1995)
  - the number of deaths and matching person-years of exposure for each gender
  - 18 age groups and 5 year age bands for each 5- or 10-year periods
  - Yıldırım (2010) organised census data for each year by applying *Preston-Bennett method*
  - Obtained mortality data for different age groups (5 years) and years (1938 to 1995) have been used

Lee-Carter Model

LC method has a linear structure and can be defined in the following way:

$$\ln(\mu_{xt}) = \alpha_x + \beta_x \kappa_t + \epsilon_{xt} \tag{1}$$

- α_x: the logarithm of the geometric mean of the empirical mortality rates
- κ_t: the underlying time trend (general mortality level)
- β_x: the sensitivity of the hazard rate at age x
- ϵ_{xt}: age and time specific effects not captured by the model (the residual of a x-years old person at the time t)
- μ_{xt}: the observed central death rate of a x-year old person at time t

Lee-Carter Model (continued)

- ϵ_{xt} is assumed to be i.i.d. random variable with $N(0, \sigma_x^2)$
- No observable quantities on the right hand side
- In order the model to be identifiable two constraints suggested by Lee and Carter (1992) are usually applied:

$$\sum_{t=t_1}^{t=t_n} \kappa_t = 0 \qquad \sum_{x=x_1}^{x=x_k} \beta_x^2 = 1 \qquad \hat{\alpha}_x = \ln \prod_{t=t_1}^{t=t_n} \mu_{xt}^{\frac{1}{h}}$$

where $h = t_n - t_1 + 1$.

Problems with LC model

- certain pattern of change in the age distribution of mortality
- not readily accommodate extraneous information about future trends
- uncertainty arising from errors in the estimation of the β_x
- uncertainty about whether the future will look like the past

Poisson Log-Bilinear Model

- Poisson assumption for the random number of deaths, Brillinger (1986)
- Weighted least squares version of the original LC approach (Wilmoth, 1993)
- Brouhns et al. (2002a) and Renshaw and Haberman (2003b) describe how to implement the LC model in a Poisson error setting, using generalized linear models (GLMs) framework
- D_{xt} is modelled as independent Poisson response variable with λ systematic component:

$$E(D_{xt}) = e_{xt} exp(\nu_{xt}) \qquad V(D_{xt}) = E(D_{xt})$$

where $\nu_{xt} = \alpha_x + \beta_x \kappa_t$ and e_{xt} is the relevant exposure

Poisson Log-Bilinear Model (continued)

- Simulate $\hat{D}_{xt}$ by using Poisson distribution.
- Set starting values for α_x, β_x and κ_t and compute $\hat{D}_{xt}$
- Update α_x by $\hat{\alpha}_{x+1} = \hat{\alpha}_x + \frac{\sum_{all\ t} D_{xt} - \hat{D}_{xt}}{\sum_{all\ t} \hat{D}_{xt}}$
- Update κ_t by $\hat{\kappa}_{t+1} = \hat{\kappa}_t + \frac{\sum_{all\ t} (D_{xt} - \hat{D}_{xt})\hat{\beta}_x}{\sum_{all\ t} \hat{D}_{xt}\beta_x^2}$
- Update β_x by $\hat{\beta}_{x+1} = \hat{\beta}_x + \frac{\sum_{all\ t} (D_{xt} - \hat{D}_{xt})\hat{\kappa}_t}{\sum_{all\ t} \hat{D}_{xt}\hat{\kappa}_t^2}$

The Two-Factor Cairns-Blake-Dowd Model

Why (CBD)?

- Easy to incorporate parameter uncertainty
- Two correlated factors: level and slope
- Robust
- Tractable
- Biologically reasonable
- Allows us to simulate the distribution of survivor index under both real world and risk-neutral measures (for pricing) (www.cbdmodel.com)

The Two-Factor Cairns-Blake-Dowd Model (continued)

$$q(t+1,x) = \frac{\exp[A_1(t+1) + A_2(t+1)(x-\bar{x})]}{1+\exp[A_1(t+1) + A_2(t+1)(x-\bar{x})]} \tag{2}$$

- $q(t+1,x)$: realized mortality rate in year $t+1$ for individual aged x at time 0
- $\bar{x}$: mean of the range of ages used in the calibration (65-90) of the model

The Two-Factor Cairns-Blake-Dowd Model (continued)

- $A(t+1) = (A_1(t+1), A_2(t+1))$ is a random walk with drift such that

$$A(t+1) = A(t) + \mu + CZ(t+1) \tag{3}$$

- μ is a constant 2×1 vector of drift parameters
- C is a constant 2×2 lower triangular Choleski square root matrix of the covariance matrix V (that is $V = CC^T$)
- $Z(t+1)$ is a 2×1 vector of independent standard normal variables

Interest-rate model: Cox-Ingersoll-Ross (CIR)

$$dr(t) = \alpha(\bar{r} - r(t))dt + \sigma\sqrt{r(t)}d\tilde{W}(t) \tag{4}$$

- α: mean-reversion parameter
- $\bar{r}$: the risk-neutral long-term mean spot interest-rate
- σ: volatility parameter of the interest-rate
- $\tilde{W}(t)$ is a standard Brownian motion under a probability measure $\mathbb{Q}$

Interest-rate model: Cox-Ingersoll-Ross (CIR) (continued)

Why CIR?

- This model allows for interest rates to be mean-reverting where the long term mean equals $\bar{r}$
- Negative interest-rates are prevented
- Easy to simulate: if $r(T)$ follows a CIR process, then $(4\alpha r(T)/(\sigma^2(1 - \exp(-\alpha T))$, for given $r(0)$, has a non-central **chi-squared distribution** with $4\alpha\bar{r}/\sigma^2$ degrees of freedom and non-centrality parameter equal to $(4\alpha r(0)/(\sigma^2(\exp(\alpha T) - 1)$

Pricing

- Inspired by the pricing methodology of EIB/BNP (2004) longevity bond
- Annual coupon payments are proportionally linked to the survivor index of the reference population
- The initial price of the bond can be expressed as below:

$$V(0) = \sum_{T=1}^{25} P(0,T) \exp(T\delta) \hat{S}(T,x) \tag{5}$$

where $P(0,T)$ is the price at time 0 of a fixed-interest zero-coupon bond, δ is a spread. $\hat{S}(T,x)$ represent the projected survival rates where $\hat{S}(T,x) = \mathbb{E}_{\mathbb{P}}\left[S(T,x)|M_0\right]$ and M_0 is the relevant filtration.

Case Study-Scenario

Timeline for 25-year annuity (longevity bond):

		$\ldots$	$V(40)$	$S(41,25)$	$\ldots$	$S(65,25)$
$T=0$	1	$\ldots$	40	41	$\ldots$	65
$x=25$	26		65	66		90
$Y=2013$	2014		2053	2054		2078

Value of the 25-year annuity at time 40:

$$V(40) = \sum_{i=1}^{25} P(40, 40+i \mid r(40)) \mathbb{E}_{\mathbb{Q}} \left[\frac{S(40+i, 25)}{S(40,25)} \mid A(40) \right] \tag{6}$$

Assumptions: interest-rates and mortality rates are independent.
$S(T) = (1-q(0)) * (1-q(1)) * \ldots * (1-q(T-1))$
$q(t)$ is the mortality rate for aged 65 between t and $t+1$
$S(40,25) = 1$

Case Study - Survivor Index

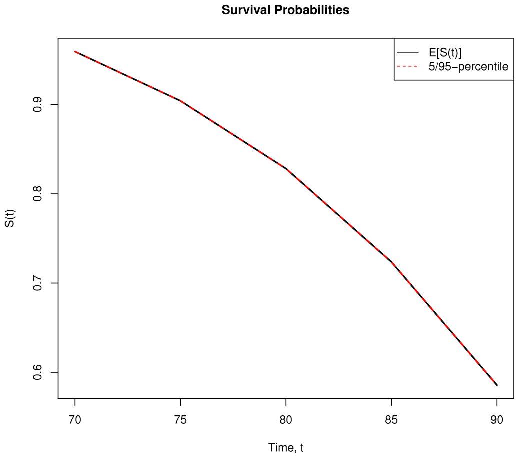

Figure 1 : *Survivor Index: Lee-Carter Model*

Case Study - Survivor Index

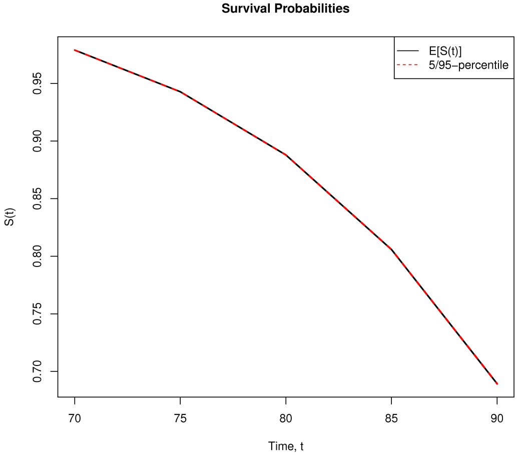

Figure 2 : *Survivor Index: Poisson-Log Bilinear Model*

Case Study - Survivor Index

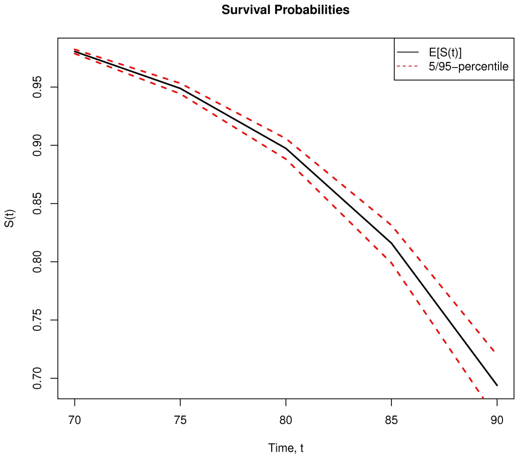

Figure 3 : *Survivor Index: CBD Model*

Case Study - Annuity Distributions

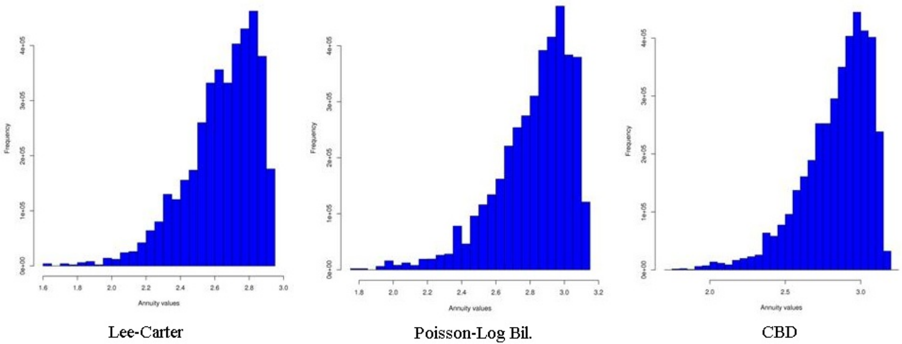

Figure 4 : *Annuity Distributions for All Models*

Case Study - Annuity Distributions

Table 1 : Descriptive Statistics and Risk Measures of Future Annuity Distributions in 40 years' Time for Different Models.

25-Year Annuities			
Measures	Lee-Carter	Poisson-Log Bil.	CBD
Mean	2.649	2.821	2.840
S.Deviation	0.209	0.223	0.226
Var.of.Coef.	0.079	0.079	0.080
Variance	0.044	0.050	0.051
Skewness	-1.071	-1.386	-1.158
VaR 95%	2.897	3.086	3.111
VaR 99.5%	2.931	3.123	3.157
ES 95%	2.914	3.105	3.133
ES 99.5%	2.934	3.125	3.167

Notes:

- In all cases, the future annuity values are obtained by taking the time 40 present values of later cash flows discounted at the relevant interest rate where these rates are obtained from the CIR interest-rate model.
- Parameter values of the mortality models are based on estimates of the Turkish mortality data over the period 1938-1995.
- The instantaneous spot interest-rate at $T=40$ is assumed to be equal to 0.04.

Conclusions

- High variability of survival probabilities under the CBD model makes that model better than other models
- Survival probabilities are overestimated under LC and PL model (less variability)
- Risk measures' sensitivity to different confidence levels show that CBD model has a fatter tail.
- The quality of the mortality data reduces the effectiveness of the results. However, under these circumstances CBD model moderately represents a better approximation.

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Thank you for your attention.