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Forward transition rates in multi-state models

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Agenda

Motivation (1)

- Forward rates are a well-known concept in the **interest rate** world.
- In the last decade: transfer to **mortality rates**.
- Forward mortality rates are discussed a lot in **literature**; e.g.
  - Bauer et al. (2012): Detailed analysis of forward mortality models.
  - Cairns et al. (2006): Discussion of forward mortality models.
  - Dahl (2004): Calculating premiums with forward mortality rates.
  - Miltersen and Persson (2005): Introduction of forward force of mortality without any dependency assumptions.
- Norberg (2010) makes the first attempt to define **forward rates in a multi-state model**.

Motivation (2)

Forward rates are a valuable concept, since forward rates ...

- ... have an **intuitive interpretation** as today's price of a future rate,
- ... are **easier to model** than e.g. the price of future probabilities,
- ... are **more practicable**, since they allow an easy and fast calculation.

Norberg (2010) shows the **limits** of forward rates:

- In contrast to the forward interest rate, the forward mortality rate is a **definition** and not a result.
- Problem: **example** where the forward mortality rate cannot be the same for a term insurance and a life annuity.
  $\Rightarrow$ Forward rates can **depend on the product**.

Goal

Our paper has **three objectives**:

(1) Formulation of a **sound definition of forward rates** in a multi-state model that takes into account the dependency on the insurance product types.

(2) Discussion of the **dependency** between mortality and interest rate in the well-known **active-dead model** to obtain unique forward rates.

(3) Discussion of the **dependency** in other models: **active-dead model with lapse**, **simple disability insurance**, and **joint life insurance**.

Definition of forward mortality rates

- E.g. Bauer et al. (2012), Cairns et al. (2006), Dahl (2004), Dahl and Møller (2006), and Milevsky and Promislow (2001) **define the forward mortality rate $\mu(t,T)$** as the $\mathcal{F}_t$-measurable solution of
$$\mathbb{E}\left(\mathrm{e}^{-\int_t^T m_u \,\mathrm{d}u}\middle|\mathcal{F}_t\right) = \mathrm{e}^{-\int_t^T \mu(t,u)\mathrm{d}u}\,.$$
- Only Miltersen and Persson (2005) define the **forward mortality rate either as** the solution of
$$\mathbb{E}\left(\mathrm{e}^{-\int_t^T r_u + m_u \,\mathrm{d}u}\middle|\mathcal{F}_t\right) = \mathrm{e}^{-\int_t^T \mu(t,u)+\rho_t(u)\mathrm{d}u} \tag{1}$$
or
$$\mathbb{E}\left(\int_t^T \mathrm{e}^{-\int_t^\tau r_u + m_u \,\mathrm{d}u}\, m_\tau \,\mathrm{d}\tau\middle|\mathcal{F}_t\right) = \int_t^T \mathrm{e}^{-\int_t^\tau \mu(t,u)+\rho_t(u)\mathrm{d}u}\,\mu(t,\tau)\mathrm{d}\tau\,. \tag{2}$$
At the same time, the **forward interest rate $\rho_t(T)$** is defined by
$$\mathbb{E}\left(\mathrm{e}^{-\int_t^T r_u \,\mathrm{d}u}\middle|\mathcal{F}_t\right) = \mathrm{e}^{-\int_t^T \rho_t(u)\mathrm{d}u}\,. \tag{3}$$

Problem with the definition of forward rates

Example 1: term insurance and life annuity

For all $T \geq t$ it should hold:

$$\mathbb{E}_Q\left(e^{-\int_t^T r_u \, du} \middle| \mathcal{F}_t\right) = e^{-\int_t^T \rho_t(u) du}$$

$$\mathbb{E}_Q\left(e^{-\int_t^T r_u + m_u \, du} \middle| \mathcal{F}_t\right) = e^{-\int_t^T \rho_t(u) + \mu(t,u) du}$$

$$\mathbb{E}_Q\left(\int_t^T e^{-\int_t^\tau r_u + m_u \, du} \, m_\tau \, d\tau \middle| \mathcal{F}_t\right) = \int_t^T e^{-\int_t^\tau \rho_t(u) + \mu(t,u) \, du} \, \mu(t,\tau) \, d\tau \, .$$

- By the first **two equations** $\rho_t(u)$ and $\mu(t,u)$ are **determined uniquely**.
- It **depends on r_u and m_u** if the the third product can also be included in the set M.
- E.g. for r_u and m_u **independent**, all three products can be included in M.
- Norberg (2010): **Example** where this does not work (r_u, m_u dependent).
- **Forward rates can depend on the product!**

Definition of general forward rates

Idea:

- We generalize the **substitution** concept.
- **Dependency** on the product type is included in the definition.

Definition: general forward rates

Let M be a set of mappings $F(t,T,r,m)$. We call $\rho : \{(t,u) : 0 \le t \le u\} \to \mathbb{R}$ the forward interest rate and $\mu : \{(t,u) : 0 \le t \le u\} \to \mathbb{R}^{|S|\times|S|}$ the forward transition rates of M with respect to r and m if $\rho(t,u)$, $\mu(t,u)$ are $\mathcal{F}_t$-measurable for all u, t with $u \ge t \ge 0$ and

$$\mathbb{E}_{\mathrm{Q}}\big(F(t,T,r,m)\big|\mathcal{F}_t\big) = F(t,T,\rho,\mu) \text{ for all } F \in M, T \ge t \ge 0\,.$$

Discussion of the active-dead model

Recall Example 1

For all $T \geq t$ it should hold:

$$\mathbb{E}_{\mathrm{Q}}\left(\mathrm{e}^{-\int_t^T r_u \,\mathrm{d}u}\middle|\mathcal{F}_t\right) = \mathrm{e}^{-\int_t^T \rho_t(u)\mathrm{d}u}$$

$$\mathbb{E}_{\mathrm{Q}}\left(\mathrm{e}^{-\int_t^T r_u + m_u \,\mathrm{d}u}\middle|\mathcal{F}_t\right) = \mathrm{e}^{-\int_t^T \rho_t(u)+\mu(t,u)\mathrm{d}u}$$

$$\mathbb{E}_{\mathrm{Q}}\left(\int_t^T \mathrm{e}^{-\int_t^\tau r_u + m_u \,\mathrm{d}u}\, m_\tau \,\mathrm{d}\tau\middle|\mathcal{F}_t\right) = \int_t^T \mathrm{e}^{-\int_t^\tau \rho_t(u)+\mu(t,u)\,\mathrm{d}u}\,\mu(t,\tau)\,\mathrm{d}\tau\,.$$

- For r_u and m_u **independent**, all three products can be included in M.
- Norberg (2010): **Example** where this does not work (r_u, m_u dependent).
- **Is independence necessary or only sufficient?**

Setting

Assumption 1

We assume that the transition intensities and the interest rate are processes of the form $m_i(t) = m_i(0) + \int_0^t \alpha^i(\tau, m_i(\tau))\,\mathrm{d}\tau + \int_0^t \beta^i(\tau, m_i(\tau))\,\mathrm{d}W^i_\tau$, where

- m_i is a Cox-Ingersoll-Ross process **or**
- α^i and β^i meet some week requirements as measurability, Lipschitz condition, linear growth bound, and an initial value condition.

Furthermore, we assume pairwise $(i \neq j)$:

(i) $[W^i, W^j]_t = \int_0^t \rho(s)\,\mathrm{d}s$, where $\rho(t)$ is continuous in $[0, T]$,

(ii) there is a constant $\epsilon > 0$ and a random variable Y with $Y \geq \mathrm{e}^{-\sum_{i=1}^n \int_{T_i} X_i(u)\mathrm{d}u}$ and $\mathbb{E}\left(|W|^{2(1+\epsilon)}\right) < \infty$ for all intervals $T_i \subseteq [0, T^*]$,

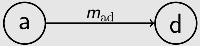
(iii) and $\mathrm{Q}(\beta_i(t, X_i(t))\,\beta_j(t, X_j(t)) = 0) < 1$ for all $i \neq j$ and $t \in [0, T^*]$.

Results

We get the following result for the active-dead model (see Example 1).

Theorem 1: necessary condition for active-dead model

We assume that r_u and $m_{\mathrm{ad}}(u)$ fulfill Assumption 1 and that the forward rates $\rho_t(T)$ and $\mu(t, T)$ fulfill equations (1), (2), and (3).



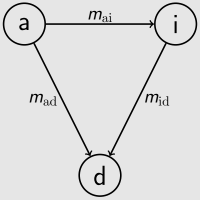
$\Rightarrow$ r and m_{ad} **must** be **independent**.

What about **other models**, as a model with lapse, a simple disability insurance, and a joint life insurance? Does this still hold?

Discussion of the independence assumption (1)

Example 2: simple disability insurance

Set M includes:



- **standardized products**

  $$\mathbb{E}_{\mathrm{Q}}\left(\mathrm{e}^{-\int_t^T r(\tau)\mathrm{d}\tau} \middle| \mathcal{F}_t\right)$$

  $$\mathbb{E}_{\mathrm{Q}}\left(\mathrm{e}^{-\int_t^T m_{\mathrm{x}}(\tau)+r(\tau)\mathrm{d}\tau} \middle| \mathcal{F}_t\right),\ x \in \{\mathrm{ad}, \mathrm{ai}, \mathrm{id}\}$$

- **benefits** in state a / i

  $$\mathbb{E}_{\mathrm{Q}}\left(\mathrm{e}^{-\int_t^T m_{\mathrm{ad}}(\tau)+m_{\mathrm{ai}}(\tau)+r(\tau)\mathrm{d}\tau} \middle| \mathcal{F}_t\right)$$

  $$\mathbb{E}_{\mathrm{Q}}\left(\int_t^T \mathrm{e}^{-\int_t^\tau m_{\mathrm{ad}}(u)+m_{\mathrm{ai}}(u)+r(u)\mathrm{d}u}\, m_{\mathrm{ai}}(\tau)\, \mathrm{e}^{-\int_\tau^T m_{\mathrm{id}}(u)+r(u)\mathrm{d}u} \mathrm{d}\tau \middle| \mathcal{F}_t\right)$$

- **benefits** for transition between states

  $$\mathbb{E}_{\mathrm{Q}}\left(\int_t^T \mathrm{e}^{-\int_t^\tau m_{\mathrm{ad}}(u)+m_{\mathrm{ai}}(u)+r(u)\mathrm{d}u}\, m_{\mathrm{ai}}(\tau)\, \mathrm{d}\tau \middle| \mathcal{F}_t\right)$$

  $$\mathbb{E}_{\mathrm{Q}}\left(\int_t^T \mathrm{e}^{-\int_t^\tau m_{\mathrm{ad}}(u)+m_{\mathrm{ai}}(u)+r(u)\mathrm{d}u}\, m_{\mathrm{ad}}(\tau)\, \mathrm{d}\tau \middle| \mathcal{F}_t\right)$$

  $$\mathbb{E}_{\mathrm{Q}}\left(\int_t^T \mathrm{e}^{-\int_t^\tau m_{\mathrm{id}}(u)+r(u)\mathrm{d}u}\, m_{\mathrm{id}}(\tau)\, \mathrm{d}\tau \middle| \mathcal{F}_t\right)$$

Discussion of the independence assumption (2)

Example 2: simple disability insurance (continued)

Set M includes the products from the last slide.

Assumption: r, m_{ad}, m_{ai}, and m_{id} are $\mathcal{F}_t$-independent.

$$\Rightarrow \mathbb{E}_{\mathrm{Q}}\left(\mathrm{e}^{-\int_t^T m_{\mathrm{id}}(u)\mathrm{d}u}\,\middle|\,\mathcal{F}_t\right) = \mathbb{E}_{\mathrm{Q}}\left(\mathrm{e}^{-\int_t^\tau m_{\mathrm{id}}(u)\mathrm{d}u}\,\middle|\,\mathcal{F}_t\right)\mathbb{E}_{\mathrm{Q}}\left(\mathrm{e}^{-\int_\tau^T m_{\mathrm{id}}(u)\mathrm{d}u}\,\middle|\,\mathcal{F}_t\right)$$

$\Rightarrow$ Under Assumption 1: m_{id} is **deterministic**!

$\Rightarrow$ Independence assumption is **not appropriate**.

$\Rightarrow$ **Other dependency** structure is needed.

Dependency structure

We consider the following dependency structure:

Dependency structure for a simple disability insurance

- r_s is conditionally independent of $m_{\mathrm{ai}}(s)$, $m_{\mathrm{id}}(s)$, and $m_{\mathrm{ad}}(s)$
- $m_{\mathrm{ai}}(s)$, $m_{\mathrm{id}}(s)$ and $m_{\mathrm{ad}}(s) - m_{\mathrm{id}}(s)$ are conditionally independent

$$\begin{pmatrix} m_{\mathrm{aa}}(u) & m_{\mathrm{ai}}(u) & m_{\mathrm{ad}}(u) \\ 0 & m_{\mathrm{ii}}(u) & m_{\mathrm{id}}(u) \\ 0 & 0 & m_{\mathrm{dd}}(u) \end{pmatrix}$$

The **independence assumption of $m_{\mathrm{id}}(s)$ and $m_{\mathrm{ad}}(s) - m_{\mathrm{id}}(s)$** is a serious **restriction** of the framework!

Result

Theorem 2: sufficient condition

With the dependency structure from above, the following products can be **in the set M at the same time** :

- standardized products,
- products with payments for transition between states,
- products with payments for sojourns in states.

Theorem 3: necessary condition

We assume that Assumption 1 holds for r, m_{ai}, m_{id}, and m_{ad}. Furthermore, we assume that $m_{\mathrm{ai}}(t) \neq 0$ for all t almost surely and that α and β fulfill some technical assumptions (basically twice differentiable).

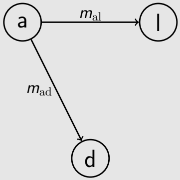
Then the dependency structure is also necessary.

Application to other models

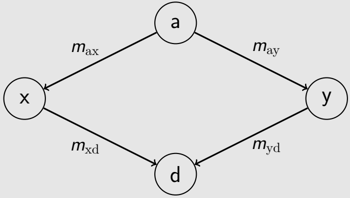
Theorem 4: necessary condition for other models

Under the assumptions of Theorem 3:

- **Active-dead model with lapse:**
  $\Rightarrow r$, m_{al} and m_{ad} must be (cond.) **independent**



- **Joint life insurance:**
  $\Rightarrow r$ (cond.) independent of transition rates;
  $m_{\mathrm{ax}}(u)$, $m_{\mathrm{xd}}(u)$, and $m_{\mathrm{ay}}(u) - m_{\mathrm{xd}}(u)$ as well as $m_{\mathrm{ay}}(u)$, $m_{\mathrm{yd}}(u)$, and $m_{\mathrm{ax}}(u) - m_{\mathrm{yd}}(u)$ (cond.) independent



Conclusion

Conclusion

(1) The idea of a **substitution rule** allows us to give a **general forward rate definition**.

(2) In the active-dead model with and without cancellation the **independence** between the transition rates and interest is not only sufficient, but also **necessary**.

(3) By **requiring a liquid market** with all common products and standardized products, some special **dependency structure is implicitly assumed** for a simple disability insurance and a joint life insurance.

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Thank you very much for your attention!

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