Pricing of Guaranteed Minimum Benefits in Variable Annuities

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Agenda

Introduction and motivation

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Variable Annuities

- **Variable Annuities** (VA) are (deferred), fund-linked annuity and insurance products allowing guaranteed payments and participation in the financial markets at the same time.
- Examples for guaranteed payments include
  - minimum interest rate guarantees
  - ratchets
- Variable annuities are often referred to as GMxB, **Guaranteed Minimum Benefits** of type x:
  - GM**D**B (Death)
  - GM**A**B (Accumulation)
  - GM**I**B (Income)
  - GM**W**B (Withdrawal)

Markets for Variable Annuities

- Motivation
  - Increasing life expectancy
  - Reduction of the state retirement pensions in several countries
- Consequences
  - VA as a major success story in the North American insurance market
  - Rapid growth of VA business in Japan - from $1.3 billion in 2001 to more than $216 billion in 2011 (assets under management)
  - Europe as the latest market for Variable Annuities
- Risks: financial, actuarial, behavioral

Existing literature

- GMDB: financial protection to dependents of the insured in case of death [Milevsky and Posner 2001], [Ulm 2008]
- GMAB: choice between fund performance and guarantee at maturity [van Haastrecht et al. 2009]
- GMIB: market value of fund account paid at once or lifelong annuity [Boyle and Hardy 2003], [Marshall et al. 2010]
- GMWB: Possibility to withdraw money from account within certain limits [Milevsky and Salisbury 2006], [Dai et al. 2008]
- General framework for pricing GMxB's, either geometric Brownian Motion or numerical valuation:
  [Bauer et al. 2008], [Bacinello et al. 2011]

Our contribution:
Explicit solutions for the prices of GMABs in a hybrid model for insurance and market risk.

Valuation model

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Financial market model

Notation and definitions

- $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$: filtered probability space
- r: short rate process adapted to filtration $\mathbb{F}$ and money-market account

$$B(t) = exp\left(\int_0^t r(s)ds\right).$$

- $\mathbb{Q}$: **risk-neutral measure**
- S: traded security with S/B a $\mathbb{Q}$-martingale:

$$S(t) = \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T r(s)ds} S(T)|\mathcal{F}_t\right].$$

- Hull-White-Black-Scholes hybrid model with time-dependent volatility (HWBS). Dynamics under $\mathbb{Q}$:

$$dr(t) = (\theta_r(t) - a_r r(t))dt + \sigma_r dW_r^{\mathbb{Q}}(t),$$
$$dY(t) = \left(r(t) - \frac{1}{2}\sigma_Y^2(t)\right) dt + \sigma_Y(t)dW_Y^{\mathbb{Q}}(t),$$

where $Y(t) = ln\left(S(t)/S(0)\right)$ and $dW_r^{\mathbb{Q}}(t)dW_Y^{\mathbb{Q}}(t) = \rho dt$.

Insurance model

Notation and definitions

- **Random lifetime** of a person aged x at $t = 0$: Stopping time τ_x of counting process $N_{x+t}(t)$ with mortality intensity $\lambda_{x+t}(t)$ adapted to filtration $\mathbb{F}$.
- Mortality intensity independent from short rate and equity price.
- Introduce filtrations $\mathbb{H} = (\mathcal{H}_t)_{t\geq 0}$ with $\mathcal{H}_t = \sigma(\mathbb{1}_{\{\tau_x \leq s\}} : s \leq t)$ and $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$.
- **Survival probability**:

  Probability that a person of age $x+t$ at time t survives at least up to time T:

$$p_{x+t}(t,T) := \mathbb{Q}(\tau_x > T | \mathcal{G}_t).$$

- For a person of age of $x+t$ at time t it holds:

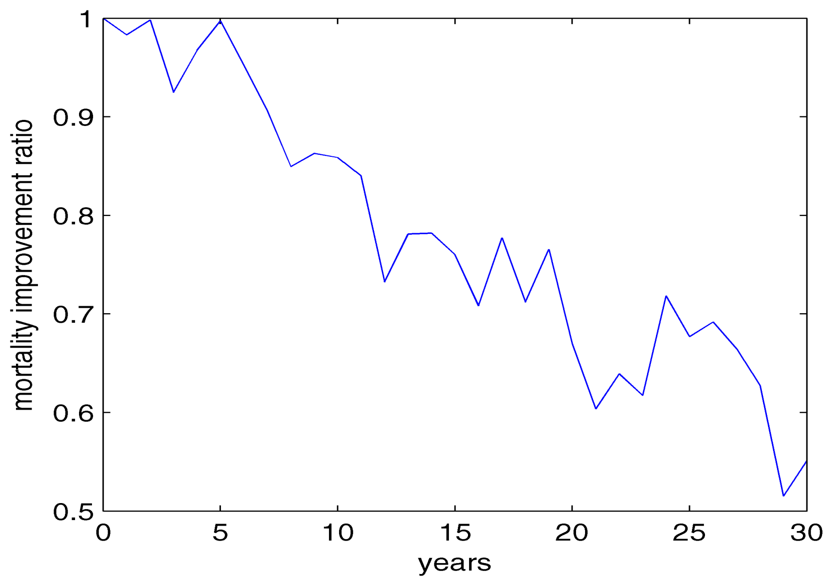
$$p_{x+t}(t,T) = \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T \lambda_{x+s}(s)ds} | \mathcal{G}_t \right] = \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T \lambda_{x+s}(s)ds} | \mathcal{F}_t \right].$$

Insurance model

Mortality improvement ratio

- Compare mortality intensity at time 0 with mortality intensity at time t
- **Mortality improvement ratio**:

$$\xi_{x+t}(t) = \frac{\lambda_{x+t}(t)}{\lambda_{x+t}(0)}$$



Sample path for the mortality improvement ratio

Insurance model
Mortality improvement ratio

- ξ_t modeled as an extended Vasicek process adapted to filtration $\mathbb{F}$:

$$d\xi(t) = k(e^{-\gamma t} - \xi(t))dt + \sigma_\xi dW^\xi(t).$$

- Initial mortality intensity described by Gompertz model:

$$\lambda_{x+t}(0) = \frac{1}{b} \cdot c^{\frac{x+t-m}{b}},$$

  calibrated to the current life table.

- Future mortality intensity can be calculated by

$$\lambda_{x+t}(t) = \lambda_{x+t}(0) \cdot \xi(t).$$

- Survival probability can be expressed as:

$$p_{x+t}(t,T) = C_\lambda(t,T)e^{-D_\lambda(t,T)\lambda_{x+t}(t)},$$

  where $C_\lambda(t,T)$ and $D_\lambda(t,T)$ satisfy two ordinary differential equations which can be solved analytically.

Pricing of variable annuities

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Guaranteed Minimum Accumulation Benefit
Definition

- IP: single premium
- $A(t)$: account value at time t, $A(0) =$ IP, 100% invested in equities.
- $G(T)$: guaranteed amount at end of the accumulation period T
- GMAB provides policyholder, who is alive at T, with a benefit $V(T)$:

$$V(T) = \mathbb{1}_{\{\tau > T\}} \cdot max(A(T), G(T))$$

- Common options for $G(T)$:
  - Return of premium: $G(T) =$ IP
  - Roll-up $G(T) =$ IP $\cdot e^{\delta T}$, with continously compounded roll-up rate δ
  - Ratchet $G(T) = \max_{t_i < T} A(t_i)$
- Fair value of GMAB at $t = 0$:

$$V(0) = \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_0^T r(s)ds}\mathbb{1}_{\{\tau > T\}} max(A(T), G(T))\right]$$

Guaranteed Minimum Accumulation Benefit

Roll-up guarantee

Theorem 1.

Explicit expression for $V(0)$ with $G(T) = \mathsf{IP} \cdot e^{\delta T}$:

$$V(0) \;=\; \mathsf{IP} \cdot p_x(0,T) \cdot \Phi\left(\frac{\mu^S_{Y(T)} - \delta T}{\sigma^S_{Y(T)}}\right) + \mathsf{IP} \cdot P^m(0,T) \cdot e^{\delta T} \cdot \Phi\left(\frac{\delta T - \mu^T_{Y(T)}}{\sigma^T_{Y(T)}}\right),$$

with

- Φ: distribution function of a standard normal distribution
- **Mortality-adjusted zero-coupon bond**:

$$P^m(0,T) = P(0,T) \cdot p_x(0,T).$$

- $\mu^S_{Y(T)}, \sigma^S_{Y(T)}$ are the moments under the equity measure $\mathbb{Q}^S$
- $\mu^T_{Y(T)}, \sigma^T_{Y(T)}$ are the moments under the forward measure $\mathbb{Q}^T$

Guaranteed Minimum Accumulation Benefit
Ratchet guarantee

Theorem 2.

Explicit expression for $V(0)$ with $G(T) = \max_{t_i<T} A(t_i)$:

$$V(0) = \text{IP} \cdot p_x(0,T) \cdot \Bigg(\Phi_{n-1}(0; -\mu^{\mathbf{S}}_{\mathbf{\Delta_k Y}}, \mathbf{\Sigma}^{\mathbf{S}}_{\mathbf{\Delta_k Y}}) + \sum_{k=1}^{n-1} \left(\Phi_{n-1}(0; -\mu^{\mathbf{S}}_{\mathbf{\Delta_k Y}} - \mathbf{\Sigma}^{\mathbf{S}}_{\mathbf{\Delta_k Y}} \mathbf{e_{n-1}}, \mathbf{\Sigma}^{\mathbf{S}}_{\mathbf{\Delta_k Y}}) \right) \cdot e^{\mu^S_{\Delta_{n,k}Y} + \frac{\left(\sigma^S_{\Delta_{n,k}Y}\right)^2}{2}} \Bigg),$$

with

- $\mathbf{e_k}$: unit vector with k-th element equal to 1
- $\mu^{\mathbf{S}}_{\mathbf{\Delta_k Y}}, \mathbf{\Sigma}^{\mathbf{S}}_{\mathbf{\Delta_k Y}}$ are the mean vector and covariance matrix under $\mathbb{Q}^S$ of

  $$\mathbf{\Delta_k Y} := \{\Delta_{i,k} Y\}_{i \in \{1,\ldots,n\} \setminus \{k\}}$$

  with

  $$\Delta_{i,k} Y := \{Y(t_k) - Y(t_i)\}_{i \in \{1,\ldots,n\} \setminus \{k\}}, \quad t_n := T$$

- $\Phi_{n-1}(\mathbf{u}, \mu, \mathbf{\Sigma})$: multivariate normal distribution function with mean vector μ and covariance matrix $\mathbf{\Sigma}$.

Guaranteed Minimum Accumulation Benefit

Ratchet guarantee

Proof.

- Separate insurance and financial parts and rewrite expectation:

$$
\begin{aligned}
V(0) &= \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_0^T r(s)ds} \cdot \mathbb{1}_{\tau>T} \cdot max\left(A(T), \max_{t_i} A(t_i)\right)\right] \\
&= \mathbb{E}_{\mathbb{Q}}\left[\mathbb{1}_{\tau>T}\right] \cdot \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_0^T r(s)ds} \cdot max\left(A(T), \max_{t_i} A(t_i)\right)\right] \\
&= p_x(0,T) \cdot \sum_{k=1}^{n} \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_0^T r(s)ds} \cdot A(t_k) \cdot \mathbb{1}_{A(t_k)\geq A(t_i), i\in\{1,\dots,n\}\setminus\{k\}}\right] \\
&= p_x(0,T) \cdot \left(\sum_{k=1}^{n} I_{t_k}\right)
\end{aligned}
$$

with

$$
I_{t_k} := \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_0^T r(s)ds} \cdot A(t_k) \cdot \mathbb{1}_{A(t_k)\geq A(t_i), i\in\{1,\dots,n\}\setminus\{k\}}\right].
$$

Guaranteed Minimum Accumulation Benefit
Ratchet guarantee

Proof (continued).

- Change to equity measure:

$$\begin{aligned} I_{t_k} &= \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_0^T r(s)ds} \cdot A(t_n) \cdot \frac{A(t_k)}{A(t_n)} \cdot \mathbb{1}_{A(T)\geq A(t_i), i\in\{1,\ldots,n\}\setminus\{k\}}\right] \\ &= A(0) \cdot \mathbb{E}_{\mathbb{Q}^S}\left[\frac{A(t_k)}{A(t_n)} \cdot \mathbb{1}_{\frac{A(t_i)}{A(t_k)}\leq 1, i\in\{1,\ldots,n\}\setminus\{k\}}\right] \\ &= A(0) \cdot \mathbb{E}_{\mathbb{Q}^S}\left[e^{Y(t_k)-Y(t_n)} \cdot \mathbb{1}_{Y(t_i)-Y(t_k)\leq 0, i\in\{1,\ldots,n\}\setminus\{k\}}\right] \\ &= A(0) \cdot \mathbb{E}_{\mathbb{Q}^S}\left[e^{\Delta_{nk}Y} \cdot \mathbb{1}_{\Delta_{ki}Y\leq 0, i\in\{1,\ldots,n\}\setminus\{k\}}\right] \end{aligned}$$

  with

$$\Delta_{ij}Y = Y(t_j) - Y(t_i),\ t_n := T.$$

- Integration over multivariate normal density function gives final formula.

Model calibration

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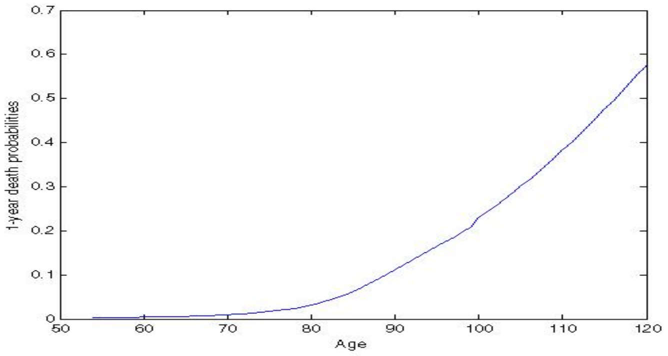
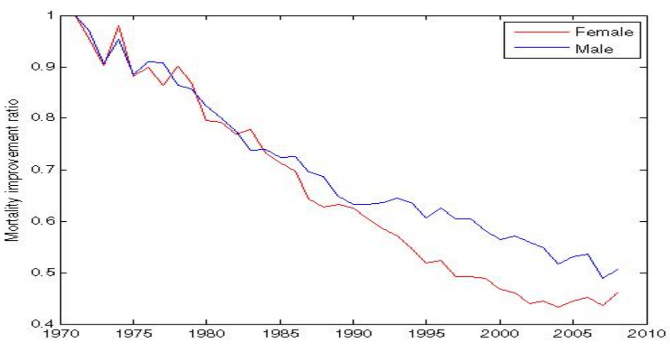
Insurance model calibration
Data

- Initial mortality table (Source: Federal Statistical Office of Germany)



- Mortality improvement ratio (Source: Federal Statistical Office of Germany)



Insurance model calibration
Algorithm and results

- Gompertz model: via least-squares method.
- Mortality improvement ratio: via maximum likelihood method.
- Log-likelihood function:

$$
\begin{aligned}
\mathcal{L}(k,\gamma,\sigma_\xi) &= \sum_{i=1}^{n} \ln(f(\xi_i|\xi_{i-1};k,\gamma,\sigma_\xi)) \\
&= \frac{n}{2}\ln(2\pi) - n\ln\hat{\sigma}_\xi \\
&- \frac{1}{2\hat{\sigma}_\xi^2}\sum_{i=1}^{n}\left(\xi_i - \xi_{i-1}e^{-k\cdot\Delta} - \frac{k}{k-\gamma}e^{-\gamma t_i}\cdot\left(1-e^{(\gamma-k)\cdot\Delta}\right)\right)^2,
\end{aligned}
$$

where

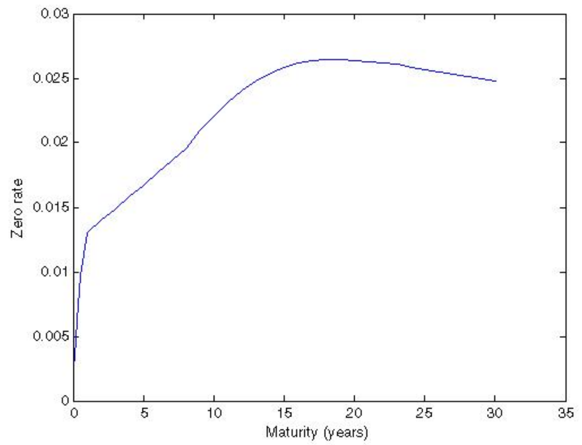
$$\hat{\sigma}_\xi = \sigma_\xi\sqrt{\frac{1-e^{-2k\cdot\Delta}}{2k}}$$

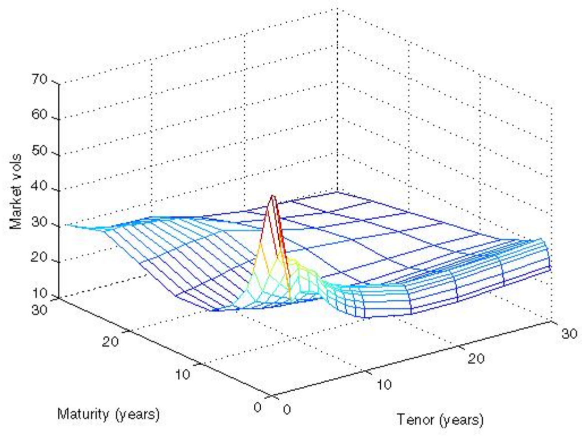
- Result:

Mortality	b	m	k	γ	σ_ξ
female	7.80	88.09	0.5529	0.0223	0.0512
male	9.57	83.89	0.4301	0.0179	0.0485

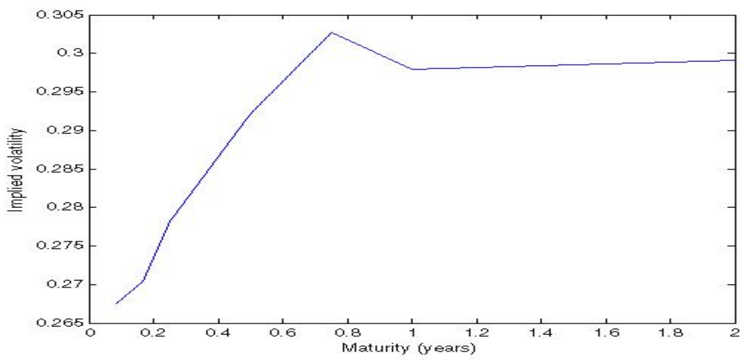
Financial model calibration
Data

- Interest rate data: deposit rates, swaps, swaptions (Source: Bloomberg)



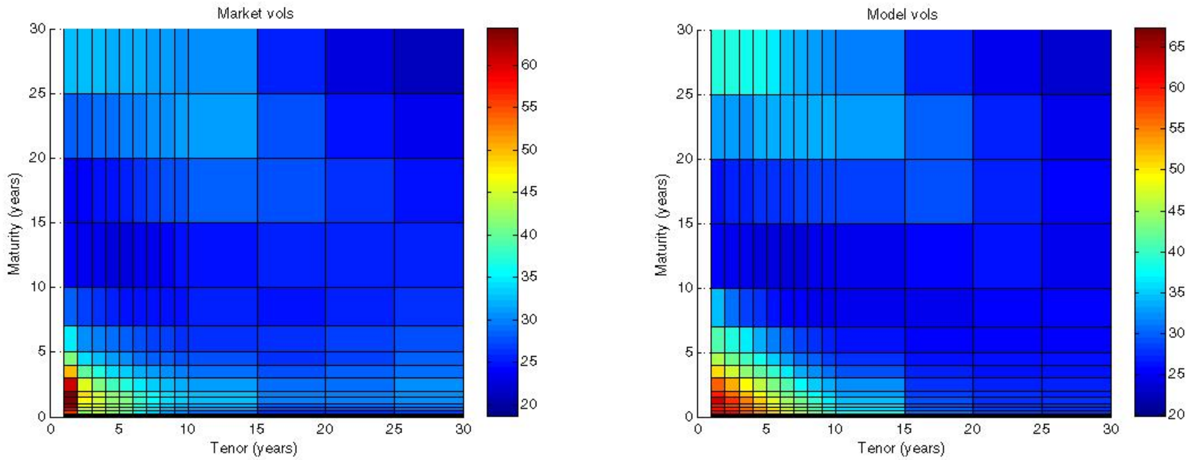


- Equity data: implied volatilities term structure (Source: Bloomberg)



Financial model calibration
Algorithm

- $\theta_r(t)$: shift to current term structure of interest rates
- Hull-White model: minimize sum of squared deviations from observed European swaption prices



- Result: $a_r = 0.0151$ and $\sigma_r = 0.009$.
- Instantaneous volatility: (piecewise) constant, extracted by recursion.
- Correlation: historical correlation between EuroStoxx50 log-returns and absolute differences in 3-month zero rates.
- Result: $\sigma_S = 0.2923$ and $\rho = 0.1209$.

Example

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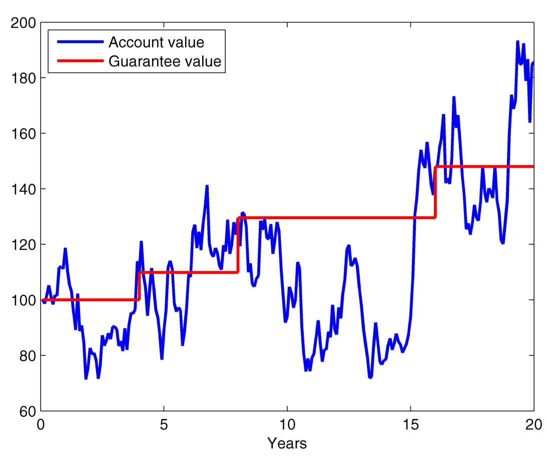
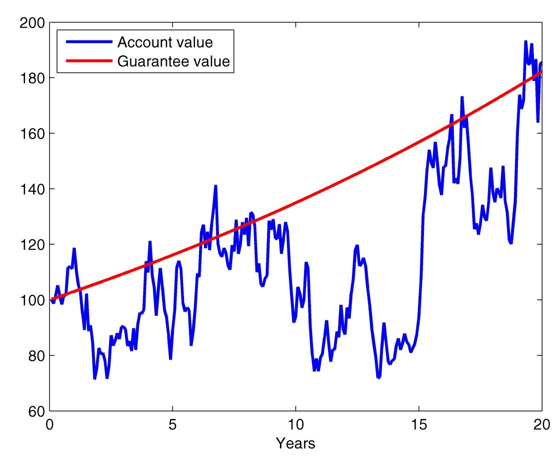
Setup

- Type of the guarantee: single premium GMAB, T = 20 years.
- Maturity of the guarantee: 20 years.
- Policyholder: male, 45 years old.
- Mortality improvement ratio: German population for period 1968-2008.
- Roll-up and ratchet considered:



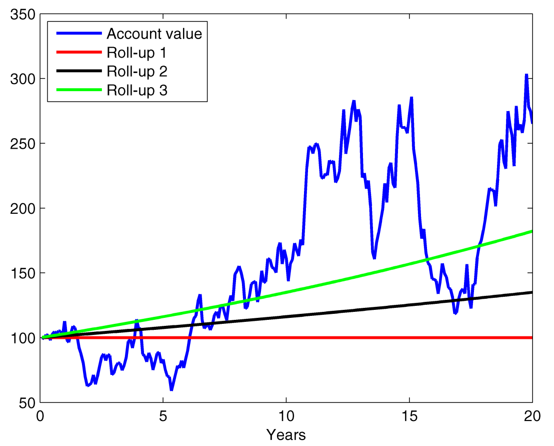
Ratchet step = 4 years



Roll-up rate = 2%

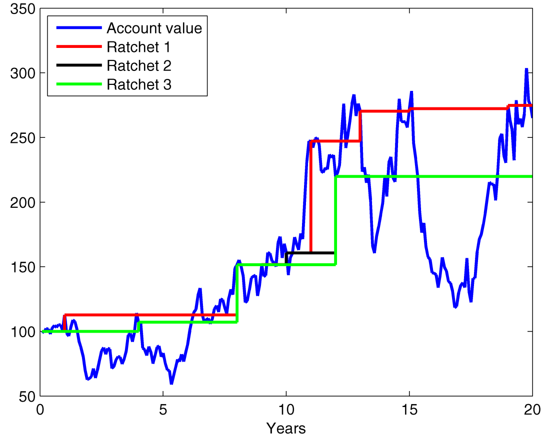
Sensitivities to product parameters
Roll-up rate

- Roll-up guarantee



Roll-up	Roll-up rate	GMAB
1	0%	102.49
2	1.5%	111.51
3	3.0%	125.64

- Ratchet guarantee



Ratchet	Ratchet step	GMAB
1	2 years	125.28
2	4 years	118.49
3	8 years	114.19

Sensitivities to financial market parameters
Equity volatility

- Sensitivities: Central difference quotient for a parallel shift of $\pm 0.01\%$.
- Stress test according to QIS5 calibration paper for Solvency II[a]: Relative increase (up stress) of 50% and decrease (down stress) of 15% from current value.
- Roll-up guarantee

ImpVol	Roll-Up 1	Roll-Up 2	Roll-Up 3
Sensitivity	0.75%	0.98%	1.19%
Current value	102.49	111.51	125.64
Up stress	111.66	122.99	139.43
Down stress	99.69	107.83	121.13

- Ratchet guarantee

ImpVol	Ratchet 1	Ratchet 2	Ratchet 3
Sensitivity	2.24%	1.81%	1.24%
Current value	125.28	118.49	114.19
Up stress	155.89	142.53	133.53
Down stress	117.14	111.93	108.74

[a] Committee of the European Insurance and Occupational Pension Supervisors, CEIOPS-SEC-40-10.

Sensitivities to financial market parameters
Interest rates

- Sensitivities: Central difference quotient for a parallel shift of $\pm 0.01\%$.
- Stress test scenarios according to QIS5 calibration paper for Solvency II[a]:
- Roll-up guarantee

IR	Roll-up 1	Roll-up 2	Roll-up 3
Sensitivity	-4.19%	-6.73%	-10.44%
Current value	102.49	111.51	125.64
Up stress	98.90	105.73	116.63
Down stress	107.96	120.13	138.80

- Ratchet guarantee

IR	Ratchet 1	Ratchet 2	Ratchet 3
Sensitivity	-6.76%	-6.16%	-4.74%
Current value	125.28	118.49	114.19
Up stress	120.95	114.35	110.49
Down stress	133.92	126.33	121.12

[a] The altered term structures are derived by multiplying the current interest rate curve by $1 + s^{up}$ and $1 + s^{down}$, where s^{up} (s^{down}) ranges from 0.70 (-0.75) for short-term maturities to 0.25 (-0.30) for long-term maturities.

Sensitivities to insurance market parameters

Mortality

- Sensitivities: one-directional difference quotient for a relative decrease of 1%.
- Stress test according to Solvency II requirements: 25% reduction applied to entire mortality table.
- Roll-up guarantee

Mortality	Roll-up 1	Roll-up 2	Roll-up 3
Sensitivity	0.11%	0.12%	0.14%
Initial	102.49	111.51	125.64
Reduced	105.32	114.56	129.10

- Ratchet guarantee

Mortality	Ratchet 1	Ratchet 2	Ratchet 3
Sensitivity	0.14%	0.13%	0.08%
Initial	125.28	118.49	114.19
Reduced	128.72	121.76	117.33

Conslustion & Outlook

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Conclusion & further research

- HWBS for the financial market.
- 2-step approach for stochastic mortality modelling.
- Explicit expressions for GMABs with different guarantee riders.
- Calibration of the presented hybrid model.
- Example with sensitivity analysis.

- Analyse other types of guarantees (GMIB, GMDB).
- Incorporate policyholder behavior risk.
  (with Escobar, M., Ramsauer, F., Saunders, D., Zagst, R.)

Thank you for your attention.

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Appendix
Zero-coupon bond

- Zero-coupon bond:

$$P(t,T) = \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T r(u)du}|\mathcal{F}_t\right] = C_r(t,T) \cdot e^{-D_r(t,T)r(t)}$$

with

$$C_r(t,T) = \frac{P^M(0,T)}{P^M(0,t)} \cdot exp\left[D_r(t,T)f^M(0,t) - \frac{\sigma_r^2}{4a_r}(1-e^{-2a_r t})D_r(t,T)^2\right]$$
$$D_r(t,T) = \frac{1}{a_r}\left[1-e^{a_r(t-T)}\right]$$

- Long-term zero-coupon rate $R(t,T)$ is a linear function of short rate $r(t)$:

$$R(t,T) = -a + br(t),$$

with

$a := log(C_r(t,T))/(T-t)$ and $b := D_r(t,T)/(T-t)$.

Appendix
Zero-coupon bond as a numeraire

- $\mathbb{Q}^T$: **T-forward measure** with zero-coupon bond $P(\cdot, T)$ as numeraire.
- Corresponding Radon-Nikodym derivative:

$$\frac{d\mathbb{Q}^T}{d\mathbb{Q}} = \frac{P(T,T)/P(t,T)}{B(T)/B(t)} = exp\left[-\frac{1}{2}\int_0^T \gamma^2(t)dt - \int_0^T \gamma(t)dW_r^{\mathbb{Q}}\right],$$

with

$$\gamma(t) = \sigma_r \cdot D_r(t,T).$$

- Dynamics under $\mathbb{Q}^T$:

$$dr(t) = (\theta_r(t) - a_r r(t) - \sigma_r^2 D_r(t,T))dt + \sigma_r dW_r^{\mathbb{Q}^T}(t),$$

$$dY(t) = \left(r(t) - \frac{1}{2}\sigma_Y^2(t) - \sigma_Y(t)\sigma_r\rho D_r(t,T)\right)dt + \sigma_Y(t)dW_Y^{\mathbb{Q}^T}(t).$$

- $r(T)$ and $Y(T)$ are normally distributed with corresponding moments

$$\mu_{r(T)}^{\mathbb{Q}^T}, \sigma_{r(T)}^{\mathbb{Q}^T} \text{ and } \mu_{Y(T)}^{\mathbb{Q}^T}, \sigma_{Y(T)}^{\mathbb{Q}^T}.$$

Appendix
Equity price as a numeraire

- $\mathbb{Q}^S$: **equity measure** with equity price S as numeraire.
- Corresponding Radon-Nikodym derivative:

$$\frac{d\mathbb{Q}^S}{d\mathbb{Q}} = \frac{S(T)/S(t)}{B(T)/B(t)} = exp\left[-\frac{1}{2}\int_0^T \sigma_Y^2(t)dt + \int_0^T \sigma_Y(t)dW^Y(t)\right],$$

- Dynamics under $\mathbb{Q}^S$:

$$dr(t) = (\theta_r(t) - a_r r(t) + \sigma_r\sigma_Y(t)\rho)dt + \sigma_r dW_r^{\mathbb{Q}^S}(t),$$
$$dY(t) = \left(r(t) + \frac{1}{2}\sigma_Y^2(t)\right)dt + \sigma_Y(t)dW_Y^{\mathbb{Q}^S}(t).$$

- $r(T)$ and $Y(T)$ are normally distributed with corresponding moments

$$\mu_{r(T)}^{\mathbb{Q}^S}, \sigma_{r(T)}^{\mathbb{Q}^S} \text{ and } \mu_{Y(T)}^{\mathbb{Q}^S}, \sigma_{Y(T)}^{\mathbb{Q}^S}.$$