Markov Chain Modeling of Policy Holder Behavior in Life Insurance and Pension *

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Abstract

We calculate reserves regarding expected policy holder behavior. The behavior is modeled to occur incidentally similarly to insurance risk. The focus is on multi-state modeling of insurance risk, e.g. in a disability model, and of behavioral risk, e.g. in a premium payment - free policy - surrender model. We discuss valuation techniques in the cases where the behavior is modeled to occur independently of insurance risk and where we take explicitly into account that e.g. disabled do not hold behavioral options, respectively. Ordinary differential equations make it easier to work with dependence between insurance risk and behavior risk. We analyze the effects of the underlying behavioral assumptions in two contracts. For a 'new' contract, i.e. low technical interest rate relative to the market interest rate, we obtain the lowest reserve by working with the correct model without inaccurate shortcut assumptions. For an 'old' contract, i.e. high technical interest rate relative to the market interest rate, the picture is more blurred, depending on assumptions on reactivation (recovery) and the route of the shortcut.

1 Introduction

We characterize reserves under finite-state Markov chain modeling of policy holder behavior and illustrate numerically the effects on values from modeling policy holder behavior in various ways. By policy holder behavior we think, in particular, of policy holder intervention like transcription to free policy and surrender. The reserves are characterized by ordinary differential equations and their more or less explicit solutions, depending on the behavior model and the underlying risk model. These solutions are particularly tractable if one assumes independence between insurance risk and behavior, although such independence is often ruled out by contract design: Disability

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and life annuitants are typically not allowed to exercise such behavior options. In the illustrations we calculate values for standard contracts in order to analyze the effects of taking into account policy holder behavior in various ways. In particular, we study the consequences of assuming independence between insurance risk and behavior.

Current developments in insurance accounting and solvency rules take an explicit approach to policy holder behavior. For calculating reserves it is to an increasing extent required to take into account policy holder behavior. Policy holder behavior should be thought of as actions taken by the policy holder that influence either the risk in the processes that drive the payment streams of an insurance contract or the payments themselves. In this paper we pay special attention to the surrender option, i.e. the option to terminate the contract in exchange for a lump sum payment, and the free policy option, i.e. the option to stop paying the premium against a reduction of benefits. Among other options held by the policy holder may be the annuitization option in case the default coverage is a pension sum that can then, on basis of technical assumptions about interest and mortality, be converted to an annuity. Although this distinction is not necessarily a clean cut in practice, annuitization is an option that can be exercised upon retirement and can therefore be thought of as a European type option. The surrender and free policy options can, in general, be exercised at any point in time and can therefore be thought of as American type options. Another option that is sometimes mentioned explicitly is the option to raise the premiums. Typically such an option is provided in connection with an occupational pension scheme, where e.g. premiums are calculated as a percentage of the salary.

There exists a range of approaches to modeling of behavior risk. One extreme position to take is to assume that the policy holder exercises his options based on an economically optimal strategy, i.e. in order to maximize the value of the payment stream from the contract. This approach is taken in Steffensen (2002) in order to characterize values in terms of so-called variational inequalities known in a financial context from American option pricing. Compared to a standard American option it is, of course, a delicate feature of the contract that, possibly, both a free policy and a surrender option exist. Furthermore, if the free policy option is exercised the contract does not vanish but continues under different terms and possibly including, still, a surrender option. This is all dealt with by Steffensen (2002). The same extreme American option approach is taken by e.g. Grosen and Jørgensen (2000) and Bacinello (2003). A primitive approximation of the value obtained from this approach is to reserve, at any point in time, the larger of the value based on no exercise and the surrender value. This is what has been called a 'now or never'-reserve since it corresponds to optimizing over two intervention strategies corresponding to exercising now or never. Due to its tractability, this is often seen as a first approach to take intervention options into account in practical accounting rules. Clearly, this approximation underestimates the true value since the optimal strategy may be to exercise somewhere between now or never.

Another extreme approach to take is to assume that the intervention options are exercised completely incidentally. Then intervention risk can be treated formalistically as diversifiable insurance risk, although they are different concepts and the treatment considerably complicates, in general, the states of the world that have to be taken into account. This approach is taken e.g. by Buchardt and Møller (2013) and Buchardt et al. (2013).

A modern approach in accounting and solvency is to base reserves on expected policy holder behavior rather than rational policy holder behavior. This approach is taken e.g. in preliminary formulations of both Solvency II and IFRS. This draws attention towards the latter of the two extremes. Then, however, the expectation to policy holder behavior is explicitly required to take into account also e.g. the economic environment and/or whether the option is beneficial. This appears to be one step back towards the first extreme without really going that far. Such intermediary modeling is an interesting object of studies with a lot of challenges concerning the statistical material available, economic intuition and mathematical tractability of the studied objects depending on the driving factors. Simple ideas are to let the intervention intensity depend on interest rates, as was done by De Giovanni (2010), or a relation between the intervention value and (some notion of) the market reserve. These ideas address the questions about regarding the economic environment and whether the option is beneficial. There exists a large amount of empirical literature discussing explanatory variables. These range from macro variables like interest rates, e.g. studied by Kuo et. al. (2003), to micro variables like for instance policy holder age. We refer to Eling and Kiesenbauer (2013) and references therein for a comprehensive literature overview.

In this paper we take the extreme approach to assume that the intervention options are exercised completely incidentally. This does not mean that we do not believe that working with interest rate or reserve dependent intervention intensities are interesting, important, challenging, or relevant. We are just focusing on something else. Also, we focus on something rather different from Buchardt and Møller (2013) who mainly concentrate on representation and calculation of cash flows in one of the special cases of our study, and Buchardt et al. (2013) who in a more theoretical framework deal with duration dependence in the risk model.

We are interested in discussing the dependence between insurance risk and behavioral risk that arises essentially from the product design. We do this in a finite state Markov chain framework. That allows us to characterize conditional expected values by ordinary differential equations and representations of solutions. Their structures make it clear in what sense one can choose between a complicated, differential equation based approach and the wrong solution. Or said in a different way: Keep it simple or keep it right! To make the good choice here, it is of course relevant to qualify this one-liner. How simple is simple? And if simple means wrong, then how wrong is wrong? These questions are discussed from a theoretical point of view throughout the first part of the paper in Sections 2 - 5 and addressed numerically in the second part in Section 6. In a thorough analysis where several aspects are taken into account, including varying over the value of the intervention options, the answer is not surprisingly: It depends! This paper illuminates on what it depends. A conclusion is that it really does matter for todays entry values, in general, whether one takes the 'simple' or the 'right' approach. This makes a case for our advanced methods. All numerical results are obtained by Actulus® Calculation Platform

2 Risk and Behavior Models

In this section we present the idea of considering a combined model for risk and behavior as being decomposed into two separate models for risk and behavior, respectively, that are or are not probabilistically dependent of each other. We think of a risk state model Z^{risk} and a behavior state model $Z_{behavior}$ and consider the state model $\left(Z^{risk}, Z_{behavior}\right)$.

Given (the whole process history of) $Z_{behavior}$, Z^{risk} is assumed to be a finite-state Markov chain taking values in $\mathbf{Z}^{risk}$. Thus, conditional on $Z_{behavior}$, there exist transition intensities $\mu^{jk}\left(t\right)$

for $j, k \in \mathbf{Z}^{risk}$ and $t \geq 0$, such that for all $k \in \mathbf{Z}^{risk}$, $\int_0^t \mu^{Z^{risk}(s)k}(s)\, ds$ is (conditional on $Z_{behavior}$) a compensator for the counting process counting the number of jumps into risk state k. The transition intensities may be independent of $Z_{behavior}$ and in that case Z^{risk} is, even unconditional on $Z_{behavior}$, a finite-state Markov chain. A canonical multi-state example of a risk model is the disability model illustrated in Figure 1. We have labeled the states {active, disabled, dead} by the letters $\{a, i, d\}$. This risk model is a key example below and in the numerical illustrations in particular.

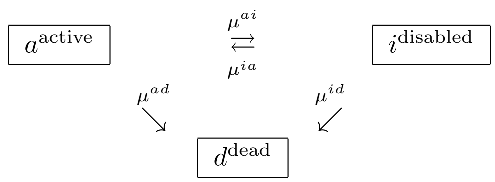

Figure 1: Disability risk model

Given (the whole process history of) Z^{risk}, $Z_{behavior}$ is assumed to be a finite-state Markov chain taking values in $\mathbf{Z}_{behavior}$. Thus, conditional on Z^{risk}, there exist transition intensities $\nu_{jk}(t)$ for $j, k \in \mathbf{Z}_{behavior}$ and $t \geq 0$, such that for all $k \in \mathbf{Z}_{behavior}$, $\int_0^t \nu_{Z_{behavior}(s)k}(s)\, ds$ is (conditional on Z^{risk}) a compensator for the counting process counting the number of jumps into behavior state k. The transition intensities may be independent of Z^{risk} and in that case $Z_{behavior}$ is, even unconditional on Z^{risk}, a finite-state Markov chain. A canonical multi-state example of a behavior model is the free policy/surrender model illustrated in Figure 2. We have labeled the states {premium payment, free policy, surrender} by the letters $\{p, f, s\}$. This behavior model is a key example below and in the numerical illustrations in particular.

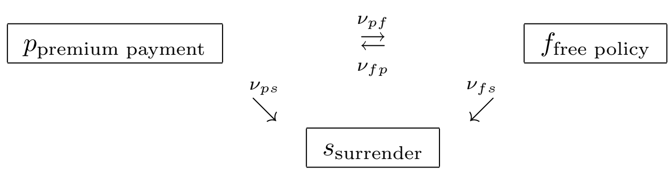

Figure 2: Behavior model

The repetitions in the two paragraphs above are not made to bore the reader but to emphasize the up-front symmetry in the two separate models. As can be seen above, we refer consequently to specifications and states in the risk model by superscripts and to specifications and states in the behavior model by subscripts. When Z^{risk} given $Z_{behavior}$ and $Z_{behavior}$ given Z^{risk} are Markov models, the combined model $Z = \left(Z^{risk}, Z_{behavior}\right)$ is a Markov model. Thus, there exist risk transition intensities $\mu_l^{jk}(t)$ for $j, k \in \mathbf{Z}^{risk}$, $l \in \mathbf{Z}_{behavior}$ and $t \geq 0$, such that for all $k \in \mathbf{Z}^{risk}$, $\int_0^t \mu_{Z_{behavior}(s)}^{Z^{risk}(s)k}(s)\, ds$ is a compensator for the counting process counting the number of jumps into risk state k. Similarly, there exist risk transition intensities $\nu_{jk}^l(t)$ for $j, k \in \mathbf{Z}_{behavior}$, $l \in \mathbf{Z}^{risk}$ and $t \geq 0$, such that for all $k \in \mathbf{Z}_{behavior}$, $\int_0^t \mu_{Z_{behavior}(s)k}^{Z^{risk}(s)}(s)\, ds$ is a compensator for the counting process counting the number of jumps into behavior state k.

We introduce the notation $p_{lm}^{jk}(t, s)$ for the transition probability that the risk model goes from

j to k and the behavior model goes from l to m over (t, s). In the case where the two sub-models for Z^{risk} and $Z_{behavior}$ are independent, i.e. the transition intensities μ do not depend on $Z_{behavior}$ and the transition intensities ν do not depend on Z^{risk}, we can simplify this probability into a product of probabilities with respect to each sub-model, i.e.

$$p_{lm}^{jk}(t, s) = p^{jk}(t, s)\, p_{lm}(t, s).$$

In case of independence we specify here the transition probabilities in the two models exemplified above. If μ^{ai} and μ^{ia} are both positive, we have no closed-form expressions for the probabilities $\left(p^{aa}, p^{ii}, p^{ai}, p^{ia}, p^{ad}, p^{id}\right)$. But in the case of no reactivation, i.e. $\mu^{ia} = 0$, we do:

$$\begin{aligned} p^{aa}(t, s) &= e^{-\int_t^s \left(\mu^{ai}(\tau) + \mu^{ad}(\tau)\right) d\tau}; p^{ii}(t, s) = e^{-\int_t^s \mu^{id}(\tau) d\tau}; \\ p^{ai}(t, s) &= \int_t^s p^{aa}(t, \tau)\, \mu^{ai}(\tau)\, p^{ii}(\tau, s)\, d\tau; p^{ia}(t, s) = 0. \end{aligned}$$

The probabilities p^{ad} and p^{id} are calculated residually by summing conditional probabilities to 1. Correspondingly, if μ_{pf} and μ_{fp} are both positive, we have no closed-form expressions for the probabilities $(p_{pp}, p_{ff}, p_{pf}, p_{fp}, p_{ps}, p_{fs})$. But in the case of no premium resumption, i.e. $\mu_{fp} = 0$, we do:

$$\begin{aligned} p_{pp}(t, s) &= e^{-\int_t^s (\nu_{pf}(\tau) + \nu_{ps}(\tau)) d\tau}; p_{ff}(t, s) = e^{-\int_t^s \nu_{fs}(\tau) d\tau}; \\ p_{pf}(t, s) &= \int_t^s p_{pp}(t, \tau)\, \nu_{pf}(\tau)\, p_{ff}(\tau, s)\, d\tau; p_{fp}(t, s) = 0. \end{aligned}$$

The probabilities p_{ps} and p_{fs} are calculated residually by summing conditional probabilities to 1.

A specific model for behavior is a model for the demand from policy holders. A probabilistic model for demand means that there is a tendency in a portfolio that policy holders hold certain types of contracts. There are many motivations for thinking of the two processes Z^{risk} and $Z_{behavior}$ as being dependent. Two classical features in risk trading represent each one direction of influence between the two sub-models. Adverse selection, on one hand, means that policy holders with certain risks tend to demand certain contracts. We can reflect this in our model by letting the transition intensities in the behavior model be more or less explicitly dependent on the risk process. Moral hazard, on the other hand, means that policy holders with certain behavior/demand tend to cause certain levels of risks. We can reflect this in our model by letting the transition intensities in the risk model be more or less explicitly dependent on the behavior model. Thus, causal effects between the models have directions and each direction may have a given economic interpretation. But at the end of the day, we observe a combined process where it may be difficult/impossible to detect the direction of causal effects from the experienced dependence.

The canonical behavior model illustrated in Figure 2 above is also such a model for demand of certain types of payment profiles. Again there may be effects of adverse selection, i.e. policy holders in different risk states tend to exercise their behavioral options, free policy and surrender, differently. Or there may be effects of moral hazard, i.e. policy holders in the premium payment or free policy states, which essentially means that they hold different insurance contracts, have different mortality/disability rates. Below we pay full attention to a simple effect in the policy design that contractualizes the dependence between the risk and behavior models. It is common practice that e.g. only policy holders in the risk state 'active' are allowed to transcribe into a free policy or surrender. A standard contractual formulation is that such exercise options fall away when the contract goes from a premium payment contract to a benefit receipt contract, either by transition of state or by transition of time. We assume throughout that the risk of policy holders taking up their premium payment after having been transcribed to free policy is zero, i.e. $\nu_{fp} = 0$. This is often a harmless assumption since such a contract is typically handled as a new contract and should therefore not be taken into account. With such a dependence coming exclusively from the behavioral options in the contract, we have illustrated the two-dimensional model in Figure 3.

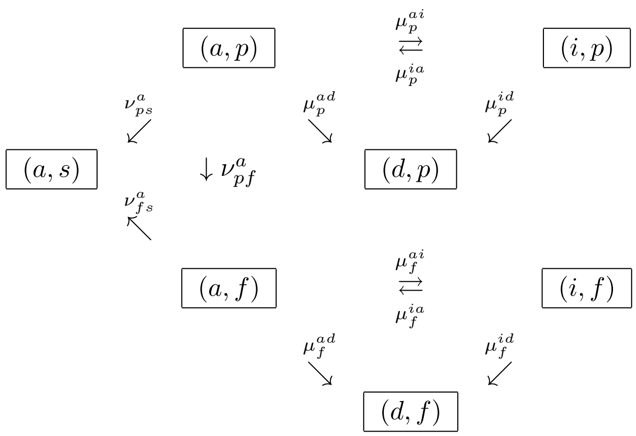

Figure 3: Combined model

We conclude this section with a general remark regarding regime shift models. The Markov structure of the model makes it relatively easy to allow for underlying regime shifts. Regime shifts underlying the risk model could be motivated by an impact from the state of the economy on disability and recovery rates. This is obtained by generalizing the model illustrated in Figure 1 by subdivision of the active and disability states corresponding to the different states of the economy. It could be even more relevant to model an impact from the state of the economy on free policy and surrender rates. This is done by a corresponding extension of the model illustrated in Figure 2. Note that if the state of the economy influences transition rates in both the risk model and the behavior model, we have introduced a dependence between the two models, even in the case where there is no dependence via the contract design. Although this constitutes an interesting source of dependence between the underlying models, we will not pursue the idea further here.

3 Values and Cash Flows in Risk and Behavior Models

In this section we describe the contractual payments and present formulas for calculation of their conditional expected present values. We assume a general risk model in combination with the canonical behavior model illustrated in Figure 2 with the premium resumption rate set to zero, i.e. $\nu_{fp} = 0$.

We take as starting point a contract that, in the first place, specifies its payments in the risk model, conditional on the behavior model being in the premium payment state p. We assume that the contract pays net benefits to the policy holder at rate b^j as long as the policy holder is in state j and a lump sum net benefit b^{jk} upon a transition from state j to state k. By net benefit we mean that premium payments are taken into account by a negative sign. We can now formalize the expected payment rate at time s given that the policy holder is in risk state k at time s as

$$c^k(s) = b^k(s) + \sum_{l:l\neq k} \mu_p^{kl}(s)\, b^{kl}(s)$$

We assume that the contract specifies that upon surrender from risk state k at time t, all future payments are canceled and a surrender sum $G^k(t)$ is paid out in return.

We assume that the contract specifies that if the policy is transcribed into a free policy while the policy holder is in risk state h at time t, the future payments are changed in the following way. The negative elements of b^j and b^{jk}, i.e. premiums, are set to zero whereas positive elements of b^j and b^{jk}, denoted by b^{j+} and b^{jk+}, are multiplied by a so-called free policy factor that is, exclusively, depending on t and h and that we denote by $f^h(t)$. First we introduce the expected payment rate of positive payments (before multiplication by f) as

$$c^{k+}(s) = b^{k+}(s) + \sum_{l:l\neq k} \mu_f^{kl}(s)\, b^{kl+}(s).$$

We can also write the expected payment rate at time s given that the policy holder is in risk state k at time s and jumped into the behavior state free policy at time t while being in risk state h as

$$f^h(t)\, c^{k+}(s) = f^h(t)\left(b^{k+}(s) + \sum_{l:l\neq k} \mu_f^{kl}(s)\, b^{kl+}(s)\right).$$

One can imagine a series of alternative recalculations of payments upon transition into the free policy state. E.g. the policy holder may want his risk coverages (like term insurance and disability annuities) to either fall away or to be fully kept upon transcription, and then the saving coverages (like deferred life annuities) are changed residually. We elaborate briefly on such alternatives in Section 5 below. But we develop the valuation formulas under the assumption that all future benefits are changed proportionally. That even goes for the surrender payment in the following sense. If the policy was transcribed into a free policy while the policy holder was in risk state h at time t, and the policy is surrendered at time $u > t$ while the policy holder is in risk state j, the policy pays out a surrender sum $f^h(t)\, G^{j+}(u)$.

It is important to note the following. Since the process Z is Markovian the intensity of making a jump at time t depends on the position of Z only. However, this does not mean that the payment rate at time t only depends on Z. Since the expected payment rate at time s, $f^h(t)\, c^{k+}(s)$, depends on t and h through the free policy factor, we have introduced a specific duration dependence in the payment process which is not present in the probability model.

3.1 Given the free policy state

In this subsection we present a differential equation characterizing the reserve defined as the expected present value of future payments at time $t > \tau$ given that the policy jumped to the free policy state while the policy holder was in risk state h at time τ. We also specify its solution. Here and throughout we refer to Steffensen (2000) for all differential equations and their solutions.

Denoting by $V_f^j(t)_{\tau h}$ the reserve if the policy holder is in risk state j at time t and became a free policy while being in risk state h at time τ, we can characterize this reserve by the differential equation

$$\begin{aligned}\frac{d}{dt}V_f^j(t)_{\tau h} &= rV_f^j(t)_{\tau h} - f^h(\tau)\,b^{j+}(t) - \sum_{k:k\neq j}\mu_f^{jk}(t)\left(f^h(\tau)\,b^{jk+}(t) + V_f^k(t)_{\tau h} - V_f^j(t)_{\tau h}\right) \\ &\quad -\nu_{fs}^j(t)\left(f^h(\tau)\,G^{j+}(t) - V_f^j(t)_{\tau h}\right).\end{aligned}$$

The solution can be written as

$$V_f^j(t)_{\tau h} = f^h(\tau)\int_t^n e^{-\int_t^s r}\sum_k p_{ff}^{jk}(t,s)\left(c^{k+}(s) + \nu_{fs}^k(s)\,G^{k+}(s)\right)ds,$$

where $p_{ff}^{jk}(t,s)$ is the probability that the policy holder moves from j to k in the risk model while staying in state f in the behavior model. This interpretation of $p_{ff}^{jk}(t,s)$ relies on the assumption that $\nu_{fp}=0$, such that $p_{ff}^{jk}(t,s) = p_{\overline{ff}}^{jk}(t,s)$. Although we speak of this as the solution, it is not in closed form, since these transition probabilities, in general, do not exist in closed form. In the integral solution we can see that the reserve consists of payments during sojourn in the free policy state (the $c^{k+}(s)$ terms) and payments paid upon surrender (the $G^{k+}(s)$ terms).

In the special case where the behavior and the risk models are independent, then we have the simple form

$$p_{ff}^{jk}(t,s) = p_{ff}(t,s)\,p^{jk}(t,s)$$

such that

$$V_f^j(t)_{\tau h} = f^h(\tau)\int_t^n e^{-\int_t^s r}p_{ff}(t,s)\sum_k p^{jk}(t,s)\left(c^{k+}(s) + \nu_{fs}(s)\,G^{k+}(s)\right)ds.$$

This formula is particularly convenient since it can be built around the 'original' expected cash flow rates $\sum_k p^{jk}(t,s)\,c^{k+}(s)$ and the rates $\sum_k p^{jk}(t,s)\,\nu_{fs}(s)\,G^{k+}(s)$. Thus, when making use of the integral solution, one may, for computational convenience, be inclined to assume probabilistic independence. This is not correct, but the numerical consequences may or may not be negligible. We elaborate on this in Section 6 below. When working with the differential equations, on the other hand, using the correct model does not introduce any additional complexity and hence, in this case, there is no excuse for not performing the right calculations.

3.2 Given the premium payment state

In this subsection we present a differential equation characterizing the reserve given that the policy is in the premium payment behavior state and in risk state j at time t. We also specify its solution. Denoting this reserve by $V^j(t)$, tacitly skipping the subscript p on all reserves below, we can characterize this reserve by the differential equation

$$\begin{aligned}\frac{d}{dt}V^j(t) &= rV^j(t) - b^j(t) - \sum_{k:k\neq j}\mu_p^{jk}(t)\left(b^{jk}(t) + V^k(t) - V^j(t)\right) \\ &\quad -\nu_{pf}^j(t)\left(V_f^j(t)_{tj} - V^j(t)\right) - \nu_{ps}^j(t)\left(G^j(t) - V^j(t)\right).\end{aligned} \tag{1}$$

Note that the reserve $V_f^j(t)_{tj}$ is obtained by solving the differential equation for $V_f^j(t)_{\tau j}$ for fixed τ and subsequently replacing τ by t. The solution can be written as

$$\begin{aligned} V^j(t) &= \int_t^n e^{-\int_t^s r} \sum_k p_{pp}^{jk}(t,s) \left(c^k(s) + G^k(s)\, \nu_{ps}^k(s) \right) ds \\ &\quad + \int_t^n e^{-\int_t^s r} \sum_k W^{jk}(t,s) \left(c^{k+}(s) + \nu_{fs}^k(s)\, G^{k+}(s) \right) ds \end{aligned} \tag{2}$$

where $p_{pp}^{jk}(t,s)$ is the probability that the policy holder moves from j to k in the risk model while staying in state p in the behavior model. This interpretation of $p_{pp}^{jk}(t,s)$ relies on the assumption that $\nu_{fp} = 0$, such that $p_{pp}^{jk}(t,s) = p_{\overline{pp}}^{jk}(t,s)$. Furthermore,

$$W^{jk}(t,s) = \int_t^s \sum_h p_{pp}^{jh}(t,\tau)\, \nu_{pf}^h(\tau)\, p_{ff}^{hk}(\tau,s)\, f^h(\tau)\, d\tau.$$

In the integral solution we can see that the reserve consists of payments during sojourn in the premium payment state (the $c^k(s)$ terms), payments due upon surrender from the premium payment state (the $G^k(s)$ terms), payments during sojourn in the free policy state (the $c^{k+}(s)$ terms), and payments upon surrender from the free policy state (the $G^{k+}(s)$ terms). The ratio $W^{jk}(t,s)/p_{pf}^{jk}(t,s)$ is the expected free policy ratio given that the policy holder jumps in risk states from j to k and in behavior states from p to f over (t,s).

In the special case where the behavior and the risk models are independent, then we have the simple form (recall the simple forms for p_{pp}, p_{ff}, and p_{pf} from Section 2)

$$\begin{aligned} p_{pp}^{jk}(t,s) &= p_{pp}(t,s)\, p^{jk}(t,s), \\ p_{ff}^{jk}(t,s) &= p_{ff}(t,s)\, p^{jk}(t,s), \\ p_{pf}^{jk}(t,s) &= p_{pf}(t,s)\, p^{jk}(t,s), \end{aligned}$$

such that

$$\begin{aligned} V^j(t) &= \int_t^n e^{-\int_t^s r} p_{pp}(t,s) \sum_k p^{jk}(t,s) \left(c^k(s) + \nu_{ps}(s)\, G^k(s) \right) ds \\ &\quad + \int_t^n e^{-\int_t^s r} \sum_k W^{jk}(t,s) \left(c^{k+}(s) + \nu_{fs}(s)\, G^{k+}(s) \right) ds, \\ W^{jk}(t,s) &= \int_t^s p_{pp}(t,\tau)\, \nu_{pf}(\tau)\, p_{ff}(\tau,s) \sum_h p^{jh}(t,\tau)\, p^{hk}(\tau,s)\, f^h(\tau)\, d\tau. \end{aligned}$$

This formula appears convenient since the first line can be built around the 'original' expected cash flow rates $\sum_k p^{jk}(t,s)\, c^k(s)$ and the rates $\sum_k p^{jk}(t,s)\, \nu_{ps}(s)\, G^k(s)$. Thus, when making use of the integral solution, one may, for computational convenience, again be inclined to assume probabilistic independence. However, this shortcut is not as appealing as it seems. In spite of the probabilistic independence, the second line is still an involved quantity. In order to really benefit from 'original' expected cash flow rates, we could even further assume that $f^h(\tau)$ does not depend on h. Then

$$W^{jk}(t,s) = p^{jk}(t,s)\, W(t,s)$$

with

$$W(t,s) = \int_t^s p_{pp}(t,\tau)\,\nu_{pf}(\tau)\,p_{ff}(\tau,s)\,f(\tau)\,d\tau, \tag{3}$$

such that the second line becomes

$$\int_t^n e^{-\int_t^s r} W(t,s) \sum_k p^{jk}(t,s) \left(c^{k+}(s) + \nu_{fs}(s)\,G^{k+}(s)\right) ds.$$

Finally, we have now reached the most elegant, but incorrect, expression since all elements are built around 'original' cash flow rates $\sum_k p^{jk}(t,s)\,c^k(s)$ and $\sum_k p^{jk}(t,s)\,c^{k+}(s)$ and the rates $\sum_k p^{jk}(t,s)\,\nu_{ps}(s)\,G^k(s)$ and $\sum_k p^{jk}(t,s)\,\nu_{fs}(s)\,G^{k+}(s)$.

4 Important special cases

In this section we specialize the main results from Section 3 to two particularly important special cases. We consider the canonical risk model illustrated in Figure 1, the disability model, and the survival model respectively. In both cases we present relevant differential equations and their more or less explicit solutions depending on what we assume about the underlying model or contract. Particular attention is paid to the various simplifying assumptions that one can make in order to ease calculations. We concentrate on the valuation of policies that are in the behavior state 'premium payment' since this is where the main calculation challenges arise.

4.1 The disability model

First we consider the disability model. We assume that all payments are zero in the state 'dead'. We label the different states according to Figure 1. Then the reserve corresponding to the policy holder being premium paying and active at time t can be characterized by a special case of (1) that becomes

$$\begin{aligned}
\frac{d}{dt}V^a(t) &= rV^a(t) - b^a(t) \\
&\quad -\mu^{ai}(t)\left(b^{ai}(t) + V^i(t) - V^a(t)\right) - \mu^{ad}(t)\left(b^{ad}(t) - V^a(t)\right) \\
&\quad -\nu^a_{pf}(t)\left(V^a_f(t)_{ta} - V^a(t)\right) - \nu^a_{ps}(t)\left(G^a(t) - V^a(t)\right).
\end{aligned}$$

Here, the second line contains the risk premia related to the state transitions in the risk model whereas the third line contains risk premia related to the state transitions in the behavior model.

The general solution represented in (2) becomes

$$\begin{aligned}
V^a(t) &= \int_t^n e^{-\int_t^s r} \begin{pmatrix} p^{aa}_{pp}(t,s)\left(c^a(s) + \nu^a_{ps}(s)\,G^a(s)\right) \\ +p^{ai}_{pp}(t,s)\left(c^i(s) + \nu^i_{ps}(s)\,G^i(s)\right) \end{pmatrix} ds \\
&\quad + \int_t^n e^{-\int_t^s r} \begin{pmatrix} W^{aa}(t,s)\left(c^{a+}(s) + \nu^a_{fs}(s)\,G^{a+}(s)\right) \\ +W^{ai}(t,s)\left(c^{i+}(s) + \nu^i_{fs}(s)\,G^{i+}(s)\right) \end{pmatrix} ds,
\end{aligned}$$

where

$$\begin{aligned} W^{aa}(t,s) &= \int_t^s \begin{pmatrix} p_{pp}^{aa}(t,\tau)\,\nu_{pf}^{a}(\tau)\,p_{ff}^{aa}(\tau,s)\,f^{a}(\tau) \\ +p_{pp}^{ai}(t,\tau)\,\nu_{pf}^{i}(\tau)\,p_{ff}^{ia}(\tau,s)\,f^{i}(\tau) \end{pmatrix} d\tau, \\ W^{ai}(t,s) &= \int_t^s \begin{pmatrix} p_{pp}^{aa}(t,\tau)\,\nu_{pf}^{a}(\tau)\,p_{ff}^{ai}(\tau,s)\,f^{a}(\tau) \\ +p_{pp}^{ai}(t,\tau)\,\nu_{pf}^{i}(\tau)\,p_{ff}^{ii}(\tau,s)\,f^{i}(\tau) \end{pmatrix} d\tau. \end{aligned}$$

We now make the realistic assumption that the contract specifies that behavioral events only take place as long as the policy holder is active. This means that $\nu_{pf}^{i}(t) = \nu_{ps}^{i}(t) = 0$ such that W^{aa} and W^{ai} reduce to

$$\begin{aligned} W^{aa}(t,s) &= \int_t^s p_{pp}^{aa}(t,\tau)\,\nu_{pf}^{a}(\tau)\,p_{ff}^{aa}(\tau,s)\,f^{a}(\tau)\,d\tau, \\ W^{ai}(t,s) &= \int_t^s p_{pp}^{aa}(t,\tau)\,\nu_{pf}^{a}(\tau)\,p_{ff}^{ai}(\tau,s)\,f^{a}(\tau)\,d\tau. \end{aligned}$$

These formulas are, of course, not explicit due to the allowance for positive reactivation rate. This makes the probabilities impossible to calculate explicitly. However, if we further do not allow for reactivation, we get simpler expressions for the probabilities, e.g.

$$\begin{aligned} p_{pp}^{aa}(t,s) &= p_{pp}^{\overline{aa}}(t,s) = e^{-\int_t^s \mu_p^{ai}+\mu_p^{ad}+\nu_{pf}^{a}+\nu_{ps}^{a}}, \\ p_{ff}^{aa}(t,s) &= p_{ff}^{\overline{aa}}(t,s) = e^{-\int_t^s \mu_f^{ai}+\mu_f^{ad}+\nu_{fs}^{a}}. \end{aligned}$$

This has simplifying consequences for calculation of $p_{ff}^{ai}(t,s)$, $W^{aa}(t,s)$, and $W^{ai}(t,s)$.

Instead of assuming that the contract allows for behavioral events from the 'active' state only, we now make the 'opposite' assumption and say that the risk and behavior models are independent. We know from the previous section that this may help making the calculations much simpler. We get the expression for the reserve,

$$\begin{aligned} V^{a}(t) &= \int_t^n e^{-\int_t^s r} p_{pp}(t,s) \begin{pmatrix} p^{aa}(t,s)\left(c^{a}(s)+\nu_{ps}(s)\,G^{a}(s)\right) \\ +p^{ai}(t,s)\left(c^{i}(s)+\nu_{ps}(s)\,G^{i}(s)\right) \end{pmatrix} ds \\ &\quad + \int_t^n e^{-\int_t^s r} \begin{pmatrix} W^{aa}(t,s)\left(c^{a+}(s)+\nu_{fs}(s)\,G^{a+}(s)\right) \\ +W^{ai}(t,s)\left(c^{i+}(s)+\nu_{fs}(s)\,G^{i+}(s)\right) \end{pmatrix} ds, \end{aligned}$$

where

$$\begin{aligned} W^{aa}(t,s) &= \int_t^s p_{pp}(t,\tau)\,\nu_{pf}(\tau)\,p_{ff}(\tau,s) \begin{pmatrix} p^{aa}(t,\tau)\,p^{aa}(\tau,s)\,f^{a}(\tau) \\ +p^{ai}(t,\tau)\,p^{ia}(\tau,s)\,f^{i}(\tau) \end{pmatrix} d\tau, \\ W^{ai}(t,s) &= \int_t^s p_{pp}(t,\tau)\,\nu_{pf}(\tau)\,p_{ff}(\tau,s) \begin{pmatrix} p^{aa}(t,\tau)\,p^{ai}(\tau,s)\,f^{a}(\tau) \\ +p^{ai}(t,\tau)\,p^{ii}(\tau,s)\,f^{i}(\tau) \end{pmatrix} d\tau. \end{aligned}$$

There are still several difficulties with this representation. Even though some elements relate to conditional expected cash flows from the original contract, we note that we cannot calculate the transition probabilities explicitly as long as we allow for reactivation. Furthermore, the expected cash flows from the free policy state are still quite complicated and do not relate to original cash flows in an easy manner. Referring to the results in the previous section, we propose the additional assumption that $f^{i} = f^{a}$. Then

$$\begin{aligned} W^{aa}(t,s) &= p^{aa}(t,s)\,W(t,s), \\ W^{ai}(t,s) &= p^{ai}(t,s)\,W(t,s), \end{aligned}$$

with W defined as in (3), such that

$$\begin{aligned} V^{a}(t) &= \int_{t}^{n} e^{-\int_{t}^{s} r} p_{pp}(t,s) \begin{pmatrix} p^{aa}(t,s)\left(c^{a}(s)+\nu_{ps}(s)G^{a}(s)\right) \\ +p^{ai}(t,s)\left(c^{i}(s)+\nu_{ps}(s)G^{i}(s)\right) \end{pmatrix} ds \\ &\quad + \int_{t}^{n} e^{-\int_{t}^{s} r} W(t,s) \begin{pmatrix} p^{aa}(t,s)\left(c^{a+}(s)+\nu_{fs}(s)G^{a+}(s)\right) \\ +p^{ai}(t,s)\left(c^{i+}(s)+\nu_{fs}(s)G^{i+}(s)\right) \end{pmatrix} ds. \end{aligned}$$

Finally, the ingredients relate to original cash flows. In these cash flows transition probabilities appear. How accessible they are, depends on whether or not we allow for reactivation. If we do not, the probabilities are explicit and we have reached the 'simplest' representation of our reserve.

4.2 The Survival model

Now we specialize to the survival model by skipping the disability state in the Subsection 4.1. We skip the superscript a in the reserve V since all quantities are conditional on the policy holder being alive. Then the reserve corresponding to the policy holder being premium paying and alive at time t can be characterized by a special case of (1) that becomes

$$\begin{aligned} \frac{d}{dt}V(t) &= rV(t) - b^{a}(t) \\ &\quad -\mu^{ad}(t)\left(b^{ad}(t) - V(t)\right) \\ &\quad -\nu_{pf}(t)\left(V_{f}^{a}(t)_{ta} - V(t)\right) - \nu_{ps}(t)\left(G^{a}(t) - V(t)\right) \end{aligned}$$

As above, the second line contains the risk premium related to the death state transition in the risk model whereas the third line contains risk premia related to state transitions in the behavior model.

The general solution represented in (2) becomes

$$\begin{aligned} V(t) &= \int_{t}^{n} e^{-\int_{t}^{s} r} p_{pp}^{aa}\left(c^{a}(s)+\nu_{ps}(s)G^{a}(s)\right) ds \\ &\quad + \int_{t}^{n} e^{-\int_{t}^{s} r} W^{aa}(t,s)\left(c^{a+}(s)+\nu_{fs}(s)G^{a}(s)\right) ds, \end{aligned}$$

where

$$W^{aa}(t,s) = \int_{t}^{s} p_{pp}^{aa}(t,\tau)\,\nu_{pf}(\tau)\,p_{ff}^{aa}(\tau,s)\,f^{a}(\tau)\,d\tau.$$

If we assume independence between the models we essentially assume that the mortality rate is not affected by the state of the behavior model and the transition intensities in the behavior model are not affected by the state of the risk model. The latter assumption is harmless since we have assumed that there are no payments in the death state. This means that it does not affect the value to allow for transcription into a free policy or surrendering among dead policy holders. If instead there were payments in the death state it would make, of course, a difference whether we allow for behavioral intervention in these payments or not. Anyway, under the assumption of independence we get the simplifications,

$$\begin{aligned} V(t) &= \int_{t}^{n} e^{-\int_{t}^{s} r} p_{pp}(t,s)\,p^{aa}(t,s)\left(c^{a}(s)+\nu_{ps}(s)G^{a}(s)\right) ds \\ &\quad + \int_{t}^{n} e^{-\int_{t}^{s} r} W^{aa}(t,s)\left(c^{a+}(s)+\nu_{fs}(s)G^{a+}(s)\right) ds, \end{aligned}$$

where

$$\begin{aligned} W^{aa}(t,s) &= \int_t^s p_{pp}(t,\tau)\,\nu_{pf}(\tau)\,p_{ff}(\tau,s)\,p^{aa}(t,\tau)\,p^{aa}(\tau,s)\,f^a(\tau)\,d\tau \\ &= p^{aa}(t,s)\int_t^s p_{pp}(t,\tau)\,\nu_{pf}(\tau)\,p_{ff}(\tau,s)\,f^a(\tau)\,d\tau. \end{aligned}$$

Then we have reached an expression based on the original cash flows. In the survival model, the final simplification, $f^h = f$, is not necessary. We stress, however, that this is true only because we have no payments in the death state.

5 The free policy ratio

In the calculations above we have assumed that all future benefits are multiplied by the same factor $f^j(t)$ upon transcription into free policy from risk state j at time t. But we have not discussed what this f^j should be and we have not discussed the situation where different ratios apply to different future benefits. If f^j applies to all future benefits, a natural idea is to let f be determined by

$$f^j(t) = \frac{V^{j*}(t)}{V^{j+*}(t)}$$

where '$*$' denotes valuation of the contractual payments corresponding to a technical basis consisting of (r^*, μ^*), possibly different from our valuation basis, (r, μ). To see why this idea is natural, consider the free policy sum at risk $V_f^j(t)_{tj} - V^j(t)$. Since $V_f^j(t)_{tj} = f^j(t)\,V^{j+}(t)$, we have that

$$V_f^j(t)_{tj} - V^j(t) = f^j(t)\,V^{j+}(t) - V^j(t).$$

Now, a particular version of our valuation basis would of course be the technical basis. In that case, as $f^j(t) = \frac{V^{j*}(t)}{V^{j+*}(t)}$, we get

$$V_f^{j*}(t)_{tj} - V^{j*}(t) = f^j(t)\,V^{j+*}(t) - V^{j*}(t) = 0.$$

In that sense we can say that under the technical basis the policy holder pays himself fully for the free policy risk. An important consequence of this approach is that one can disregard the free policy option for technical valuation purposes, e.g. for setting an equivalence premium. We can keep this as a constraint on the free policy ratio, that under the technical basis the free policy sum at risk is zero. Then we can consider situations where different reduction factors apply to different benefits.

If a group of benefits ('keep') is fully kept during transcription while another group of benefits is deleted ('delete'), a residual group is reduced by the factor $f^j(t)$. It is found by solving the equation

$$0 = V^{j+(keep)*}(t) + f^j(t)\left(V^{j+*}(t) - V^{j+(keep)*}(t) - V^{j+(delete)*}(t)\right) - V^{j*}(t). \quad (4)$$

Hence

$$f^j(t) = \frac{V^{j*}(t) - V^{j+(keep)*}(t)}{V^{j+*}(t) - V^{j+(keep)*}(t) - V^{j+(delete)*}(t)} \quad (5)$$

If there is a prioritized order in which the benefits are to be kept and the rest deleted, one could start filling the group 'keep' as long as $V^{j*}(t) \geq V^{j+(keep)*}(t)$. The first benefit that violates this inequality should be broken up by the f^j above with the residual benefits left as deleted. Two special cases of this are when either the group 'keep' or the group 'delete' is empty. If the group 'keep' is empty and the group 'delete' is not, then

$$f^j(t) = \frac{V^{j*}(t)}{V^{j+*}(t) - V^{j+(delete)*}(t)}$$

for the residual benefits. Note that this may lead to $f > 1$. If the group 'delete' is empty and the group 'keep' is not, then

$$f^j(t) = \frac{V^{j*}(t) - V^{j+(keep)*}(t)}{V^{j+*}(t) - V^{j+(keep)*}(t)},$$

for the residual benefits. Note that this leads to $f \leq 1$. However, in principle we may now end up with $f < 0$.

In all these cases the free policy sum at risk is

$$V_f^j(t)_{tj} - V^j(t) = V^{j+(keep)}(t) + f^j(t)\left(V^{j+}(t) - V^{j+(keep)}(t) - V^{j+(delete)}(t)\right) - V^j(t). \quad (6)$$

Plugging f^j defined in (5) into (6) under the technical basis gives that the desired technical free policy sum at risk is equal to zero, cf. (4).

6 Numerical results and discussion

We consider here the situation in Section 4.1, i.e. where the risk chain is a disability model with states 'active' (a), 'disabled' (i) and 'dead' (d) (cf. Figure 1) and the behavioral chain consists of the states 'premium payment' (p), 'free policy' (f) and 'surrender' (s) (cf. Figure 2). For the various setups (modeling of dependent/independent chains, using the same/different free policy ratios in risk states and including/disregarding reactivation from disability) mentioned in Section 4.1, regarding the disability model, we compute the reserve conditional on the policy holder being in the risk state 'active' and the behavioral state 'premium payment'. The purpose of this is to quantify the implications of using the various alternatives to the correct model, which is the dependent model where reactivation is included. Here 'dependent' corresponds to including policy holder options only from the risk state 'active', cf. Section 6.1 below.

6.1 Model parameters

There are two computational bases in play; the technical (sometimes referred to as 'first order') basis and the market (sometimes referred to as 'third order') basis. In other words, we omit including a separate so-called 'second order' basis (which is sometimes used to model bonus distribution schemes). As usual, the payments initiated by exercise of policy holder options (i.e. surrender and free policy) are defined such that the corresponding sums at risk under the technical basis are zero (cf. Section 6.2). Hence we disregard them there, see also the discussion in Section 5.

We start off by defining the transition intensities in the risk chain under the bases; throughout this section they are independent of the behavioral chain implying e.g. that the mortality of a premium paying policy holder is the same as the mortality of a free policy holder of the same age.

From	To	Technical basis	Market basis
active	dead	$\mu^{*ad}(age) = 0.0005 + 10^{5.728-10+0.038(age)}$	$\mu^{ad} = \mu^{*ad}$
active	disabled	$\mu^{*ai}(age) = 0.0006 + 10^{4.71609-10+0.06(age)}$	$\mu^{ai} = \mu^{*ai}$
disabled	dead	$\mu^{*id}(age) = \mu^{*ad}(age)$	$\mu^{id} = \mu^{*id}$
disabled	active	$\mu^{*ia}(age) \equiv 0$	$\mu^{ia}(age) = e^{-0.06(age)}$ or 0

Table 1: Transition intensities, risk chain

We remark that for the technical basis we use the standard intensities for a female occurring in the Danish G82 risk table. We let the mortality and disability intensities of the two bases be the same; the interest rates, on the other hand, are different (cf. Section 6.2 below). Furthermore, we consider both the case when the market reactivation intensity, μ^{ia}, is non-zero and when it is zero (as the latter assumption generally simplifies the semi-closed formulae, cf. Section 4.1).

Finally, we introduce the transition intensities in the behavioral model. We consider two situations.

1. The behavioral intensities are dependent of the policy holder's current state in the risk chain in the sense that they are zero unless the policy holder is in the risk state 'active'; this corresponds to stopping policy holder options in the risk states 'disabled' and 'dead'.
2. The behavioral intensities are independent of the risk chain; this corresponds to continuing the policy holder options in the risk states 'disabled' and 'dead' (since there are no payments in the state 'dead', the latter consequence is irrelevant).

Transition intensities not mentioned in the table are zero.

From	To	Independent model	Dependent model
premium payment	free policy	$\nu_{pf}(age) = e^{-0.07(age)}$	$\tilde{\nu}_{pf} = 1_{\{\text{active}\}} \cdot \nu_{pf}$
premium payment	surrender	$\nu_{ps}(age) = \nu_{pf}(age)$	$\tilde{\nu}_{ps} = 1_{\{\text{active}\}} \cdot \nu_{ps}$
free policy	surrender	$\nu_{fs}(age) = \nu_{ps}(age)$	$\tilde{\nu}_{fs} = 1_{\{\text{active}\}} \cdot \nu_{fs}$

Table 2: Transition intensities, behavioral chain

Hence, as is the general assumption throughout this paper, we do not model the option of 're-entering' the premium paying state from the free policy state, and furthermore the state 'surrender' is absorbing. There does not seem to exist a standard parameterization for the behavioral transition intensities; the idea behind their forms used here is simply that the inclination to exercise policy holder options decreases with age. This reflects a certain loyalty effect that is usually seen among policy holders.

6.2 Contracts

6.2.1 The surrender and free policy options

The surrender option gives the policy holder the choice of abandoning the contract in exchange for a lump sum payment, $G^j(t)$, where j is the risk state in which the policy holder resides at time t. In what follows, we let $G^j(t)$ be the value of the contract under the technical basis, i.e. the technical reserve, $G^j(t) := V^{j*}(t)$ (in reality the sum received is sometimes reduced by some factor; we disregard that in what follows). Note that this choice of G^j makes the sum at risk upon surrender equal to zero under the technical basis. The free policy option gives the policy holder the choice of stopping the premium payment; the contract is then kept but the benefits are scaled by a certain factor, $f^j(t)$, where, again, j denotes the risk state from which the transcription took place.

Before we turn to specifying the f, we note that the size of the technical reserve in relation to the market reserve is what determines if the surrender option increases or decreases the market value of the contract; in a situation where the technical reserve is higher than the market reserve, it is to be considered profitable for the policy holder to surrender and conversely when it is lower (as mentioned above, under the technical basis the sum at risk is zero by definition). In order to account for both situations, we consider two different contracts as specified below.

In all realistic scenarios the policy holder options are only allowed from the risk state 'active'. It is then standard to define $f^a(t) = \frac{V^{a*}(t)}{V^{a*+}(t)}$, i.e. the quotient between the technical reserve and the technical benefit reserve (premium removed, all benefits kept); this causes the sum at risk upon transcription to free policy, under the technical basis, to be zero, cf. Section 5. Other alternations of payment streams upon premature stopping of premium payment can also be studied numerically, cf. Section 5, but we focus on the proportional benefit reduction in this numerical example. When allowing policy holder options also from the 'disabled' state, the most natural choice appears to be $f^i(t) = \frac{V^{i*}(t)}{V^{i*+}(t)} \equiv 1$ (again, this causes the sum at risk, under the technical basis, upon transcription to free policy to be zero). However, as is mentioned in Section 4.1 above, setting $f^i := f^a$ yields even simpler closed-form solutions, and we therefore consider both these variants of f^i. We refer to the situations as using 'Separate f' and 'Same f', respectively.

6.2.2 Common

We outline the common features of the two contracts considered.

- Contract expiry age 65
- Premium payment of intensity 20,000 USD p.a.
- Disability annuity of intensity 100,000 USD p.a.
- Term insurance at 400,000 USD
- Pure endowment at expiry (corresponding to a life annuity) determined at initiation time of the contract such that it gives the contract a technical value (i.e. V^{a*}) of zero at the time of initiation

Furthermore, the market interest rate, r, is the forward rate equivalent to the yield curve as published by the Danish FSA at 2013-04-08 (cf. appendix A)

6.2.3 New contract

We consider here the situation where a relatively young policy holder has just signed the contract, and where the technical interest rate (r^*) is low relative to the current market interest rate; this makes the technical reserve higher than the market reserve. More precisely we have the following additional parameters.

- Contract initiation age 30
- Age 30 at the time ($t = 0$) of calculation
- $r^* = 1\%$ p.a. (continuously compounded)
- Pure endowment at expiry of 552,796 USD (corresponding to the reserve of a life annuity, at expiry, of 38.070 USD p.a. computed under the market value basis)

6.2.4 Old contract

We consider here the situation where an older policy holder signed the contract 20 years earlier, and where the technical interest rate (r^*) is high relative to the current market interest rate; this makes the technical reserve lower than the market reserve. The rationale for this is that when the contract was signed, r^* was indeed low compared to the contemporary market interest rate. More precisely we have the following additional parameters.

- Contract initiation age 30
- Age 50 at the time ($t = 0$) of calculation
- $r^* = 5\%$ p.a. (continuously compounded)
- Pure endowment at expiry of 1,597,593 USD (corresponding to the reserve of a life annuity, at expiry, of 110,023 USD p.a. computed under the market value basis)

6.2.5 Remarks

We comment on the significant difference in the pure endowments of the two contracts. The endowment sum computed at initiation of the contract should be viewed as what is then guaranteed. If the insurance company can obtain a higher interest than the technical interest rate, this sum is typically increased via surplus bonus as time goes by. Assuming, in the context of the new contract, that the insurance company realizes an interest rate of 5% p.a. and uses all surplus contributions to immediately and continuously increase the guaranteed pure endowment sum, it will be exactly the same as that computed for the old contract when the policy holder reaches the age of 50.

6.3 Numerical results

We now display and discuss the numerical results obtained in the two contractual contexts.

We mention first that model variations occur in two 'dimensions'; on the one hand regarding whether or not we include reactivation from disability and on the other hand whether or not we model the risk and behavioral chains independently. Finally, in the case when the chains are modeled independently, we consider the two situations when the free policy ratios are the same, $f^i(t) := f^a(t)$, from the states 'active' and 'disabled', and when they are not (they are precisely defined in Subsection 6.2.1 above).

Figure 4 below illustrates the various computational setups that we consider; for each box we have computed the corresponding reserve for both contracts. The obtained numerical results allow us to find variations in the reserve by alternating the model 'one step at a time'. In the figure below the arrows illustrate a situation where we start off by using a model without reactivation and independent chains and then additionally regard dependence and reactivation, one at a time. Hence, we have a means of decomposing the change between two models; in the case below the parts being a consequence by regarding dependence and regarding reactivation, respectively. We emphasize that the top left box, corresponding to the correct model, is always considered as the benchmark result in comparisons.

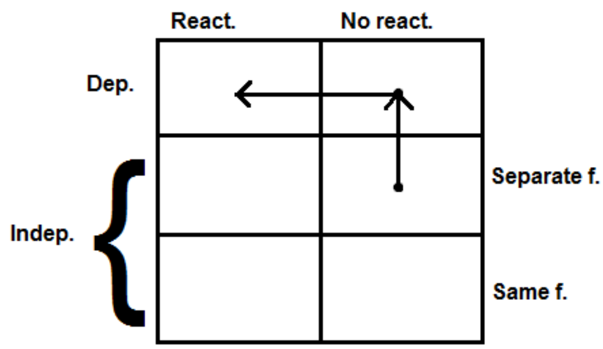

Figure 4: The included model variations, and a possible path between two of them.

Some inequalities between reserves computed under different setups can be obtained in general by theoretical considerations. For example it is fairly obvious that using, in the independent models, $f^i(t) := f^a(t) \leq 1$ rather than $f^i(1) \equiv 1$ yields, fixing the remaining parameters, a smaller market reserve. We show, however, by means of examples that other inequalities are indeed dependent on the concrete setup (e.g. the reactivation intensity).

6.3.1 New contract

Figure 5 below displays two reserves; the technical reserve ('Technical'), and the reserve in the model including reactivation from disability and the aforementioned dependence between the risk and behavioral Markov chains ('Dep., react.', the correct model) as functions of time, conditional on being in the states 'active' and 'active; premium payment', respectively. Recall that at time zero, the policy holder is 30 years old. As expected, the magnitude of the reserves are at different levels, primarily due to the difference in interest rate levels of the technical and market bases.

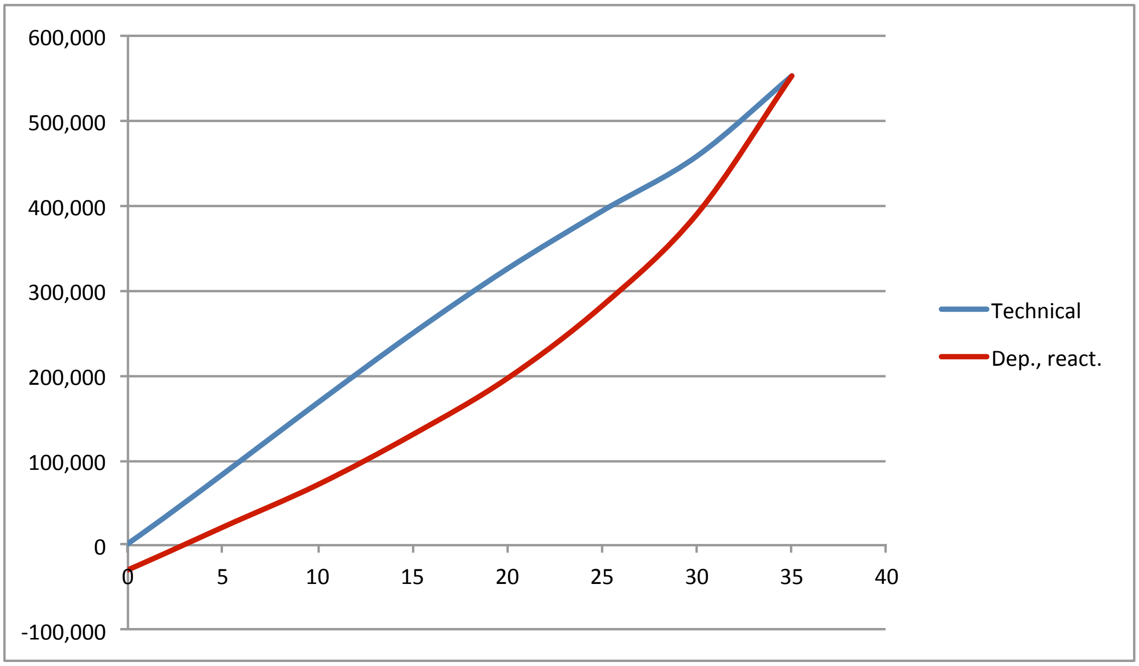

Figure 5: The technical reserve and the 'dependent' reserve (including reactivation)

We now present all computed reserves evaluated at a few time points and furthermore plot the difference between the market reserves and the correct market reserve ('Dep., react.') where we model reactivation and dependence between the chains.

t	0	5	10	15	20	25	30	35
Technical	0	83,621	167,653	249,401	325,518	393,614	458,275	552,796
Indep., no react., separate f	-19,493	34,300	85,716	143,915	206,457	287,067	391,278	552,796
Dep., no react.	-21,625	31,852	83,269	141,833	205,131	286,489	391,171	552,796
Indep., no react., same f	-23,212	30,728	82,425	141,259	204,800	286,340	391,145	552,796
Indep., react., separate f	-26,450	25,026	74,769	133,055	198,205	282,578	390,216	552,796
Indep., react., same f	-28,341	22,876	72,484	130,987	196,794	281,913	390,088	552,796
Dep., react.	-30,014	21,198	71,130	130,124	196,436	281,852	390,094	552,796

Figure 6: Computed reserves in descending order (in USD)

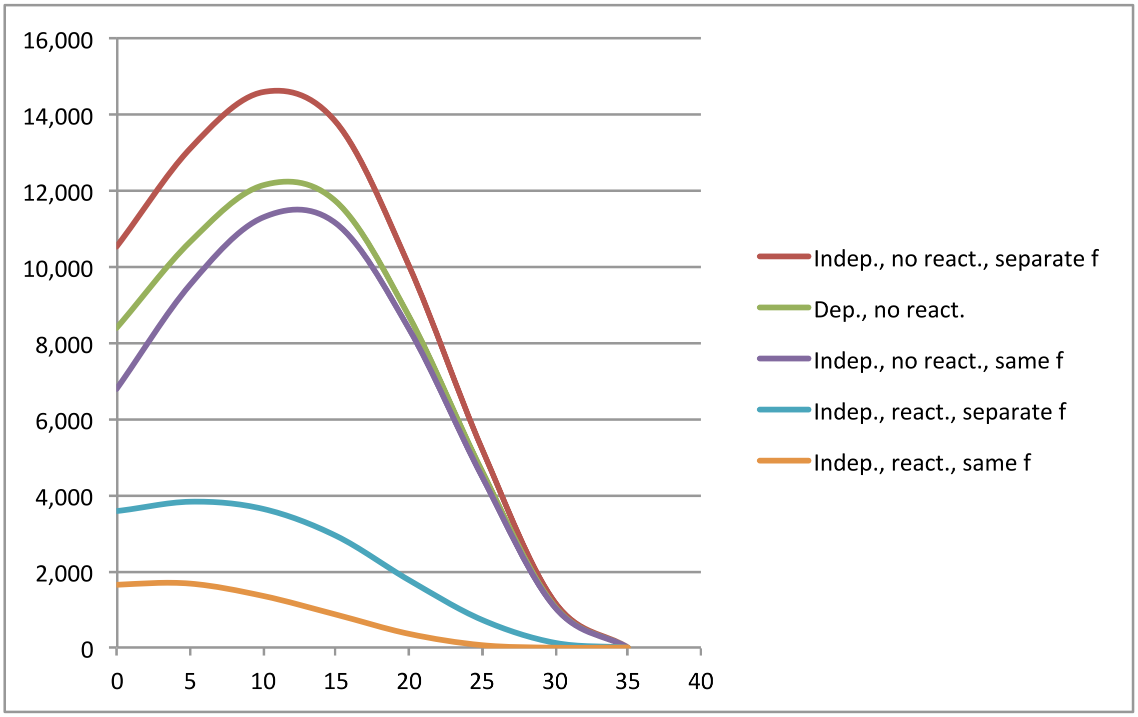

Figure 7: Differences compared to 'Dep., react.'

More precisely, for a given reserve V^{Model}, where Model is e.g. 'Dep., no react.', we have plotted the function

$$t \mapsto V^{\text{Model}}(t) - V^{\text{Dep., react.}}(t).$$

We draw a few conclusions from the results above.

1. For this particular setup an insurance company would in fact benefit from using the correct model; it yields the smallest reserve.

2. One cannot conclude from the above that using the model yielding the simplest closed form expression for the reserve, 'Indep., no react., same f', is 'at least on the safe side' of the correct reserve. Namely, from the numbers above it is clear that letting the reactivation intensity approach zero, the reserve in 'Dep., react' converges to that in 'Dep., no react.', which is larger than the one in 'Indep., no react., same f '. Hence the order relation between the two reserves is in fact intensity dependent.

3. It is by no means a surprise that the largest market reserve is 'Indep., no react., separate f'. We omit a formal argument but note that when assuming independent chains we have a surrender and free policy option in the disability state. When using separate f, i.e. $f^i \equiv 1$, exercising the free policy option from the disability state gives the policy holder $V^{i+}(t)$, in other words the value of his own contract with premium payment streams removed from all states. The surrender option, when exercised from the disability state, gives the policy holder an amount equal to the value of his contract computed under an interest rate lower than the market interest rate and furthermore disregarding reactivation and free policy (namely, the

technical reserve, $V^{i*}(t)$). Hence the disability state is as expensive as possible and, clearly, further omitting reactivation from this state additionally increases the reserve (the policy holder can never resume the premium payment instead of receiving the disability annuity).

4. Note that the difference between the reserves 'Indep., no react., separate f' and 'Dep., no react.' is caused solely by the surrender option. To see this, note that transcribing to free policy from the disability state gives the corresponding benefit reserve, $V^{i+}(t) = V^{i}(t)$ (no reactivation), scaled by the free policy ratio $f^{i}(t) \equiv 1$.

6.3.2 Old contract

Figure 8 below displays two reserves; the technical reserve ('Technical'), and the reserve in the model including reactivation from disability and the aforementioned dependence between the risk and behavioral Markov chains ('Dep., react.', the correct model) as functions of time, conditional on being in the states 'active' and 'active; premium payment', respectively. Recall that at time zero, the policy holder is 50 years old. As expected, the technical reserve is now below the market reserve as opposed to the situation in Section 6.3.1.

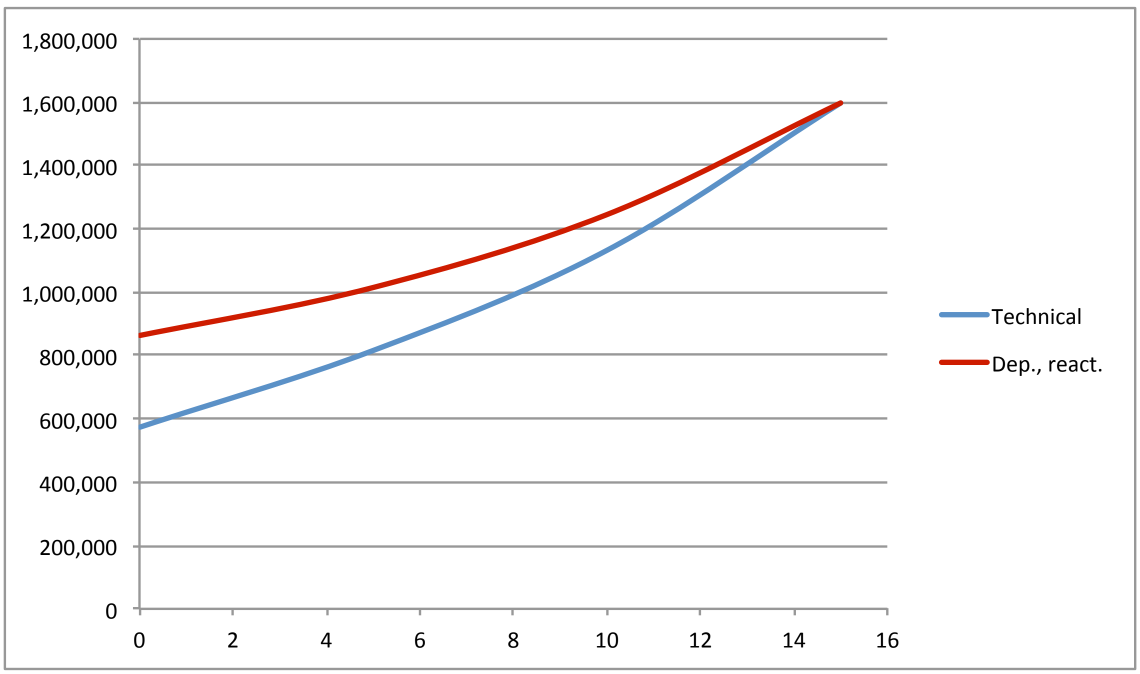

Figure 8: The technical reserve and the 'dependent' reserve (including reactivation)

We now present all computed reserves evaluated at a few time points and furthermore plot the difference between the market reserves and the correct market reserve ('Dep., react.') where we model reactivation and dependence between the chains.

t	0	5	10	15
Technical	573,984	815,950	1,132,248	1,597,593
Dep., no react.	872,815	1,020,138	1,246,235	1,597,593
Indep., no react., separate f	870,710	1,019,318	1,246,089	1,597,593
Indep., no react., same f	869,112	1,018,675	1,245,971	1,597,593
Dep., react.	861,537	1,014,869	1,245,103	1,597,593
Indep., react., separate f	860,342	1,014,298	1,244,981	1,597,593
Indep., react., same f	858,947	1,013,702	1,244,867	1,597,593

Figure 9: The technical reserve and the market reserves in descending order (in USD)

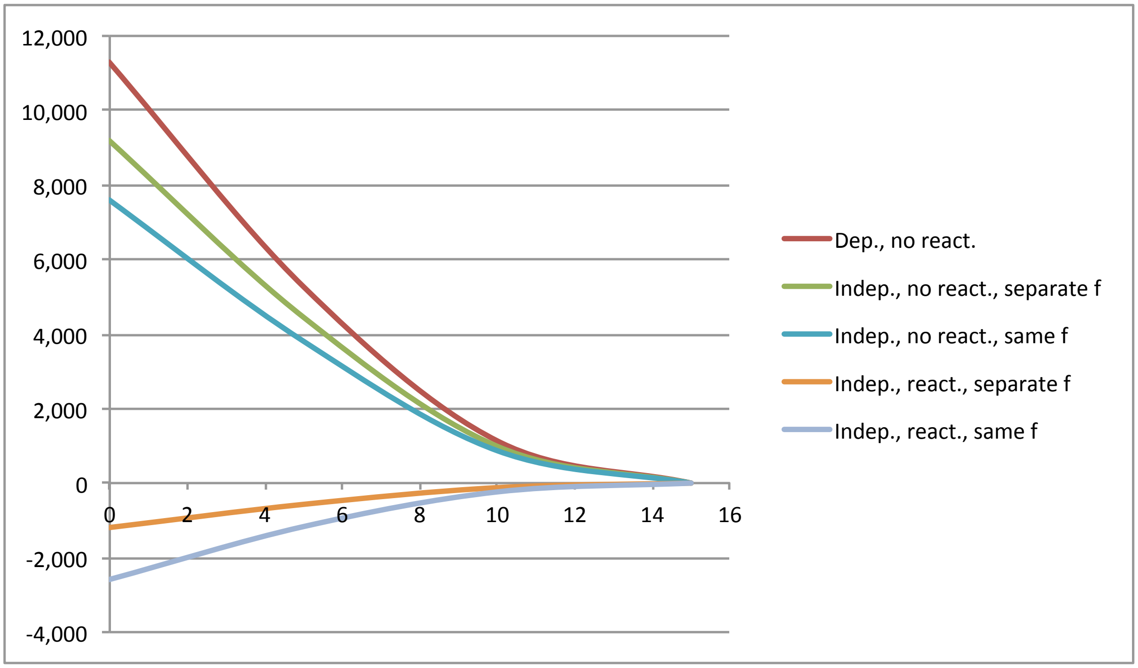

Figure 10: Differences compared to 'Dep., react.'

More precisely, for a given reserve V^{Model}, where Model is e.g. 'Dep., no react.', we have plotted the function

$$t \mapsto V^{\text{Model}}(t) - V^{\text{Dep., react.}}(t).$$

We draw a few conclusions from the results above.

1. The fact that we model the inclination to exercise policy holder options as decreasing with age makes the percentual differences between the various reserves significantly smaller here than in Section 6.3.1 since we here consider a policy holder that is 20 years older.

2. The reserve obtained when using the correct model 'Dep., react.' is below those in the models yielding the simplest semi-closed form solutions, 'Indep., no react., separate f' and 'Indep., no react., same f'. Note, however, that when making the reactivation intensity small the reserve in 'Dep., react.' tends to that in 'Dep., no react.' which is larger than both. Hence also in this situation these order relations are intensity dependent.

3. For precisely the same reason as in in Section 6.3.1, we note that the difference between the reserves 'Indep., no react., separate f' and 'Dep., no react.' is caused solely by the surrender option.

A The Danish FSA yield curve used in the market basis

We plot the (discretely compounded) yield curve, R as published on 2013-04-08 by the Danish FSA, from which we extracted the equivalent continuous forward rate used in the market basis above. We also plot the technical interest rates used in the two contracts considered in Section 6.2, here discretely compounded ($R^* = e^{r^*} - 1$).

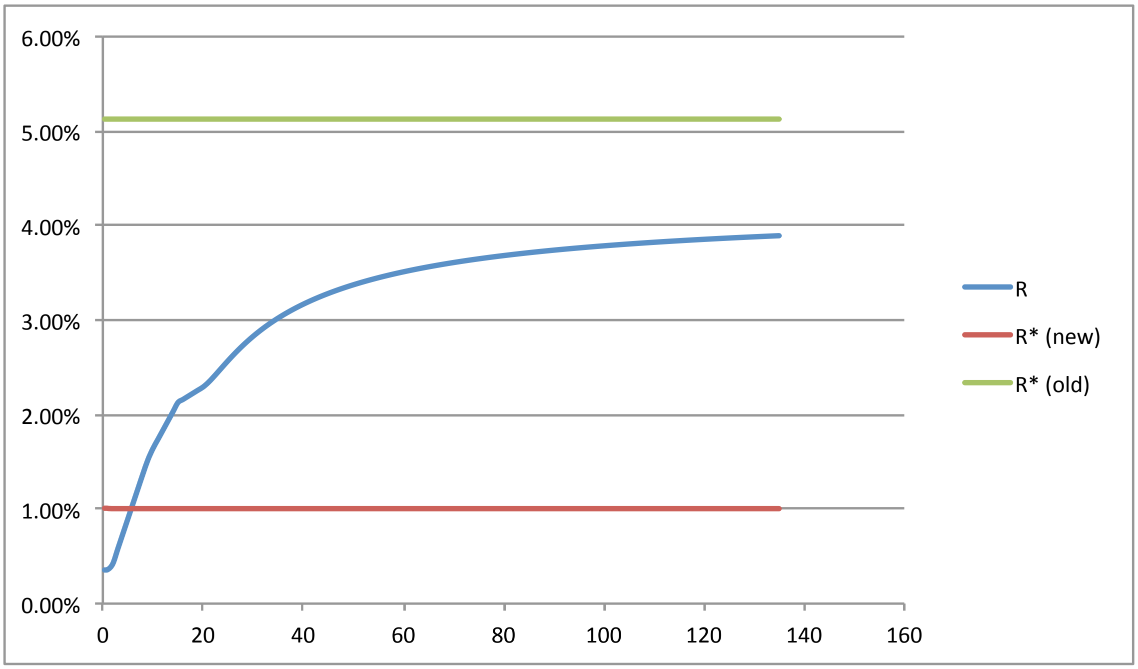

Figure 11: The interest rates used in the bases of the examples

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