An Optimal Asset Allocation in the Pre–Retirement Accumulation Period, Conditional on the Post–Retirement Deccumulation

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Abstract – *We investigate optimal asset allocation in the pre–retirement period, when the individual accumulate his assets, with the respect of the chosen optimal asset allocation and annuitization strategies in the post–retirement period. We assume that the individual has access to the three assets: risk–free asset i.e. one year bond, low–risk asset i.e. 10–years rolling bond and risky asset i.e. equities. We assume that the individual is going to retire at age 65. Before retirement the individual has income from salary and has no access to the annuities and after retirement the individual has no salary but social security income. We investigate three cases regarding the availability of the annuities after retirement. The first one is when the individual has no access to annuities, the second one is when the individual has access to the annuities at age 65 only, and the third one is when the individual can annuities at any age. If the individual has access to annuities, we assume that he will annuitize optimally. We assume that the individual draws utility from consumption and that the criterion for optimization is maximizing expected discounted utility. We use the results for post–retirement optimal consumption, asset allocation and annuitization from Gavranovic (2012). These results are characterized in the utility function at age 65. In this paper, we start from that utility function and investigate optimal consumption and asset allocation in pre–retirement period. We compare the resulting expected discounted utilities in pre–retirement period and make the conclusions about the pre–retirement optimal consumption and asset allocation with respect to the availability of the access to annuities in the post–retirement period assuming optimal consumption, asset allocation and annuitization.*

Keywords*: optimal asset allocation; utility from consumption; interest rate; computer modeling; discrete time/state spaces, three assets model*

1. Introduction

1.1 Main Features of Defined Contribution Pension Scheme

The member of the Defined Contribution Pension Scheme (DCPS) joins the scheme in the early years of his employment, and stays involved up to the end of his life. In a pre–retirement period, the prospective pensioner contributes into his pension account and that period of the member's life is referred to as the accumulation phase. Contributions are invested into appropriate assets yielding investment returns.

At the end of a DCPS member's active working period of life, he has certain assets that are then used for income in retirement. In many countries the state provides certain income to the pensioners in the form of social security. Additionally, the pensioner will receive an income from his defined contribution pension schemes.

Throughout this paper, we assume that our investigation is done on the microeconomic level. In other words, we assume that the market is exogenously given and the market is perfectly competitive. In that environment, the member is a price–taker, i.e. the member's decision cannot influence the market itself. We assume that the financial market is frictionless meaning that all costs and restraints associated with transactions are non–existent. Taxation is ignored. We allow that trading and flow of money into or out of a fund are done only at distinct time–points, thus choosing a discrete time environment for the model.

1.1.1 Pre–Retirement Period

In the pre–retirement period, the DCPS member contributes the new amounts on a regular basis during the whole period, invests any new contributions, and also reinvests any amount earned from investments. Usually no outflow, i.e. no consumption of the pension wealth, is allowed in the accumulation period. However, we can observe DCPS member's overall assets and assume that he consumes part of his wealth and saves the remaining part.

The member, together with the investments advisers, will manage the assets available in the portfolio. The higher return and the lower risk are often stated as the most important requirements of the asset allocation strategy. Many other criteria can be set up, and asset allocation can be managed and assessed in accordance with these criteria as well (for example Haberman and Vigna (2002)).

The investment strategy usually adopted by actuaries and investment managers of DCPS in pre–retirement period is the "lifestyle strategy" (Vigna and Haberman (2001)). The lifestyle strategy in the accumulation period means that the member switches from more to less risky assets when he is close to retirement. In practice, it means a higher proportion of stocks in earlier years and a gradual switch towards bonds and maybe cash in the years before retirement. The time when this switch begins is usually less than ten years before retirement. The switch is usually implemented gradually throughout the last five to ten years in the pre–retirement period. If the decrease of the percentage invested in the risky asset and increase of the percentage invested in less risky asset is a deterministic function of the time left to retirement, then it is referred to as a deterministic lifestyle strategy. On the other hand, if these percentages are stochastic processes, then it is referred to as a stochastic lifestyle strategy.

1.1.2 Post–Retirement Period

In the post–retirement period, the member's contributions into the pension fund terminate, and the consumptions of the assets accumulated prior to the time of retirement commence. We differentiate income and consumption in retirement. In this paper, we assume that income in retirement comes from social security and from annuities bought earlier in retirement. Consumption is the amount that the pensioner actually consumes. The amounts used for purchasing annuities are deemed as change of the form of the pension wealth, and purchasing annuities is neither income nor consumption. If the income is larger than the consumption in certain periods then the difference between income and consumption is simply added to the pension wealth. Otherwise, the positive difference between consumption and income is deducted from the pension wealth. We can categorize income in retirement in three main groups: annuities, income drawdown and the combination of these two.

The annuity is a financial contract, usually offered by an insurance company, to provide a given income on a regular basis from the moment when an annuity is bought until the annuitant's death. Bequeathing some assets on death can be specially arranged.

On the other side of the spectrum of income plans in retirement is income drawdown, sometimes also referred to as self–investment in retirement or self–annuitization. By taking income drawdown, the member keeps the control of the allocation of his pension wealth in retirement. In order to provide income in retirement, he deducts certain amounts from the pension fund from time to time. In contrast to annuitization, self–annuitization involves a positive probability that the member will run out of pension wealth while still alive. Income in retirement is the combination of social security income, income from annuities and self–annuitization.

1.2 Asset Allocation in Pre–retirement Period in DCPS

The analysis of DCPS is usually done separately for the accumulation period and for the decumulation period. One reason for this approach could come from the real life experience. The time of retirement is a turning point in life, the end of salary earning and accumulation, i.e. end of a saving strategy for the retirement period and the beginning of the decumulation and income from the social security and from the assets in possession, i.e. beginning of the pension consumption strategy. The other reason lies in the complexity of the models investigating both phases at the same time.

In this paper we want to investigate optimal asset allocation strategies in the pre–retirement period for the DCPS member retiring at age 65, with a certain initial starting wealth at that age 25, with a stochastic salary process from age 25 to age 65, with a certain replacement rate at age 65, with a calculated utility function at age 65. The DCPS member in this paper wishes to maximize utility drawn from consumptions during pre–retirement conditional on the different utility functions at age 65. We want to develop optimal asset allocation strategies for the DCPS member wishing to maximize expected discounted utility drawn from future consumption and bequest.

Our work in this paper can be deemed as an extension of known models and results of optimal asset allocation in pre–retirement in the directions of adding dependence on the optimal asset allocation and annuitization strategies in the post–retirement period.

1.2.1 Asset Allocation

According to the model developed in this paper, the DCPS member can invest in equities as a high–risk asset with a random return, in long–term bonds as a low–risk asset with a random return and a one–year bond as risk free asset. We develop the optimal asset allocation strategy so that expected discounted utility from consumption is maximized. We assume no borrowing constraints in our models and results.

The DCPS member can choose the asset allocation for the whole wealth in his possession. We will not make the difference between pension and the remaining wealth. We develop optimal asset allocation as function of the state variables, where the state variables are known values of the variables which influence future developments. Once knowing those functions, we can also make a sample of random realizations (simulations) and investigate behaviors of optimal asset allocation paths for the DCPS member.

1.3 Utility function at the end of Pre–retirement Period

In this paper, we assume that at age 65 the DCPS member has the utility function calculated numerically in Gavranovic (2012). This utility function is developed assuming optimal asset allocation and annuitization strategies for the DCPS member retiring at age 65, with a certain pension wealth at that age, with a certain last salary received at age 65, with a certain replacement rate at age 65, with a certain income from social security during retirement period, with certain personal preferences towards risk and bequest, and with certain limitations on his asset allocation and annuitization strategies. The DCPS member during his retirement period wishes to maximize utility drawn from consumptions during retirement and also from bequeathing assets to his heirs if the pensioner has a bequest motive. Under these assumptions and the assumptions of the optimal asset allocation and annuitisation strategies for the pensioner wishing to maximize expected discounted utility drawn from future consumption and bequest in retirement, in this paper we want to investigate different optimal asset allocation strategies in the pre–retirement period.

1.4 Structure of the Paper

After the introduction in Chapter 1, we present the review of literature relevant for the investigation done in this paper in Chapter 2. In Chapter 3, we develop the pre–retirement model with three assets, and present the numerical solution to the problem. We also state the problem for post–retirement period in order to have a complete problem at one place. However, the problem for the –post–retirement period is solved in Gavranovic (2012) and we use these results. In Chapter 4, we investigate the results using the pre–retirement period model developed in Chapter 3. The most important findings of the developed models and the conclusions drawn from the numerical results based on the model are presented in Chapter 5. We provide a discussion on possible future research based on the results obtained in this paper. In Appendix, we present the technique of decreasing the number of state variables from four to three variables.

2 Literature Review

Lifecycle models follow an individual throughout his lifetime and investigate income and consumption patterns. In this paper we investigate the pre–retirement period with a particular emphasis on the dependence of the pre–retirement asset allocation strategy on the different annuitization programs in the post–retirement period.

The basic idea of lifecycle consumption can be given as follows. People generate income applying their labor and have desires and needs to consume. However, income and consumption do not match each other throughout the whole of life. In their early working ages, people usually spend more than they earn and generally not much is saved. The salary growth is the fastest for this age group. The early working–age period is followed by ages 40 to 50 years, when earning is higher than needs for consumption and the worker is aware of his lifecycle. This age group saves the most. Then, near the end of the working age, salary growth slows down and or even a salary decrease is experienced. However, the worker is fully aware of the approaching retirement period of life and tends to save more for old age. A retiree does not earn any more, but still has needs and desires to spend. This is financed from the assets accumulated throughout the working period of life and from social security income, or in other words from consumption given up in the working period of life. Figure 2.1 graphically shows this process.

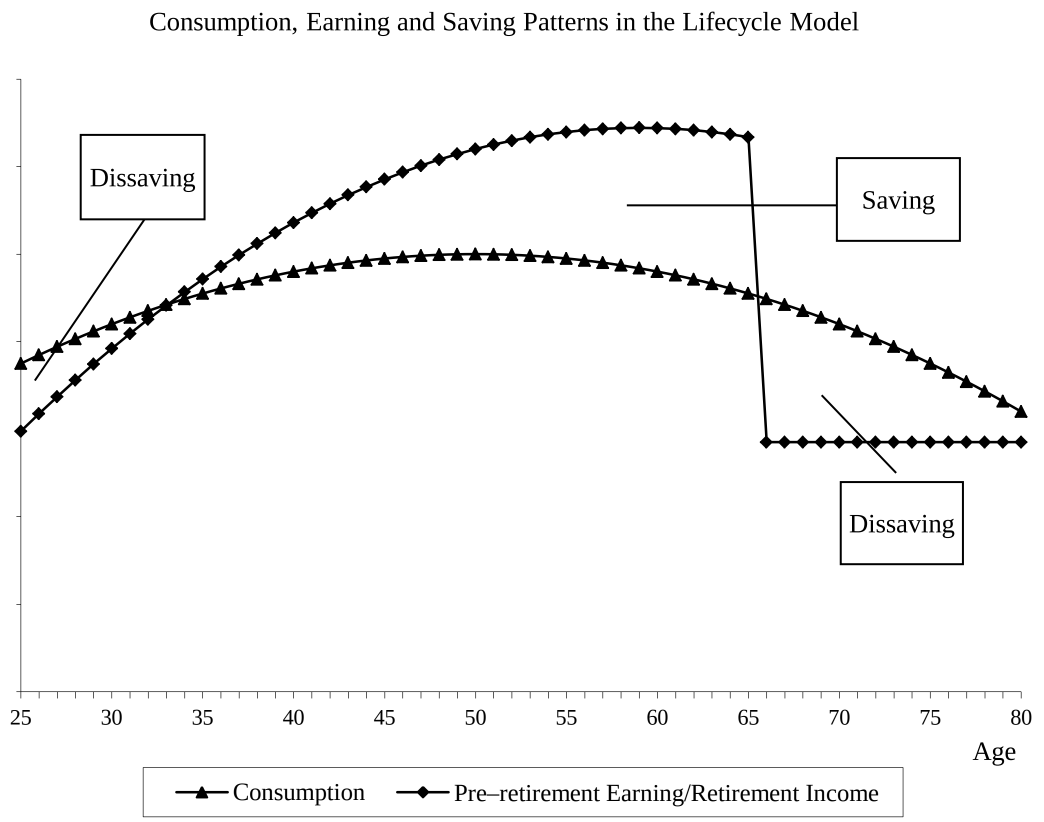


Figure 2.1 Lifecycle patterns

DCPS models for the pre–retirement period have characteristics of long term investment models with no or little initial wealth, periodic contributions and consumptions. We assume here a fixed date of retirement. On the other hand, the models for the post–retirement period will have characteristics of asset allocation and annuitization with a single contribution at the beginning and the stream of consumption afterwards. Income and consumption is needed as long as the member is alive and the time of death is not certain.

2.1. Risks Faced by the DCPS Member before Retirement

Two main risks in the pre–retirement period are the investment risk and the risk of inadequate contributions. The member bears the risks of the high volatility of return and lower than expected investment returns. The investment risk is particularly important a few years prior to the time of retirement, because if DCPS member wishes to purchase annuities not much time is left for asset prices to recover and income in retirement is lower compared to pre–retirement period.

The properly chosen asset allocation strategy can decrease or eliminate some of these risks. However, the criteria for properly chosen asset allocation and annuitization strategies will not be related to the different risks to the same extent. Optimizing to a certain criterion usually means handling one or more risks, but not all. So, we should always think of the optimal asset allocation strategy as dependent on the particular criterion or criteria.

2.2 Models and Results in Pre–Retirement Period

The models for pension wealth development in the pre–retirement period are characterized by income from investment and from contribution, and outflow due to consumption. The asset allocation strategy objective is to provide the appropriate wealth at the moment of retirement. The appropriate wealth at the moment of retirement means that the member will be in a position to obtain a satisfactory income in retirement from the accumulated pension wealth.

2.2.1 In Discrete Time

The model for DCPS fund value developed by Ludvik (1994) incorporates the most important variables and develops a closed form formula for pension benefit as a fraction of the final salary. Pension benefit is modeled as an annuity after withdrawing a lump sum at retirement. Numerical investigations are done using Wilkie (1986) model. The Wilkie (1986) model is based on modeling financial variables using time series. He finds that bonds and cash are a superior strategy to the equity and deterministic lifestyle, although with a lower median.

Booth and Yakoubov (2000) analyses deterministic lifestyle investment strategy close to retirement based on historical datasets. They use Wilkie’s simulation model, where parameters are determined by historical values from the available databases. A number of asset allocation strategies are analyzed with respect to the post–retirement preferences towards the decumulation choice of the pension wealth. Funding is analyzed for cash, for purchasing a fixed annuity and for purchasing an index–linked annuity at retirement. They find no evidence for supporting the superiority of a lifestyle investment strategy. However, they find strong evidence for supporting a well–diversified investment strategy until retirement rather than a one–off switch to low risk asset. They also conclude that the investment strategy close to retirement should be dependent on the required decumulation strategy.

The dynamic programming approach in a discrete time framework is applied by Vigna and Haberman (2001), and Haberman and Vigna (2002). Vigna and Haberman (2001) investigate the model with the two assets, one low–risk and the other high–risk. The assets are modeled by assuming that annual asset returns are iid log–normally distributed, and that returns from different assets are uncorrelated. They develop a multi–period model for DCPS asset accumulation and determine the optimal investment strategy that minimizes member's discounted future costs.

2.2.2 In Continuous Time

Boulier, Huang and Taillard (2001) set up the model for DCPS where the guarantee in the form of the minimal fund value is given on the benefit. The rate of interest is modeled using the Vasicek framework, and the guarantee is a bond like liability. They assume two sources of randomness: one from the interest rate and the other from the stock itself. The assets available for investments are cash, bonds with the constant time to maturity and stocks. The rate of contribution is assumed to follow a simple exponential function. They maximize the expected utility of the excess of the fund over guarantee, where CRRA utility function is taken.

Deelstra, Grasselli and Koehl (2000) investigate optimal investment problem with initial wealth only and no further contribution, and where the stochastic interest rate follows the Cox–Ingersoll–Ross model. They explicitly expressed the asset allocation strategy which maximizes the expected utility of the terminal wealth. They use the Cox, Huang (1989) methodology and find the explicit solution in the form of optimal proportions that should be invested in each asset in order to maximize CRRA utility drawn from the final wealth. The maximization problem in this paper is closely related to the modified maximization problem stated and solved by Boulier et al (2001). The difference is the model for stochastic interest rate, with the CIR framework probably being less easy to manage.

In related paper Deelstra, Grasselli and Koehl (2003) exploit their model and results from Deelstra et al (2000), now in the continuous time framework of the accumulation period for DCPS. Deelstra et al (2003) tackle the problem of optimal asset allocation in order to maximize the expected utility of the excess of the terminal wealth over the minimum guarantee. They assume the complete market, investing in cash, bonds and stock, CRRA utility function, and affine dynamics of the stochastic interest rate. An explicit solution of the optimal asset allocation is found under the assumption that a contribution process and the guarantee are not subject to its own sources of risk. The results include Vasicek as well as CIR stochastic interest rate models as special cases. Appling the model from Deelstra et al (2000), Deelstra et al (2003) move in the direction of obtaining the optimal guarantee that maximizes the expected utility function of the benefit in DCPS.

2.3 Lifecycle Models and Results

If we optimize asset allocation in order to maximize member's utility at the end of the accumulation period, drawn from the post–retirement consumption, we in fact take into account post–retirement asset allocation and possibly annuitization.

Cocco, Gomes and Maenhout (2005) develop a lifecycle model of consumption and portfolio choice with non–tradable uncertain labor income and borrowing constraints. They assume CRRA utility function and one risk free and one risky asset and also allow for the presence of the bequest motive of the member. They calibrate the model realistically and analyses a number of realistic labor income possibilities. Given the quantitative focus of the article, they investigate what can reduce the average allocation to stocks and thus bring the empirical predictions of the model closer to what is observed in the data. They give a number of results regarding optimal asset allocation and optimal consumption depending on many different changes in the model set up. In terms of the lifecycle pattern of optimal asset allocation, the share invested in equities is roughly decreasing with age. With an increase in age, labor income becomes less important and the investor reacts optimally to this by shifting his financial portfolio towards the risk free asset. There is no annuity option in this model, but they realistically model pension fund, income and optimal consumption and asset allocation in post–retirement period.

Horneff, Maurer, Mitchell and Stamos (2009) and Horneff, Maurer, and Stamos (2008) are two similar papers. Basically, the authors use the same models, with the difference that in Horneff el at (2009) they assume that the investor has access to variable annuities and in Horneff et al (2008) they assume access to the constant real payout lifetime annuities. Other assumptions are almost the same and we will concentrate on Horneff et al (2008) as it is more relevant to the work in this paper. They observe the investor over lifecycle facing uninsurable income risk, ruin risk, equity investment risk and uncertain lifetime. They introduce an incomplete annuity market into the lifecycle model assuming that the investor has access to annuities anytime during his lifetime. The investor can convert his available assets into one risky, one riskless asset, and into annuities. Each year, he optimally chooses the allocation into equities, bonds, annuities and optimally chooses consumption. The investor has subjective survival probabilities, while annuities are calculated using objective survival probabilities. He aims to maximize his discounted utility drawn from future consumption and bequest, if a bequest motive is present. They use Epstein–Zin preferences as in Epstein and Zin (1989). The model for income and parameterization is mostly the same as the ones used by Cocco et al (2005) and we use the same technique in this paper as well. Due to untradeable labor income, the irreversibility of annuity purchases and the short selling restrictions, the problem cannot be solved analytically, and they adopt the standard approach of dynamic stochastic programming to solve the investor's optimization problem. They find that over time the annuity demand increases (age effect) for the following reasons. The mortality credit of annuities, the excess return above the bond return, increases with age. The sinking value of human capital results in a lower stock demand, as human capital is perceived as a closer substitute to a bond investment than to equity. Liquidity is also required to rebalance the portfolio. The demand for annuities also increases with the level of wealth on hand (wealth effect) because the investor does not require a high stock position in financial wealth in order to compensate for the investment in bond like human capital. In addition, the higher is the wealth in hand, the lower is the need for liquidity. They were not able to explain a limited annuitization in the market. Utility gains from purchasing annuities are still substantial. They suggest that behavioral factors might explain the remaining part of the “annuity puzzle”.

2.4 Our Position in Literature

In this paper, we investigate the pre–retirement period only but using the result from investigation of the post–retirement period done in Gavranovic (2012).

Besides PhD thesis Gavranovic (2012), the two main articles used as a starting point for the development of the models in this paper are the models developed by Cocco, Gomes and Maenhout (2005) and Horneff, Maurer and Stamos (2008). These authors investigate the lifecycle model. We develop the model for the DCPS member, retiring at age 65 with an uncertain and limited life time. If we observe the model investigated by Cocco et al (2005), and if this individual has access to annuities and to the three assets, then we get the model in this paper. Also, we observe the model investigated by Horneff, Maurer and Stamos (2008), and if this individual has constant relative risk aversion utility function, access to annuities after age 65 only and access to three assets then we again get to the models in this paper.

Our model and results can be compared with the results by Boulier et al (2001) and Deelstra et al (2000). We actually make discrete time and space approximation of the bond market developed by Boulier et al (2001), and similar reasoning could be applied to the work of Deelstra et al (2000). We derive a discrete time and space stochastic interest rate model and develop the bond market and derive the model such that the DCPS member has access to three assets and annuities after. The main difference in this paper compared to work done by Boulier et al (2001) and Deelstra et al (2000) is in the utility function at age 65. We use the three different utility function derived from the consumption in retirement, each under a certain assumption regarding availability of annuities in retirement.

3 The model

We investigate pre–retirement and post–retirement periods of the DCPS member's. In our lifecycle model, we use the results from Gavranovic (2012) for post–retirement period and use these results to investigate optimal consumption and asset allocation in pre–retirement period depending on the different optimal consumption, asset allocation and annuitization strategies in the post–retirement period. These different optimal strategies are results of the different assumptions regarding access to annuities in retirement. Thus, we actually investigate how important is access to annuities in retirement to the optimal consumption and asset allocation in pre–retirement period. We also measure the gains from the access to annuities in retirement in term of the increase in expected discounted utility at the start of the lifecycle model for the DCPS member.

In pre–retirement period, we model the market consisting of three possible investment options. Firstly, there are three assets: risk free assets – one year bond, low risk asset – Υ_t year rolling bond, and high risk asset – equities. We emphasize that all amounts in this model are in real terms, i.e. we assume that inflation has no influence to our results.

Regarding post–retirement period assumptions, we assume that the retirement age is 65 and that the DCPS member receives his last salary at that age. At age 66 he receives the first income from social security which continues at the beginning of each year of DCPS member's life until his death. We assume that income from social security is constant.

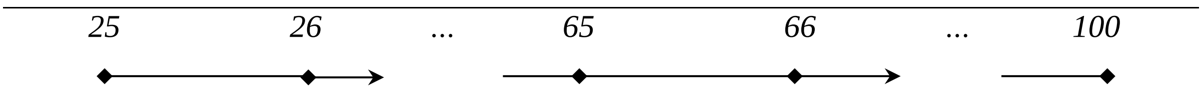
Throughout lifecycle, the DCPS member draws utility from consuming part of his available assets at the beginning of the year. Available assets consist of his wealth and received income. If a bequest motive exists then besides drawing utility from consuming the DCPS member draws utility from bequeathing assets to heirs. We assume that the remaining assets are bequeathed to heirs at the end of the year in which the member dies.

At the beginning of the year, the DCPS member receives income and interest, and then he consumes part of his available assets and invests the rest into three assets. The DCPS member allocates his assets (and annuitize in post–retirement period) at his discretion apart from no borrowing constraint.

In our model, we assume two sources of randomness: random interest rate and random rate on equity investment. On the other side, we have bonds and equities.

The graphical presentation of the most important variables in this problem is given as follows

State (information) variables						
W_t is pension wealth, Y_t is income, r_{t-1} is known interest rate during previous year						
W_{25}	W_{26}	$\dots$	W_{65}	W_{66}	$\dots$	W_{100}
$\tilde{Y}_{25}$	$\tilde{Y}_{26}$	$\dots$	$\tilde{Y}_{65}$	$\tilde{Y}_{66}$	$\dots$	$Y_{100} = 0$
r_{24}	r_{25}	$\dots$	r_{64}	r_{65}	$\dots$	r_{99}
Random variables						
$\tilde{r}_t$ is random interest rate, $\tilde{r}_t^e$ is random rate on stock investment						
$\tilde{r}_{25}$	$\tilde{r}_{26}$	$\dots$	$\tilde{r}_{65}$	$\tilde{r}_{66}$	$\dots$	–
$\tilde{r}_{25}^e$	$\tilde{r}_{26}^e$	$\dots$	$\tilde{r}_{65}^e$	$\tilde{r}_{66}^e$	$\dots$	–
Control (decision) variables						
C_t is consumption, α_t^e is proportion invested into equities, α_t^b is proportion invested into bonds, m_t is proportion used for purchasing annuities						
C_{25}	C_{26}	$\dots$	C_{65}	C_{66}	$\dots$	–
α_{25}^e	α_{26}^e	$\dots$	α_{65}^e	α_{66}^e	$\dots$	–
α_{25}^b	α_{26}^b	$\dots$	α_{65}^b	α_{66}^b	$\dots$	–
0	0	$\dots$	m_{65}	m_{66}	$\dots$	–
Age during the decumulation process						



We work in the discrete time. We assume that the pre–retirement accumulation period starts at age $t = 25$, post–retirement decumulation process starts at age $t = 65$, and finishes at age $t = 100$. The accumulation period lasts for 40 years and the decumulation process lasts for 35 years. If a bequest motive exists, then the DCPS member aged 99 will consume part of his assets and the rest will be invested and bequeathed when he dies during that year. Otherwise, he will consume everything at age 99 and nothing will be left for investing. In the earlier periods, the DCPS member consumes part of his available assets, uses one part for purchasing annuities (in retirement only) and invests the rest into three available assets. As we will see, the solution to the problem follows the same pattern for different periods. Hence, it is useful to investigate one representative period and then the solution to the whole problem can be derived from the solution of one representative period.

3.1 Constraints on Annuitization

We assume that the member annuitizes part of the available pension wealth. The member aims to maximize the expected discounted utility derived from consumption and a possible bequest by choosing the optimal consumption, asset allocation and annuitization strategies. Regarding annuitization, we distinguish the assumptions for the proportions of the pension wealth m_t to be annuitized. We group these assumptions into three groups of constraints on annuitization to be investigated as follows:

3.1 Annuitizing m_t part of pension wealth exogenously for all ages $65 \le t \le 99$. Under this assumption, the pensioner firstly chooses in a predetermined way how much to annuitize and for a given m_t he consumes and invests optimally the remaining part of pension wealth. The control variables in post–retirement period are $\{C_t, \alpha_t^e, \alpha_t^b\}$, m_t is determined exogenously and is suboptimal. The model can handle any assumption about predetermined values of m_t for $65 \le t \le 99$. We use the results of Gavranovic (2012) from this type of constraint where the DCPS member has no access to annuities. Thus, we assume no annuitization as a special case of exogenous annuitization. For the no annuities assumption we will have $m_t = 0$ for $65 \le t \le 99$. This model is similar to the one developed by Cocco et al (2005) but in this paper we assume three assets.

3.2 Annuitizing m_t part of pension wealth exogenously for some ages and endogenously for the others. In this case, the control variables are $\{C_t, \alpha_t^e, \alpha_t^b\}$ for ages where annuities are chosen exogenously and $\{C_t, \alpha_t^e, \alpha_t^b, m_t\}$ for ages where annuities are chosen endogenously. The model allows us to calculate the results for any combination of exogenous/endogenous annuitization. All we need to know is for which age annuitization is endogenous, and for which it is exogenous, and for exogenous annuitization ages we need to know the value of m_t. We use the results of Gavranovic (2012) from this type of constraint where the DCPS member optimally annuitizes at age 65 and no annuities are available afterwards. This model is similar to the one developed by Cocco et al (2005) but in this paper we assume three assets and optimal annuitization at age 65.

3.3 m_t is the optimally chosen proportion for all ages $65 \le t \le 99$. In this case, the member maximizes the value function with respect to the four control variables, and control variables are $\left\{C_t, \alpha_t^e, \alpha_t^b, m_t\right\}$. We use the results of Gavranovic (2012) from this type of constraint where the DCPS member optimally annuitizes at any age after 65. This model is similar to the one developed by Cocco et al (2005) but in this paper we assume three assets and optimal annuitization after age of 65.

The constraints on annuitization influence post–retirement period only. All results in post–retirement period are developed by Gavranovic (2012). The model and constraints in pre–retirement period are unique. The different results for pre–retirement period and for the overall lifecycle model come from the different constraints on annuitization in post–retirement period. Using this technique, we are in a position to compare different results and to make conclusions about the dependence of the optimal consumption, asset allocation in pre–retirement period and the dependence of the expected discounted utility from the whole lifecycle period on the access to annuities in post–retirement period.

3.2 The Model

Let us define the model that will be investigated in this paper. We present the model for both pre–retirement and post–retirement period and these two models are simply joined at age 65. The utility function, numerically derived by Gavranovic (2012) is all we need to join these two periods. Thus, we use the resulting numerical utility function derived by Gavranovic (2012) and do not investigate post–retirement period in this paper any more. We solve the model for pre–retirement period under assumption that we know utility function at age 65.

3.2.1 Definitions and Notation

We use the following definitions and notation:

- W_t is the pension wealth at time *t*, just before income Y_t is received;
- X_t is the defined as $X_t = W_t + Y_t$ and is referred to as cash-in-hand;
- $\tilde{U}_t$ is transitory (temporary) income shock during the period $\left[t-1, t\right]$ for $t = 20, 21, ..., 65$.
- $\tilde{N}_t$ is permanent (persistent) income shock during the period $\left[t-1, t\right]$ for $t = 20, 21, ..., 65$.
- Y_t is the variable denoting income at time *t*. We model income as

$$Y_t = e^{f(t)} P_t \tilde{U}_t \tag{3.1}$$

$$\tilde{P}_t = P_{t-1} \tilde{N}_t \tag{3.2}$$

$P_{19}=1$, for $t=20,21,...,65$, $Ln\tilde{U}_t \sim N\left(0,\sigma_U^2\right)$ and $Ln\tilde{N}_t \sim N\left(0,\sigma_N^2\right)$.

- C_t is consumption during the period $\left[t,t+1\right]$ for $t=20,21,...,65$, just receiving income Y_t. For the reason of simplicity, we assume that the consumption is done at the beginning of the period $\left[t,t+1\right]$;
- b_t is the factor which controls the pensioner's strength of the bequest motive. If no bequest motive exists then $b_t=0$, for $t=20,21,...,65$;
- $\tilde{r}_t$ is the random real interest rate during the period $\left[t,t+1\right]$ for $t=20,21,...,65$. We model the real interest rate as autoregressive process

$$\tilde{r}_{t+1}-r_t=(a_d-b_d r_t)\Delta t-\sigma_{dr}\tilde{\varepsilon}_r(t) \tag{3.3}$$

where a_d, b_d and σ_{dr} are constants and random variable $\tilde{\varepsilon}_r(t)$ is defined via its transitional matrix $\left\{p_{jk}\right\}_{(j,k)=(1,1)}^{(N,N)}$, as explained in Section 4.2. r_{64} is known interest rate during the year prior to retirement. The value of real interest rate r_t during the period $\left[t-1,t\right]$ is known at time t.;

- p_t – probability that the member aged t will survive until the age of $t+1$;
- $\bar{r}_t$ – variable denoting deterministic rate of return on one year risk free investment during the period $\left[t,t+1\right]$, for $t=20,21,...,65$;
- $\tilde{r}_t^e$ – random variable denoting random real rate on equities during the period $\left[t,t+1\right]$, for $t=20,21,...,65$. We assume that $\left[t,t+1\right]$ is one year period, and that

$$Ln(\tilde{r}_t^e)=\mu_e+\sigma_e\tilde{\varepsilon}_e\left(t\right) \tag{3.4}$$

where μ_e and σ_e are constants and $\tilde{\varepsilon}_e\left(t\right)\sim N\left(0,1\right)$;

- $\tilde{r}_t^b$ – random variable denoting random real rate on bond investment during the period $\left[t,t+1\right]$, for $t=20,21,...,65$;
- α_t^e – the proportion of the wealth invested in the equities during the period $\left[t,t+1\right]$, for $t=20,21,...,65$;
- α_t^b – the proportion of the wealth invested in the bonds during the period $\left[t,t+1\right]$, for $t=20,21,...,65$;
- m_t – the proportion of the pension wealth used for purchasing annuity at time t, for $t=20,21,...,65$;
- $B\left(T,r_{t-1}\right)$ – the price of the zero–coupon bond at time t maturing after T years and with r_{t-1} being experienced interest rate during the period $\left[t-1,t\right]$, for $t=20,21,...,65$. $B\left(T,r_{t-1}\right)$ is defined earlier;

The control variables of the most general type of the problem are $\left\{c_t,\alpha_t^e,\alpha_t^b\right\}_{t=20}^{65}$, and the state variables of the problem are $\left\{t,W_t,Y_t,r_{t-1}\right\}_{t=20}^{65}$. We will skip explicitly writing the state variable t and write state variables as $\left\{W_t,Y_t,r_{t-1}\right\}_{t=20}^{65}$. As we will see, we will decrease the number of control variables from three to two and observe control variables $\left\{X_t,r_{t-1}\right\}_{t=20}^{65}$.

Regarding risk free investment, we will assume that the member invest in risk free deposit with duration of one year. The rate on one–year risk free investment is calculated as follows

$$\overline{r}_t=\frac{1}{B\left(1,r_{t-1}\right)}-1.$$

Regarding low risk investment, we will assume that the pensioner aged t invests in bonds with the duration of Υ_t years, for $20\le t\le 65$. It means that at age t, the pensioner invests in Υ_t–years bonds at the beginning of the year and at the end of year he sells the bonds with Υ_t-1 years to maturity, rebalances his portfolio and then again purchases bonds with the duration of Υ_t years, and so on. According to this strategy, at the beginning of the period $\left[t,t+1\right]$, the member invests the amount of $\alpha_t^b\left(W_t+Y_t-C_t\right)$ into bonds and purchases them for the price of $B\left(\Upsilon_t,r_{t-1}\right)$, where r_{t-1} is real interest rate during the previous year. At the end of year, he possesses in his bond portfolio the amount of

$$\frac{\alpha_t^b\left(X_t-C_t\right)}{B\left(\Upsilon_t,r_{t-1}\right)}B\left(\Upsilon_t-1,r_t\right)=\alpha_t^b\left(X_t-C_t\right)\frac{B\left(\Upsilon_t-1,r_t\right)}{B\left(\Upsilon_t,r_{t-1}\right)}.$$

Thus, we can write that, observed at time t, the rate of return on bond investment during the year $\left[t,t+1\right]$ is

$$1+\tilde{r}_t^b=\frac{B\left(\Upsilon_t-1,\tilde{r}_t\right)}{B\left(\Upsilon_t,r_{t-1}\right)}. \tag{3.5}$$

In the main results we will assume that $\Upsilon_t=10$, for $20\le t\le 65$. It means that we will assume that the pensioner invests in 10–year rolling bonds. However, we make it more general in the model such that it is possible to use the model with the assumption of different duration of rolling bonds and that duration can depend on age.

Let us now introduce the random variable $\tilde{r}_t^P$, representing rate of return on portfolio investment during the year $\left[t,t+1\right]$

$$\begin{aligned}\tilde{r}_t^P &= \left(1-\alpha_t^e-\alpha_t^b\right)\overline{r}_t+\alpha_t^e\tilde{r}_t^e+\alpha_t^b\tilde{r}_t^b \\ &= \overline{r}_t+\alpha_t^e\left(\tilde{r}_t^e-\overline{r}_t\right)+\alpha_t^b\left(\tilde{r}_t^b-\overline{r}_t\right) \\ &= \overline{r}_t+\alpha_t^e\left(\tilde{r}_t^e-\overline{r}_t\right)+\alpha_t^b\left(\frac{B\left(\Upsilon_t-1,\tilde{r}_t\right)}{B\left(\Upsilon_t,r_{t-1}\right)}-1-\overline{r}_t\right)\end{aligned} \tag{3.6}$$

for $t=20,21,...,65$.

In interest rate risk model, we assume that all variables are in real terms. Real interest rate is modelled based on Vasicek model, and from this model we develop the market of bonds providing return in real terms. We assume in this paper that the real interest rate, and also derived bond market, is not correlated with the stock market. This assumption is a simplification of the real world in order to have more compact set of results. Introduction of the correlation between the market of bonds providing real return and the market of stock providing real return would bring the new results. We acknowledge that investigating correlation between real interest rate and stock return is important. We also acknowledge that introduction of correlation between real interest rate and stock return in the interest rate risk model is possible and computationally feasible. We leave this analysis for further research and hope that the results in this paper will be a good basis for the further research in this direction.

We assume that the member wishes to maximise expected utility from his future consumption and possibly a bequest. The utility function is CRRA function, given by

$$u(x)=\frac{x^\gamma}{\gamma} \text{ for } \gamma<1,\gamma\neq 0 \text{ and,}$$
$$u(x)=Log(x) \text{ for } \gamma=0.$$

3.2.2 Income Process

In this section we present all details of the income process. We define income process until age 65 while income process is subject to random shocks.

We assume that income at age $t=20$ is equal to $\tilde{Y}_{20}=e^{f(20)}\tilde{N}_{20}\tilde{U}_{20}$. Once, we know realisation of the random variable $\tilde{N}_{20}$, we also know $P_{20}=N_{20}$. Then, at age $t=21$ income is equal to $\tilde{Y}_{21}=e^{f(21)}P_{20}\tilde{N}_{21}\tilde{U}_{21}$. Generally, income at ages $20\le t\le 65$ is given with

$$\tilde{Y}_t=e^{f(t)}P_{t-1}\tilde{N}_t\tilde{U}_t \tag{3.7}$$

where factor $P_{t-1}=\prod_{i=20}^{t-1}N_i$.

At age $t=65$, the last salary

$$Y_{65} = e^{f(65)} P_{64} N_{65} U_{65} \tag{3.8}$$

is received. Afterwards, for ages $66 \le t \le 99$, the member's income is explained in Gavranovic (2012). It is useful to define the following variable

$$G_{t+1} = \frac{U_t Y_{t+1}}{U_{t+1} Y_t} \tag{3.9}$$

Knowing that $Y_t = e^{f(t)} P_t U_t$ and that $Y_{t+1} = e^{f(t+1)} P_t N_{t+1} U_{t+1}$, equation (3.9) can be written as

$$G_{t+1} = \frac{e^{f(t+1)}}{e^{f(t)}} N_{t+1} \tag{3.10}$$

3.2.3 Mathematical Model for the Problem

We will assume that the member's pension wealth is always non–negative, i.e. $W_t \ge 0$ for $20 \le t \le 99$. We assume that the member knows his utility function as well optimal consumption and optimal asset allocation and annuitisation for ages $65 \le t \le 99$. The problem is stated and solved in Gavranovic (2012). In this paper, we want to investigate member's optimal consumption and asset allocation before retirement, for ages $20 \le t < 65$, under assumption that after retirement his the member will behave optimally.

Gavranovic (2012) developed numerical value function $V_{65}(W_{65}, Y_{65}, r_{64})$ for $W_{65} \ge 0$ and $Y_{65} \ge 0$, r_{64} in the domain of the interest rate. Gavranovic (2012) assumes that the pensioner firstly receives income from investments, then annuitizes his pension wealth, then receives income from social security and from previously bought annuities, then consumes the part of the remaining pension wealth and income and the at the end the remaining amount is invested. In this paper, we assume that in the preretirement period the individual firstly receives investment income, then salary income, then consumes part of his wealth and then invest the remaining amount. Thus, in the preretirement period, cash-in-hand, $X_t = W_t + Y_t$ is important variable and there is no need to differentiate W_t and X_t. That's why, we will first calculate $V_{64}(X_{64}, r_{63})$ where we will use value function $V_{65}(W_{65}, Y_{65}, r_{64})$ and for lower values of t, we will calculate $V_t(X_t, r_{t-1})$ as a function of $V_{t+1}(X_{t+1}, r_t)$. In the main text of this paper we define the problem and explain its solution for $20 \le t \le 63$, while very similar solution for $t = 64$ is defined and solved in Appendix 1.

We define the problem of optimal consumption, optimal asset allocation for any $20 \le t \le 64$ as follows

$$V_t\left(X_t, r_{t-1}\right)=\max_{\left\{C_t,\alpha_t^e,\alpha_t^b\right\}} E_t\left[u\left(C_t\right)+\delta\left(1-p_t\right)b_t u\left(\tilde{W}_{t+1}\right)+\delta p_t V_{t+1}\left(\tilde{W}_{t+1}+\tilde{Y}_{t+1},\tilde{r}_t\right)\right] \quad (3.11)$$

where

$$\tilde{W}_{t+1}=\left(X_t-C_t\right)\left(1+\tilde{r}_t^P\right) \quad (3.12)$$

$$\tilde{Y}_{t+1}=e^{f(t+1)}P_t\tilde{N}_{t+1}\tilde{U}_{t+1} \quad (3.13)$$

$$\tilde{r}_t^P=\overline{r}_t+\alpha_t^e\left(\tilde{r}_t^e-\overline{r}_t\right)+\alpha_t^b\left(\frac{B\left(\Upsilon_t-1,\tilde{r}_t\right)}{B\left(\Upsilon_t,r_{t-1}\right)}-1-\overline{r}_t\right) \quad (3.14)$$

with the constraints

$$0\le C_t\le X_t \quad (3.15)$$

$$0\le\alpha_t^e\le 1,\ 0\le\alpha_t^b\le 1,\text{ and } 0\le\alpha_t^e+\alpha_t^b\le 1 \quad (3.16)$$

3.2.4 Solution to the Problem

Let us present the solution to the problem defined in the previous section.

The analytical solution to the problem (3.11)–(3.16) cannot be found in the current literature. Further, the random real interest rate and random rate of return on equity investment can be correlated.

The usual approach to this type of problems nowadays is a numerical solution using computers. We approach this problem by finding the maximum in equation (3.11) using numerical mathematics.

By observing equations (3.11)–(3.16) and the constraints accompanying them, one can see that we need to solve the problem of nonlinear optimization with constraints. In this particular problem we have three control variables. The constraints are analytical functions. We solve this problem in Mathematica 8.0 using the Gauss Quadrature for approximating the interest rate and rate on equity investment and cubic splines for interpolating the value function.

Let us assume that we have a solution for ages $t+1$ and onwards and we need to go one step backward aiming to find the solution for time t. It means that we have obtained

$$\left\{\left(C_i^*\left(X_i,r_{i-1;m}\right);\alpha_i^{e*}\left(X_i,r_{i-1;m}\right);\alpha_i^{b*}\left(X_i,r_{i-1;m}\right);V_i\left(X_i,r_{i-1;m}\right)\right)\right\}_{i=t+1}^{64} \quad (3.17)$$

for $X_i\ge 0$ and $r_{i-1;m}$ in the domain of interest rate values, where $C_i^*\left(X_i,r_{i-1;m}\right)$, $\alpha_i^{e*}\left(X_i,r_{i-1;m}\right)$ and $\alpha_i^{b*}\left(X_i,r_{i-1;m}\right)$ are optimal consumptions, optimal equity and bond allocations, and $V_i\left(X_i,r_{i-1;m}\right)$ is the value function for those optimal control variables. Having this solution in hand, we want to derive the solution at time t. It means that we want to determine $C_t^*\left(X_t,r_{t-1;j}\right)$, $\alpha_t^{e*}\left(X_t,r_{t-1;j}\right)$ and $\alpha_t^{b*}\left(X_t,r_{t-1;j}\right)$ which maximises the value function below

$$V_t\left(X_t, r_{t-1;j}\right) = \max_{\left\{C_t, \alpha_t^e, \alpha_t^b\right\}} \left[u\left(C_t\right) + E_t\left[\delta\left(1-p_t\right) b_t u\left(\tilde{W}_{t+1}\right) + \delta p_t V_{t+1}\left(\tilde{W}_{t+1} + \tilde{Y}_{t+1}, \tilde{r}_t\right)\right]\right]$$

which can be written in more explicit form as

$$\begin{aligned} V_t\left(X_t, r_{t-1;j}\right) = \max_{\left\{C_t, \alpha_t^e, \alpha_t^b\right\}} \Big[u\left(C_t\right) + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Big(\delta\left(1-p_t\right) b_t u\left(W_{t+1}\left(r_t, r_{t+1}^e\right)\right) + \\ \delta p_t V_{t+1}\left(W_{t+1}\left(r_t, r_{t+1}^e\right) + Y_{t+1}\left(N_{t+1}, U_{t+1}\right), r_t\right)\Big) dF\left(N_{t+1}\right) dF\left(U_{t+1}\right) dF\left(r_t\right) dF\left(r_{t+1}^e\right)\Big] \end{aligned} \quad (3.18)$$

Using the relation (A.2.27) from Appendix, we can write

$$V_t\left(X_t, r_{t-1;j}\right) = \left(\frac{Y_t}{U_t \bar{y}}\right)^{\gamma} V_t\left(X_t \frac{U_t \bar{y}}{Y_t}, r_{t-1;j}\right) \quad (3.19)$$

and also

$$V_{t+1}\left(X_{t+1}, r_t\right) = \left(\frac{Y_{t+1}}{U_{t+1} \bar{y}}\right)^{\gamma} V_{t+1}\left(X_{t+1} \frac{U_{t+1} \bar{y}}{Y_{t+1}}, r_t\right) \quad (3.20)$$

for any constant $\bar{y} > 0$. Introducing this relation into equation (3.18) one get

$$\begin{aligned} \left(\frac{Y_t}{U_t \bar{y}}\right)^{\gamma} V_t\left(X_t \frac{U_t \bar{y}}{Y_t}, r_{t-1;j}\right) = \max_{\left\{C_t, \alpha_t^e, \alpha_t^b\right\}} \Big[u\left(C_t\right) + \\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Big(\delta\left(1-p_t\right) b_t u\left(W_{t+1}\left(r_t, r_t^e\right)\right) + \\ \delta p_t \left(\frac{Y_{t+1}\left(N_{t+1}, U_{t+1}\right)}{U_{t+1} \bar{y}}\right)^{\gamma} V_{t+1}\left(W_{t+1}\left(r_t, r_t^e\right) \frac{U_{t+1} \bar{y}}{Y_{t+1}\left(N_{t+1}, U_{t+1}\right)} + U_{t+1} \bar{y}, r_t\right)\Big) \cdot \\ dF\left(N_{t+1}\right) dF\left(U_{t+1}\right) dF\left(r_t\right) dF\left(r_t^e\right)\Big] \end{aligned}$$

Using (3.12) and skipping writing dependent variables one get

$$\begin{aligned} \left(\frac{Y_t}{U_t \bar{y}}\right)^{\gamma} V_t\left(X_t \frac{U_t \bar{y}}{Y_t}, r_{t-1;j}\right) = \max_{\left\{C_t, \alpha_t^e, \alpha_t^b\right\}} \Big[u\left(C_t\right) + \\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Big(\delta\left(1-p_t\right) b_t u\left(\left(X_t - C_t\right)\left(1 + r_t^P\right)\right) + \\ \delta p_t \left(\frac{Y_{t+1}}{U_{t+1} \bar{y}}\right)^{\gamma} V_{t+1}\left(\left(X_t - C_t\right)\left(1 + r_t^P\right) \frac{U_{t+1} \bar{y}}{Y_{t+1}} + U_{t+1} \bar{y}, r_t\right)\Big) \cdot \\ dF\left(N_{t+1}\right) dF\left(U_{t+1}\right) dF\left(r_t\right) dF\left(r_t^e\right)\Big] \end{aligned}$$

where

$$1+r_t^P=1+\overline{r}_{t;j}+\alpha_t^e\left(r_t^e-\overline{r}_{t;j}\right)+\alpha_t^b\left(\frac{B\left(\Upsilon_t-1,r_t\right)}{B\left(\Upsilon_t,r_{t-1;j}\right)}-1-\overline{r}_{t;j}\right)$$

and rearranging terms in this equation and using (3.19) we have

$$\left(\frac{Y_t}{U_t\overline{y}}\right)^{\gamma}V_t\left(\frac{U_tX_t}{Y_t}\overline{y},r_{t-1;j}\right)=\max_{\left\{C_t,\alpha_t^e,\alpha_t^b\right\}}\left[\left(\frac{Y_t}{U_t\overline{y}}\right)^{\gamma}u\left(\frac{U_tC_t}{Y_t}\overline{y}\right)+\right.$$
$$\delta\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{Y_{t+1}}{U_{t+1}\overline{y}}\right)^{\gamma}\left((1-p_t)b_tu\left(\left(\frac{U_tX_t}{Y_t}\overline{y}-\frac{U_tC_t}{Y_t}\overline{y}\right)\left(1+r_t^P\right)\frac{U_{t+1}Y_t}{U_tY_{t+1}}\right)+p_t\cdot\right.$$
$$\left.\left.V_{t+1}\left(\left(\frac{U_tX_t}{Y_t}\overline{y}-\frac{U_tC_t}{Y_t}\overline{y}\right)\left(1+r_t^P\right)\frac{U_{t+1}Y_t}{U_tY_{t+1}}+U_{t+1}\overline{y},r_t\right)\right)dF\left(N_{t+1}\right)dF\left(U_{t+1}\right)dF\left(r_t\right)dF\left(r_t^e\right)\right]$$

Let us define

$$x_t=\frac{U_tX_t}{Y_t}\overline{y},\ w_t=\frac{U_tW_t}{Y_t}\overline{y}\ \text{and}\ c_t=\frac{U_tC_t}{Y_t}\overline{y} \tag{3.21}$$

It is easy to derive from (3.21) and from $X_t=W_t+Y_t$ that

$$x_t=\frac{U_tW_t}{Y_t}\overline{y}+U_t\overline{y}=w_t+U_t\overline{y}. \tag{3.22}$$

Multiplying both sides by $\left(\frac{U_t\overline{y}}{Y_t}\right)^{\gamma}$ and introducing (3.21) into the previous equation we have

$$V_t\left(x_t,r_{t-1;j}\right)=\max_{\left\{c_t,\alpha_t^e,\alpha_t^b\right\}}\left[u\left(c_t\right)+\right.$$
$$\delta\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{U_tY_{t+1}}{U_{t+1}Y_t}\right)^{\gamma}\left((1-p_t)b_tu\left(\left(x_t-c_t\right)\left(1+r_t^P\right)\frac{U_tY_t}{U_tY_{t+1}}\right)+p_t\cdot\right.$$
$$\left.\left.V_{t+1}\left(\left(x_t-c_t\right)\left(1+r_t^P\right)\frac{U_{t+1}Y_t}{U_tY_{t+1}}+U_{t+1}\overline{y},r_t\right)\right)dF\left(N_{t+1}\right)dF\left(U_{t+1}\right)dF\left(r_t\right)dF\left(r_t^e\right)\right]$$

Using (3.10), we have

$$G_{t+1}\left(N_{t+1}\right)=\frac{U_tY_{t+1}}{U_{t+1}Y_t}=\frac{e^{f(t+1)}}{e^{f(t)}}N_{t+1}$$

Introducing this relation into the previous equation one get

$$V_t\left(x_t, r_{t-1;j}\right) = \max_{\left\{c_t, \alpha_t^e, \alpha_t^b\right\}} \Big[u\left(c_t\right) +$$

$$\delta \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(G_{t+1}\left(N_{t+1}\right)\right)^{\gamma} \left(\left(1-p_t\right) b_t u \left(\left(x_t - c_t\right) \frac{\left(1+r_t^P\right)}{G_{t+1}\left(N_{t+1}\right)} \right) + p_t \cdot \right. \quad (3.23)$$

$$\left. V_{t+1}\left(\left(x_t - c_t\right) \frac{\left(1+r_t^P\right)}{G_{t+1}\left(N_{t+1}\right)} + U_{t+1}\bar{y}, r_t \right) \right) dF\left(N_{t+1}\right) dF\left(U_{t+1}\right) dF\left(r_t\right) dF\left(r_t^e\right) \Big]$$

Thus, we will actually derive our solution for some constant $\bar{y}$ and $x_t \geq 0$, and find control variables c_t, α_t^e and α_t^b. Then we use the transformation (3.21) and the results from Appendix to get the solutions C_t, α_t^e and α_t^b for any $X_t \geq 0$, and also for any $Y_t \geq 0$ and $W_t \geq 0$.

When finding numerical solution on the computer we need to approximate each continuous variable with a discrete one. We use the Gauss Quadrature method in order to approximate the continuous random variable $\tilde{r}_t^e$ with the appropriate discrete random variable as follows

$$^{dis}\tilde{r}_t^e \sim \begin{pmatrix} r_{t;1}^e & r_{t;2}^e & \cdots & r_{t;n_{re}-1}^e & r_{t;n_{re}}^e \\ p_{re;1} & p_{re;2} & \cdots & p_{re;n_{re}-1} & p_{re;n_{re}} \end{pmatrix} \quad (3.24)$$

Similarly, we approximate random variables $\tilde{N}_t$ and $\tilde{U}_t$ as follows

$$^{dis}\tilde{N}_t \sim \begin{pmatrix} N_{t;1} & N_{t;2} & \cdots & N_{t;n_N-1} & N_{t;n_N} \\ p_{N;1} & p_{N;2} & \cdots & p_{N;n_N-1} & p_{N;n_N} \end{pmatrix} \quad (3.25)$$

and

$$^{dis}\tilde{U}_t \sim \begin{pmatrix} U_{t;1} & U_{t;2} & \cdots & U_{t;n_U-1} & N_{t;n_U} \\ p_{U;1} & p_{U;2} & \cdots & p_{U;n_U-1} & p_{U;n_U} \end{pmatrix} \quad (3.26)$$

Let us assume that cash-on-hand takes only the values on the cash-on-hand grid $\left(x_{t;i}\right)_{i=1}^{n_x}$. We model the interest rate as a discrete state autoregressive process. We denote the states for the real interest rate as $\left(r_{t;k}\right)_{k=1}^{n_r}$ and the transitional matrix as $\left(p_{r;j,k}\right)_{(j,k)=(1,1)}^{(n_r,n_r)}$, such that $p_{r;j,k}$ is the probability that during one year period the interest rate will move from state $r_{t;j}$ to state $r_{t+1;k}$.

Thus, we actually find and save into the file the solution

$$\left\{ \left(C_t^*\left(x_{t;i}, r_{t-1;j}\right); \alpha_t^{e*}\left(x_{t;i}, r_{t-1;j}\right); \alpha_t^{b*}\left(x_{t;i}, r_{t-1;j}\right); V_t\left(x_{t;i}, r_{t-1;j}\right) \right) \right\}_{(i,j)=(1,1)}^{(n_x,n_r)} \quad (3.27)$$

of the following equation

$$V_t\left(x_{t;i},r_{t-1;j}\right)=\max_{\left\{c_{t;i,j},\alpha^e_{t;i,j},\alpha^b_{t;i,j}\right\}}\Bigg[u\left(c_{t;i,j}\right)+$$

$$\delta\sum_{m_{re}=1}^{n_{re}}\sum_{m_r=1}^{n_r}\sum_{m_U=1}^{n_U}\sum_{m_N=1}^{n_N}G^{\gamma}_{t+1;m_N}\left(\left(1-p_t\right)b_t u\left(\left(x_{t;i}-c_{t;i,j}\right)\frac{\left(1+r^P_{t;j,k,m_r;m_{re}}\right)}{G^{\gamma}_{t+1;m_N}}\right)+\right. \tag{3.28}$$

$$\left.p_t V_{t+1}\left(\left(x_{t;i}-c_{t;i,j}\right)\frac{\left(1+r^P_{t;j,k,m_r;m_{re}}\right)}{G^{\gamma}_{t+1;m_N}}+U_{t+1;m_N}\bar{y},r_{t;k}\right)\right)\cdot p_{r;j,k;m_r}\,p_{re;m_{re}}\,p_{U;m_U}\,p_{N;m_N}\Bigg]$$

where

$$1+r^P_{t;j,k,m_r;m_{re}}=1+\bar{r}_{t;j}+\alpha^e_{t;i,j}\left(r^e_{t;m_{re}}-\bar{r}_{t;j}\right)+\alpha^b_{t;i,j}\left(\frac{B\left(\Upsilon_t-1,r_{t;m}\right)}{B\left(\Upsilon_t,r_{t-1;j}\right)}-1-\bar{r}_{t;j}\right)$$

and

$$G_{t+1;m_N}=\frac{e^{f(t+1)}}{e^{f(t)}}N_{m_N}$$

and

$$\bar{r}_{t;j}=\frac{1}{B\left(1,r_{t-1;j}\right)}-1.$$

Having the set of solutions (3.27) in hands, for each $i=1,..,n_x$ we use cubic splines to interpolate the consumption through the points $\left\{c^*_t\left(x_{t;i},r_{t-1;j}\right)\right\}_{i=1}^{n_x}$, optimal asset allocation through the points $\left\{\alpha^{e*}_t\left(x_{t;i},r_{t-1;j}\right)\right\}_{i=1}^{n_x}$ and $\left\{\alpha^{b*}_t\left(x_{t;i},r_{t-1;j}\right)\right\}_{i=1}^{n_x}$ and the value function $\left\{V_{x,t}\left(x_{t;i},r_{t-1;j}\right)\right\}_{i=1}^{n_x}$ calculated in these optimal points. Thus, we have

$$\left\{\left(c^*_t\left(x_t,r_{t-1;j}\right);\alpha^{e*}_t\left(x_t,r_{t-1;j}\right);\alpha^{b*}_t\left(x_t,r_{t-1;j}\right);V_{x,t}\left(x_t,r_{t-1;j}\right)\right)\right\}_{j=1}^{n_r} \tag{3.29}$$

for $x_t\geq 0$ and $r_{t-1,j}$ taking discrete values for $j=1,..,n_r$. Now, for any $x_t\geq 0$ and U_t in the domain of U_t, using equation (3.21) the results from Appendix we can calculate

$$\left\{\left(c^*_t\left(X_t,r_{t-1;j}\right);\alpha^{e*}_t\left(X_t,r_{t-1;j}\right);\alpha^{b*}_t\left(X_t,r_{t-1;j}\right);V_t\left(X_t,r_{t-1;j}\right)\right)\right\}_{j=1}^{n_r} \tag{3.30}$$

4 The Results

In this section, we present the numerical results of the problem (3.11)–(3.16).

4.1 Criteria for Comparing Results

4.2 *CEC* and *REW* Measures Applied

5 Conclusions

5.2 Main Results and Future Research

6 Appendices

A.1 Appendix – Income as State Variable in the Interest Rate Risk Model

We will now prove the relation between solutions **Error! Reference source not found.**–**Error! Reference source not found.** of the problem (3.11)–(3.16) for different values of income variable. We will prove the relations amongst the solutions if we change the value of the income variable. Using this result we will show that it is possible to transform the solution for constant income into any value of income. Using this result it is possible to solve the problem for one value of income. Thus, we decrease the number of states variable for one.

We exclude writing index j that appears in **Error! Reference source not found.**–**Error! Reference source not found.** as subscript in interest rate variable and just assume that interest rate variable takes values in the domain of the interest rate variable.

Gavranovic (2012) proved that if $k \in \mathbb{R}^+$ and

Cash-in-hand: $$\bar{X}_{65} = kX_{65} \tag{A.2.1}$$

then the solution to the problem (3.11)–(3.16) satisfies the following rules

optimal consumption: $$\bar{C}^*_{65}\left(\bar{X}_{65}, r_{64}\right) = kC^*_{65}\left(X_{65}, r_{64}\right) \tag{A.2.2}$$

optimal equity allocation: $$\bar{\alpha}^{e*}_{65}\left(\bar{X}_{65}, r_{64}\right) = \alpha^{e*}_{65}\left(X_{65}, r_{64}\right) \tag{A.2.3}$$

optimal bond allocation: $$\alpha^{b*}_{65}\left(\bar{X}_{65}, r_{64}\right) = \alpha^{b*}_{65}\left(X_{65}, r_{64}\right) \tag{A.2.4}$$

value function: $$V_{65}\left(\bar{X}_{65}, r_{64}\right) = k^{\gamma}V_{65}\left(X_{65}, r_{64}\right) \tag{A.2.5}$$

for $X_{65} = W_{65} + Y_{65}$, $W_{65} \geq 0$ and $Y_{65} \geq 0$, r_{64} in the domain of the interest rate and value function $V_{65}\left(X_{65}, r_{64}\right)$ is derived value function as a result of the numerical solution of the interest rate risk model in Gavranovic (2012).

Using mathematical induction, we will prove that the relations equivalent to the relations (A.2.1)–(A.2.5) are valid for ages $20 \leq t \leq 64$.

We will prove that if for some $k \in \mathbb{R}^+$ we define $\bar{X}_{64} = kX_{64}$ then the relations (A.2.1) –(A.2.5) are valid. We have

$$V_{64}\left(X_{64},r_{63}\right)=\max_{\left\{C_{64},\alpha_{64}^{e},\alpha_{64}^{b}\right\}} E_{64}\left[u\left(C_{64}\right)+\delta\left(1-p_{64}\right)b_{64}u\left(\tilde{W}_{65}\right)+\delta p_{64}V_{65}\left(\tilde{X}_{65},\tilde{r}_{i}\right)\right] \quad \text{(A.2.6)}$$

and

$$V_{64}\left(\bar{X}_{64},r_{63}\right)=\max_{\left\{\bar{C}_{64},\bar{\alpha}_{64}^{e},\bar{\alpha}_{64}^{b}\right\}} E_{i}\left[u\left(\bar{C}_{64}\right)+\delta\left(1-p_{64}\right)b_{64}u\left(\tilde{\bar{W}}_{65}\right)+\delta p_{64}V_{65}\left(\tilde{\bar{X}}_{65},\tilde{r}_{64}\right)\right] \quad \text{(A.2.7)}$$

Equation (A.2.6) can be written as

$$\begin{aligned} V_{64}\left(X_{64},r_{63;j}\right)=&\max_{\left\{C_{64},\alpha_{64}^{e},\alpha_{64}^{b}\right\}}\Big[u\left(C_{64}\right)+\\ &\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Big(\delta\left(1-p_{64}\right)b_{64}u\left(\left(X_{64}-C_{64}\right)\left(1+\tilde{r}_{64}^{P}\right)\right)+\\ &\delta p_{64}V_{65}\left(\left(X_{64}-C_{64}\right)\left(1+r_{64}^{P}\right)+Y_{65}\left(N_{65},U_{65}\right),r_{64}\right)\Big)\cdot\\ &dF\left(N_{65}\right)dF\left(U_{65}\right)dF\left(r_{65}\right)dF\left(r_{65}^{e}\right)\Big] \end{aligned} \quad \text{(A.2.8)}$$

Regarding equation (A.2.7), one can rewrite in the following form

$$\begin{aligned} V_{64}\left(\bar{X}_{64},r_{63;j}\right)=&\max_{\left\{\bar{C}_{64},\bar{\alpha}_{64}^{e},\bar{\alpha}_{64}^{b}\right\}}\Big[u\left(\bar{C}_{64}\right)+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Big(\delta\left(1-p_{64}\right)b_{64}u\left(\left(\bar{X}_{64}-\bar{C}_{64}\right)\left(1+r_{64}^{P}\right)\right)+\\ &\delta p_{64}V_{65}\left(\left(\bar{X}_{64}-\bar{C}_{64}\right)\left(1+r_{64}^{P}\right)+\bar{Y}_{65}\left(N_{65},U_{65}\right),r_{64}\right)\Big)dF\left(N_{65}\right)dF\left(U_{65}\right)dF\left(r_{65}\right)dF\left(r_{65}^{e}\right)\Big] \end{aligned}$$

and now, using (A.2.5) we have

$$\begin{aligned} V_{64}\left(\bar{X}_{64},r_{63;j}\right)=k^{\gamma}&\max_{\left\{\bar{C}_{64},\bar{\alpha}_{64}^{e},\bar{\alpha}_{64}^{b}\right\}}\left[u\left(\frac{\bar{C}_{64}}{k}\right)+\right.\\ &\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\delta\left(1-p_{64}\right)b_{64}u\left(\left(\bar{X}_{64}-\frac{\bar{C}_{64}}{k}\right)\left(1+r_{64}^{P}\right)\right)+\right.\\ &\left.\delta p_{64}V_{65}\left(\left(\bar{X}_{64}-\frac{\bar{C}_{64}}{k}\right)\left(1+r_{64}^{P}\right)+Y_{65}\left(N_{65},U_{65}\right),r_{64}\right)\right)\cdot\\ &\left.dF\left(N_{65}\right)dF\left(U_{65}\right)dF\left(r_{65}\right)dF\left(r_{65}^{e}\right)\right] \end{aligned} \quad \text{(A.2.9)}$$

Knowing that k^{γ} is positive constant and that the control variables $\left\{\bar{C}_{64}^{*},\bar{\alpha}_{64}^{e*},\bar{\alpha}_{64}^{b*}\right\}$ which provide the optimal solution are unique, from equations (A.2.8) and (A.2.9) we can conclude that $\left\{\bar{C}_{64}^{*},\bar{\alpha}_{64}^{e*},\bar{\alpha}_{64}^{b*}\right\}=\left\{kC_{64}^{*},\alpha_{64}^{e*},\alpha_{64}^{b*}\right\}$ are optimal control variables for equation (A.2.7).

Thus, we proved that equations (A.2.1)-(A.2.5) are valid for $t=64$. Let us now assume that equations (A.2.1)-(A.2.5) are valid for $t=i+1$ and prove that these equations are then valid for $t=i$ as well. Thus, we assume that if $k\in\mathbb{R}^{+}$ and

wealth: $$\bar{X}_{i+1}=kX_{i+1} \quad \text{(A.2.10)}$$

then the solution to the problem (3.11)–(3.16) satisfies the following rules

optimal consumption: $\bar{C}^*_{i+1}\left(\bar{X}_{i+1},r_i\right)=kC^*_{i+1}\left(X_{i+1},r_i\right)$ (A.2.11)

optimal equity allocation: $\bar{\alpha}^{e*}_{i+1}\left(\bar{X}_{i+1},r_i\right)=\alpha^{e*}_{i+1}\left(X_{i+1},r_i\right)$ (A.2.12)

optimal bond allocation: $\bar{\alpha}^{b*}_{i+1}\left(\bar{X}_{i+1},r_i\right)=\alpha^{b*}_{i+1}\left(X_{i+1},r_i\right)$ (A.2.13)

value function: $V_{i+1}\left(\bar{X}_{i+1},r_i\right)=k^{\gamma}V_{i+1}\left(X_{i+1},r_i\right)$ (A.2.14)

for $X_{i+1}=W_{i+1}+Y_{i+1}$, $W_{i+1}\geq 0$ and $Y_{i+1}\geq 0$, and r_i in the domain of the interest rate.

Let us now assume that $t=i$ for some $20\leq i\leq 64$. We will prove that if for some $k\in\mathbb{R}^+$ we define $\bar{X}_i=kX_i$ then the relations (A.2.1)–(A.2.5) are valid. We have the following equations

$$V_i\left(X_i,r_{i-1}\right)=\max_{\left\{C_i,\alpha_i^e,\alpha_i^b\right\}}E_i\left[u\left(C_i\right)+\delta\left(1-p_i\right)b_iu\left(\tilde{W}_{i+1}\right)+\delta p_iV_{i+1}\left(\tilde{X}_{i+1},\tilde{r}_i\right)\right]\quad\text{(A.2.15)}$$

and

$$V_i\left(\bar{X}_i,r_{i-1}\right)=\max_{\left\{\bar{C}_i,\bar{\alpha}_i^e,\bar{\alpha}_i^b\right\}}E_i\left[u\left(\bar{C}_i\right)+\delta\left(1-p_i\right)b_iu\left(\tilde{\bar{W}}_{i+1}\right)+\delta p_iV_{i+1}\left(\tilde{\bar{X}}_{i+1},\tilde{r}_i\right)\right]\quad\text{(A.2.16)}$$

Equation (A.2.15) can be written as

$$\begin{aligned}V_i\left(X_i,r_{i-1;j}\right)=&\max_{\left\{C_i,\alpha_i^e,\alpha_i^b\right\}}\Big[u\left(C_i\right)+\\&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Big(\delta\left(1-p_i\right)b_iu\left(\left(X_i-C_i\right)\left(1+r_i^P\right)\right)+\\&\delta p_iV_{i+1}\left(\left(X_i-C_i\right)\left(1+r_i^P\right)+Y_{i+1}\left(N_{i+1},U_{i+1}\right),r_i\right)\Big)\cdot\\&dF\left(N_{i+1}\right)dF\left(U_{i+1}\right)dF\left(r_{i+1}\right)dF\left(r_{i+1}^e\right)\Big]\end{aligned}\quad\text{(A.2.17)}$$

and (A.2.16) can be written as

$$\begin{aligned}V_i\left(\bar{X}_i,r_{i-1;j}\right)=k^{\gamma}&\max_{\left\{\bar{C}_i,\bar{\alpha}_i^e,\bar{\alpha}_i^b\right\}}\left[u\left(\frac{\bar{C}_i}{k}\right)+\right.\\&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\delta\left(1-p_i\right)b_iu\left(\left(X_i-\frac{\bar{C}_i}{k}\right)\left(1+r_i^P\right)\right)+\right.\\&\left.\delta p_iV_{i+1}\left(\left(X_i-\frac{\bar{C}_i}{k}\right)\left(1+r_i^P\right)+Y_{i+1}\left(N_{i+1},U_{i+1}\right),r_i\right)\right)\cdot\\&\left.dF\left(N_{i+1}\right)dF\left(U_{i+1}\right)dF\left(r_{i+1}\right)dF\left(r_{i+1}^e\right)\right]\end{aligned}\quad\text{(A.2.18)}$$

Knowing that k^{γ} is positive constant and that the control variables $\left\{C_i^*,\alpha_i^{e*},\alpha_i^{b*}\right\}$ which provide the optimal solution are unique, from equations (A.2.17) and (A.2.18) we can conclude that $\left\{\bar{C}_i^*,\bar{\alpha}_i^{e*},\bar{\alpha}_i^{b*}\right\}=\left\{kC_i^*,\alpha_i^{e*},\alpha_i^{b*}\right\}$ are optimal control variables for equation (A.2.16). It means the solution to the problem (3.11)–(3.16) for $\bar{X}_i = kX_i$ for some $k \in \mathbb{R}^+$ is given by

$$\frac{\bar{C}_i^*\left(\bar{X}_i, r_{i-1}\right)}{k} = C_i^*\left(X_i, r_{i-1}\right),\ \bar{\alpha}_i^{e*}\left(\bar{X}_i, r_{i-1}\right) = \alpha_i^{e*}\left(X, r_{i-1}\right) \text{ and } \bar{\alpha}_i^{b*}\left(\bar{X}_i, r_{i-1}\right) = \alpha_i^{b*}\left(X_i, r_{i-1}\right).$$

Based on the mathematical induction we have just proved that if $k \in \mathbb{R}^+$ and if

wealth: $\bar{X}_t = kX_t$ (A.2.19)

then the solution to the problem (3.11)–(3.16) satisfies the following rules

optimal consumption: $\bar{C}_t^*\left(\bar{X}_t, r_{t-1}\right) = kC_t^*\left(X_t, r_{t-1}\right)$ (A.2.20)

optimal equity allocation: $\bar{\alpha}_t^{e*}\left(\bar{X}_t, r_{t-1}\right) = \alpha_t^{e*}\left(X_t, r_{t-1}\right)$ (A.2.21)

optimal bond allocation: $\bar{\alpha}_t^{b*}\left(\bar{X}_t, r_{t-1}\right) = \alpha_t^{b*}\left(X_t, r_{t-1}\right)$ (A.2.22)

value function: $V_t\left(\bar{X}_t, r_{t-1}\right) = k^{\gamma} V_t\left(X_t, r_{t-1}\right)$ (A.2.23)

for $X_t = W_t + Y_t$, $W_t \geq 0$ and $Y_t \geq 0$, and r_{t-1} in the domain of the interest rate and for any t such that $20 \leq t \leq 65$.

Also, if y is positive constant and $x_t = X_t \dfrac{U_t y}{Y_t}$ then the solution to the problem (3.11)–(3.16) satisfies the following rules

optimal consumption: $C_t^*\left(X_t, r_{t-1}\right) = \dfrac{Y_t}{U_t y} C_t^*\left(x_t + U_t y, r_{t-1}\right)$ (A.2.24)

optimal equity allocation: $\alpha_t^{e*}\left(X_t, r_{t-1}\right) = \alpha_t^{e*}\left(x_t + U_t y, r_{t-1}\right)$ (A.2.25)

optimal bond allocation: $\alpha_t^{b*}\left(X_t, r_{t-1}\right) = \alpha_t^{b*}\left(x_t + U_t y, r_{t-1}\right)$ (A.2.26)

value function: $V_t\left(X_t, r_{t-1}\right) = \left(\dfrac{Y_t}{U_t y}\right)^{\gamma} V_t\left(x_t + U_t y, r_{t-1}\right)$ (A.2.27)

for $X_t \geq 0$, $U_t \geq 0$, and r_{t-1} in the domain of the interest rate and for $20 \leq t \leq 64$. Adding the term U_t in the last equation is useful because we need to include t in order to decrease the number of state variables.

A.2 Appendix – Income as State Variable in the Interest Rate Risk Model

We define $X_{64} = W_{64} + Y_{64}$ and we want to derive $V_{64}\left(X_{64}, r_{63}\right)$ as a function of $V_{65}\left(W_{65}, Y_{65}, r_{64}\right)$, for $X_{64} = W_{64} + Y_{64} \geq 0$. The problem is defined as

$$V_{64}\left(X_{64}, r_{63}\right) = \max_{\left\{C_{64}, \alpha_{64}^e, \alpha_{64}^b\right\}} E_{64}\left[u\left(C_{64}\right) + \delta\left(1 - p_{64}\right) b_{64} u\left(\tilde{W}_{65}\right) + \delta p_{64} V_{65}\left(\tilde{W}_{65}, \tilde{Y}_{65}, \tilde{r}_{64}\right)\right] \quad (6.1)$$

where

$$\tilde{W}_{65} = \left(X_{64} - C_{64}\right)\left(1 + \tilde{r}_{64}^{P}\right) \tag{6.2}$$

$$\tilde{Y}_{65} = e^{f(65)} P_{64} \tilde{N}_{65} \tilde{U}_{65} \tag{6.3}$$

$$\tilde{r}_{64}^{P} = \overline{r}_{64} + \alpha_{64}^{e}\left(\tilde{r}_{64}^{e} - \overline{r}_{64}\right) + \alpha_{64}^{b}\left(\frac{B\left(\Upsilon_{64} - 1, \tilde{r}_{64}\right)}{B\left(\Upsilon_{64}, r_{63}\right)} - 1 - \overline{r}_{64}\right) \tag{6.4}$$

with the constraints

$$0 \le C_{64} \le X_{64} \tag{6.5}$$

$$0 \le \alpha_{64}^{e} \le 1,\ 0 \le \alpha_{64}^{b} \le 1, \text{ and } 0 \le \alpha_{64}^{e} + \alpha_{64}^{b} \le 1 \tag{6.6}$$

A.2.1 Solution to the Problem (6.1)-(6.6)

We will closely follow the solution presented in Section 3.2.4, but now for age 64. We assume that we have the solution (derived in Gavranovic (2012))

$$\left(C_{65}^{*}; \alpha_{65}^{e*}; \alpha_{65}^{b*}; m_{65}; V_{65}\right) \tag{6.7}$$

depending on $W_i \ge 0$, $Y_i \ge 0$ and $r_{i-1;m}$ in the domain of interest rate values, where $C_{65}^{*}\left(W_{65}, Y_{65}, r_{64;m}\right)$, $\alpha_{65}^{e*}\left(W_{65}, Y_{65}, r_{64;m}\right)$, $\alpha_{65}^{b*}\left(W_{65}, Y_{65}, r_{64;m}\right)$ and $m_{65}\left(W_{65}, Y_{65}, r_{64;m}\right)$ are optimal consumptions, optimal equity, optimal bond allocations, optimal annuitisation and $V_{65}\left(X_{65}, Y_{65}, r_{64;m}\right)$ is the value function for those optimal control variables. Having this solution in hand, we want to derive the solution at age 64. All we actually need is $V_{65}\left(X_{65}, Y_{65}, r_{64;m}\right)$. It means that we want to determine $C_{64}^{*}\left(X_{64}, r_{63;j}\right)$, $\alpha_{64}^{e*}\left(X_{64}, r_{63;j}\right)$ and $\alpha_{64}^{b*}\left(X_{64}, r_{63;j}\right)$ which maximises the value function below

$$V_{64}\left(X_{64}, r_{63;j}\right) = \max_{\left\{C_{64}, \alpha_{64}^{e}, \alpha_{64}^{b}\right\}}\left[u\left(C_{64}\right) + E_{64}\left[\delta\left(1 - p_{64}\right) b_{64} u\left(\tilde{W}_{65}\right) + \delta p_{64} V_{65}\left(\tilde{W}_{65}, \tilde{Y}_{65}, \tilde{r}_{64}\right)\right]\right]$$

which can be written in more explicit form as

$$\begin{aligned} V_{64}\left(X_{64}, r_{63;j}\right) = \max_{\left\{C_{64}, \alpha_{64}^{e}, \alpha_{64}^{b}\right\}}\Bigg[u\left(C_{64}\right) + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Big(\delta\left(1 - p_{64}\right) b_{64} u\left(W_{65}\left(r_{64}, r_{65}^{e}\right)\right) + \\ \delta p_{64} V_{65}\left(W_{65}\left(r_{64}, r_{65}^{e}\right), Y_{65}\left(N_{65}, U_{65}\right), r_{64}\right)\Big) dF\left(N_{65}\right) dF\left(U_{65}\right) dF\left(r_{64}\right) dF\left(r_{65}^{e}\right)\Bigg] \end{aligned} \tag{6.8}$$

Using the relation (A.2.27) from Appendix, we can write

$$V_{64}\left(X_{64}, r_{65;j}\right) = \left(\frac{Y_{64}}{U_{64}\overline{y}}\right)^{\gamma} V_{64}\left(X_{64}\frac{U_{64}\overline{y}}{Y_{64}}, r_{63;j}\right) \tag{6.9}$$

and also similar relation from Gavranovic (2012)

$$V_{65}\left(W_{65}, Y_{65}, r_{64}\right) = \left(\frac{Y_{65}}{U_{65}\overline{y}}\right)^{\gamma} V_{65}\left(W_{65}\frac{U_{65}\overline{y}}{Y_{65}}, \overline{y}, r_{t}\right) \tag{6.10}$$

for any constant $\bar{y} > 0$. Introducing this relation into equation (6.8), after rearranging the terms one get

$$\left(\frac{Y_{64}}{U_{64}\bar{y}}\right)^{\gamma} V_{64}\left(\frac{U_{64}X_{64}}{Y_{64}}\bar{y}, r_{63;j}\right) = \max_{\left\{C_{64},\alpha_{64}^{e},\alpha_{64}^{b}\right\}}\left[\left(\frac{Y_{64}}{U_{64}\bar{y}}\right)^{\gamma} u\left(\frac{U_{64}C_{64}}{Y_{64}}\bar{y}\right)+\right.$$
$$\delta\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{Y_{65}}{U_{65}\bar{y}}\right)^{\gamma}\left((1-p_{64})b_{64}u\left(\left(\frac{U_{64}X_{64}}{Y_{64}}\bar{y}-\frac{U_{64}C_{64}}{Y_{64}}\bar{y}\right)\left(1+r_{64}^{P}\right)\frac{U_{65}Y_{64}}{U_{64}Y_{65}}\right)+p_{64}\cdot\right.$$
$$\left.\left. V_{65}\left(\left(\frac{U_{64}X_{64}}{Y_{64}}\bar{y}-\frac{U_{64}C_{64}}{Y_{64}}\bar{y}\right)\left(1+r_{64}^{P}\right)\frac{U_{65}Y_{64}}{U_{64}Y_{65}}, U_{65}\bar{y}, r_{64}\right)\right) dF\left(N_{65}\right) dF\left(U_{65}\right) dF\left(r_{64}\right) dF\left(r_{64}^{e}\right)\right]$$

Let us define

$$x_{64} = \frac{U_{64}X_{64}}{Y_{64}}\bar{y} \text{ and } c_{64} = \frac{U_{64}C_{64}}{Y_{64}}\bar{y} \tag{6.11}$$

Multiplying both sides by $\left(\frac{U_{64}\bar{y}}{Y_{64}}\right)^{\gamma}$ and introducing (3.21) into the previous equation we have

$$V_{64}\left(x_{64}, r_{63;j}\right) = \max_{\left\{c_{64},\alpha_{64}^{e},\alpha_{64}^{b}\right\}}\left[u\left(c_{64}\right)+\right.$$
$$\delta\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{U_{64}Y_{65}}{U_{65}Y_{64}}\right)^{\gamma}\left((1-p_{64})b_{64}u\left(\left(x_{64}-c_{64}\right)\left(1+r_{64}^{P}\right)\frac{U_{65}Y_{64}}{U_{64}Y_{65}}\right)+p_{64}\cdot\right.$$
$$\left.\left. V_{65}\left(\left(x_{64}-c_{64}\right)\left(1+r_{64}^{P}\right)\frac{U_{65}Y_{64}}{U_{64}Y_{65}}, U_{65}\bar{y}, r_{64}\right)\right) dF\left(N_{65}\right) dF\left(U_{65}\right) dF\left(r_{64}\right) dF\left(r_{64}^{e}\right)\right]$$

Using (3.10), we have

$$G_{65}\left(N_{65}\right) = \frac{U_{64}Y_{65}}{U_{65}Y_{64}} = \frac{e^{f(65)}}{e^{f(64)}}N_{65}$$

Introducing this relation into the previous equation one get

$$V_{64}\left(x_{64}, r_{63;j}\right) = \max_{\left\{c_{64}, \alpha_{64}^{e}, \alpha_{64}^{b}\right\}} \Big[u\left(c_{64}\right) +$$

$$\delta \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(G_{65}\left(N_{65}\right)\right)^{\gamma} \left(\left(1-p_{64}\right) b_{64} u \left(\left(x_{64}-c_{64}\right) \frac{\left(1+r_{64}^{P}\right)}{\left(G_{65}\left(N_{65}\right)\right)} \right) + p_{64} \cdot \right. \tag{6.12}$$

$$\left. V_{65}\left(\left(x_{64}-c_{64}\right) \frac{\left(1+r_{64}^{P}\right)}{\left(G_{65}\left(N_{65}\right)\right)}, U_{65}\bar{y}, r_{64} \right) \right) dF\left(N_{65}\right) dF\left(U_{65}\right) dF\left(r_{64}\right) dF\left(r_{64}^{e}\right) \Big]$$

The remaining part of the solution for $t = 64$ is the same as the solution for $t < 64$, we just write

$$V_{65}\left(\left(x_{64}-c_{64}\right) \frac{\left(1+r_{64}^{P}\right)}{\left(G_{65}\left(N_{65}\right)\right)}, U_{65}\bar{y}, r_{64} \right)$$

instead of

$$V_{t+1}\left(\left(x_{t}-c_{t}\right) \frac{\left(1+r_{t}^{P}\right)}{\left(G_{t+1}\left(N_{t+1}\right)\right)}, U_{t+1}\bar{y}, r_{t} \right).$$

As a result we get and save into the file the solution

$$\left\{ \left(C_{64}^{*}\left(x_{64;i}, r_{63;j}\right); \alpha_{64}^{e*}\left(x_{64;i}, r_{63;j}\right); \alpha_{64}^{b*}\left(x_{64;i}, r_{63;j}\right); V_{64}\left(x_{64;i}, r_{63;j}\right) \right) \right\}_{(i,j)=(1,1)}^{(n_x, n_r)} \tag{6.13}$$

Having the this solutions in hands, for each $i = 1,..,n_x$ we use cubic splines to get

$$\left\{ \left(c_{64}^{*}\left(x_{64}, r_{63;j}\right); \alpha_{64}^{e*}\left(x_{64}, r_{63;j}\right); \alpha_{64}^{b*}\left(x_{64}, r_{63;j}\right); V_{64}\left(x_{64}, r_{63;j}\right) \right) \right\}_{j=1}^{n_r} \tag{6.14}$$

for $x_{64} \geq 0$ and $r_{63,j}$ taking discrete values for $j = 1,..,n_r$. Now, for any $x_{64} \geq 0$ and U_{64} in the domain of U_{64}, using equations (6.11) and the results from Appendix we calculate

$$\left\{ \left(c_{64}^{*}\left(X_{64}, r_{63;j}\right); \alpha_{64}^{e*}\left(X_{64}, r_{63;j}\right); \alpha_{64}^{b*}\left(X_{64}, r_{63;j}\right); V_{64}\left(X_{64}, r_{63;j}\right) \right) \right\}_{j=1}^{n_r} \tag{6.15}$$

for any $X_{64} \geq 0$ and $r_{64,j}$ in the domain of the real interest rate.

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