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Skew t-copula and tail dependence

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Outline

- Introduction
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- Simulation
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Introduction

- Multivariate skew-normal distribution - Azzalini & Dalla Valle (1996)
- Multivariate skew *t*-distribution – several definitions in Kotz & Nadarajah (2004)
- We follow Azzalini & Capitanio (2003)
- Skew *t*-copula - Demarta & McNeil (2005)
- Skew *t*-copula - Kollo & Pettere (2010)

Skew *t*-distribution

Azzalini & Capitanio (2003)

A random p-vector $\mathbf{X} = (X_1, \ldots, X_p)^T$ *has p-variate skew* t*-distribution with parameters* $\boldsymbol{\mu}$, $\boldsymbol{\alpha}$ *and* $\boldsymbol{\Sigma}$ *, if its density function is of the form*

$$g_{p,\nu}(\mathbf{x}; \boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\alpha}) = 2 \cdot t_{p,\nu}(\mathbf{x}; \boldsymbol{\mu}, \boldsymbol{\Sigma}) \cdot T_{1,\nu+p}\left[\boldsymbol{\alpha}^T \mathbf{W}^{-1} (\mathbf{x} - \boldsymbol{\mu}) \left(\frac{\nu + p}{Q + \nu} \right)^{\frac{1}{2}} \right],$$

where Q *denotes the quadratic form*

$$Q = (\mathbf{x} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1} (\mathbf{x} - \boldsymbol{\mu}).$$

$T_{1,\nu+p}(\cdot)$ *denotes the distribution function of the central univariate* t *- distribution with* $\nu + p$ *degrees of freedom and* $\mathbf{W} = (\delta_{ij} \sqrt{\sigma_{ij}})$.

Construction of skew *t*-copula

If $X_i,\ i \in \{1, 2, \ldots, p\}$ are continuous random variables, the density $f(x_1, x_2, \ldots, x_p)$ of their joint distribution can be presented through a copula density $c(u_1, u_2, \ldots, u_p)$ and marginal densities $f_i(x_i)$:

$$f(x_1, x_2, \ldots, x_p) = c(F_1(x_1), \ldots, F_p(x_p)) \cdot f_1(x_1) \cdot \ldots \cdot f_p(x_p).$$

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Construction of skew *t*-copula

DEFINITION. *A copula is called skew t-copula, if its density function is*

$$c_{p,\nu}(\mathbf{u};\boldsymbol{\mu},\boldsymbol{\Sigma},\boldsymbol{\alpha}) = \frac{g_{p,\nu}[\{G_{1,\nu}^{-1}(u_1;0,\sigma_{11},\alpha_1),\ldots,G_{1,\nu}^{-1}(u_p;0,\sigma_{pp},\alpha_p)\};\boldsymbol{\mu},\boldsymbol{\Sigma},\boldsymbol{\alpha}]}{\prod_{i=1}^{p} g_{1,\nu}[G_{1,\nu}^{-1}(u_i;\mu_i,\sigma_{ii},\alpha_i);\mu_i,\sigma_{ii},\alpha_i]}$$

where $g_{p,\nu}(\cdot;\boldsymbol{\mu},\boldsymbol{\Sigma},\boldsymbol{\alpha}): R^p \to R$ *is the density function of the p-variate skew* t*-distribution and function* $G_{1,\nu}^{-1}(\cdot;\mu_i,\sigma_{ii},\alpha_i): R^1 \to I$, $i \in \{1,\ldots,p\}$ *denotes the inverse of the univariate* $t_{1,\nu}$*-distribution function.*

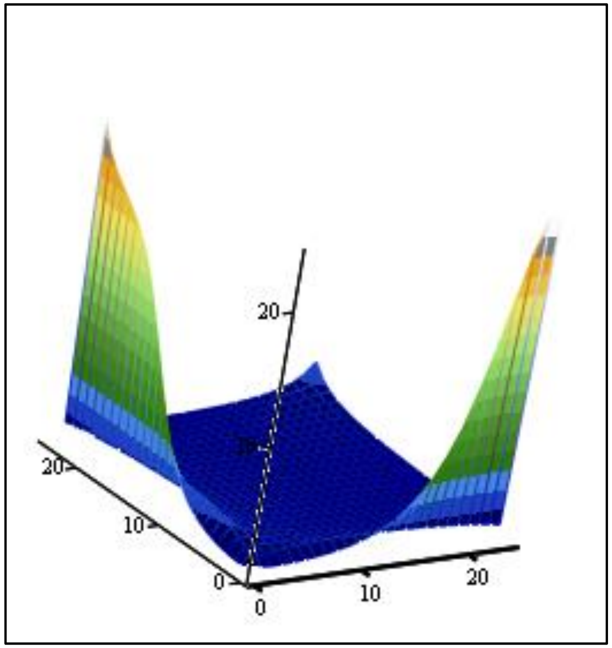
Construction of skew *t*-copula

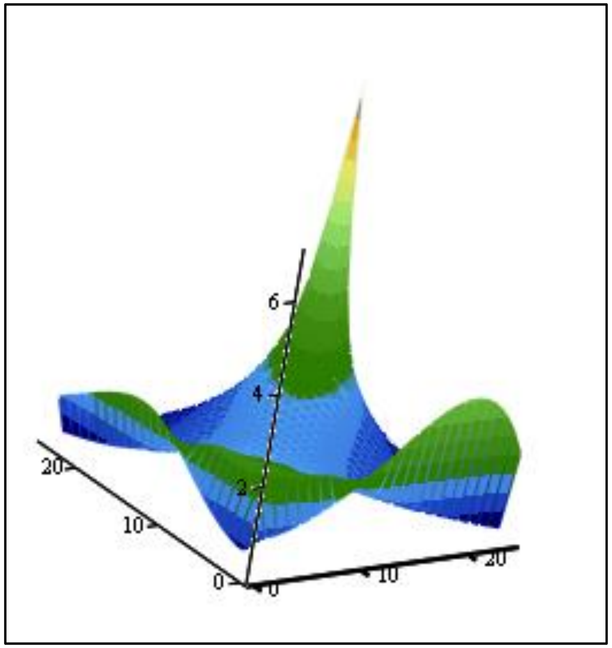
Multivariate skew $t_{p,\nu}$-copula can be presented of the form:

$$C_{t,\nu}(\mathbf{u};\boldsymbol{\mu},\boldsymbol{\Sigma},\boldsymbol{\alpha}) = 2\int_{-\infty}^{G_{1,\nu}^{-1}(u_1;\mu_1,\sigma_1,\alpha_1)} \cdots \int_{-\infty}^{G_{1,\nu}^{-1}(u_p;\mu_p,\sigma_p,\alpha_p)} t_{p,\nu}(\mathbf{x};\boldsymbol{\mu},\boldsymbol{\Sigma}) \cdot T_{\nu+p}\left[\boldsymbol{\alpha}^T \mathbf{W}^{-1}(\mathbf{x}-\boldsymbol{\mu})\left(\frac{\nu+p}{Q+\nu}\right)^{\frac{1}{2}}\right] d\mathbf{x}$$

where $Q = (\mathbf{x}-\boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})$ and $\mathbf{W} = (\delta_{ij}\sqrt{\sigma_{ij}})$.

Construction of skew *t*-copula





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$$\text{a)}\ \nu = 4, r = 0.4, \boldsymbol{\alpha} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}, \sigma_{11} = 3, \sigma_{22} = 4 \quad \text{b)}\ \nu = 6, r = 0.8, \alpha = \begin{pmatrix} 2 \\ 5 \end{pmatrix}, \sigma_{11} = 3, \sigma_{22} = 4$$

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Tail dependence

- Paola Bortot (2010).

Tail dependence in bivariate skew-normal and skew-t distributions (15 pp)

Asymptotic values of tail dependence coefficients for bivariate *t*-distribution and skew *t*-distribution have been obtained.

Tail dependence coefficients

- Let (X_1, X_2) be a bivariate random vector with marginal distribution functions $F_1(x)$ and $F_2(x)$
- Define the upper tail dependence coefficient $\lambda_{up} = \lim_{u \to 1} \lambda_{up}(u)$
- where

$$\lambda_{up}(u) = P(F_1(X_1) > u | F_2(X_2) > u).$$

Tail dependence coefficients

- Similarly the lower tail dependence coefficient

$$\lambda_{lo} = \lim_{u \to 0} \lambda_{lo}(u)$$

- where

$$\lambda_{lo}(u) = P(F_1(X_1) < u | F_2(X_2) < u).$$

In the case of bivariate normal distribution both coefficients are equal to zero.

Tail coefficients via copulas

- The joint distribution function can be presented through a copula:

$$F(x_1, x_2)=C(F_1(x_1), F_2(x_2)).$$

In terms of the copula:

$$\lambda_{up} = \lim_{u \to 1} \frac{1 - 2u + C(u,u)}{1 - u}$$

$$\lambda_{lo} = \lim_{u \to 0} \frac{C(u,u)}{u}$$

Tail coefficients of *t*-distribution

- In the case of bivariate *t*-distribution

$$\lambda_{lo} = \lambda_{up} = 2T_{1,\nu+1}\left(\frac{\sqrt{(\nu+1)(1-\rho)}}{\sqrt{(1+\rho)}}\right)$$

where $T_{1,\nu}(t)$

is the distribution function of standard *t*-distribution.

Tail coefficient of skew *t*-distribution

- Take Σ equal to the correlation matrix with the correlation ρ

Upper tail coefficient for skew *t*-distribution is

$$\lambda_{up,St} \geq \lambda_{up} \frac{T_{1,\nu+2}\left((\alpha_1+\alpha_2)\sqrt{\frac{(\nu+2)(1+\rho)}{2}}\right)}{T_{1,\nu+1}(\beta\sqrt{\nu+1})}$$

$$\beta = \frac{\alpha_2+\alpha_1\rho}{\sqrt{1+\alpha_1^2(1-\rho^2)}}$$

Tail coefficient of skew *t*-distribution

- If

$$\alpha_1 = \alpha_2 = \alpha$$

- then

$$\lambda_{up,St} = \lambda_{up} \frac{T_{1,\nu+2}\left(2\alpha\sqrt{\frac{(\nu+2)(1+\rho)}{2}}\right)}{T_{1,\nu+1}\left(\frac{\alpha(1+\rho)\sqrt{\nu+1}}{1+\alpha^2(1-\rho^2)}\right)}$$

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Tail coefficient of skew *t*-distribution

- If $\alpha > 0$

  then the ratio of distribution functions is >1 and skew *t*-distribution has stronger tail dependence than *t*-distribution.

- If $\alpha < 0$

  then the corresponding *t*-distribution has stronger tail dependence than the skew one.

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Tail coefficient of skew *t*-distribution

- The biggest values of the upper tail coefficient are obtained when

$$\alpha_1 > 0, \ \ \alpha_2 < 0$$

- For instance, with correlation 0.5 and

$$\alpha_1 = 10, \ \ \alpha_2 = -9$$

the tail coefficient is more than 5 times bigger than the corresponding *t*-distribution has.

Tail coefficient of skew *t*-distribution

ν	ρ	α_1	α_2	$\lambda_{up.St}$	$\lambda_{up,St} - \lambda_{up}$
3	0.5	5	5	0.3169	0.0044
3	0.5	5	0	0.3728	0.0603
3	0.5	5	-5	0.9660	0.6535
3	0.5	10	-10	0.9911	0.6786
3	0.85	5	5	0.6003	0.0008
3	0.85	5	0	0.6115	0.0120
3	0,85	5	-5	0.9628	0.3633
3	0.85	10	-10	0.9898	0.3903

Simulation

- In simulations the number of degrees of freedom was 3

- 1 000 000 values from bivariate skew *t*-copula were generated and tail dependence coefficients estimated at different argument values

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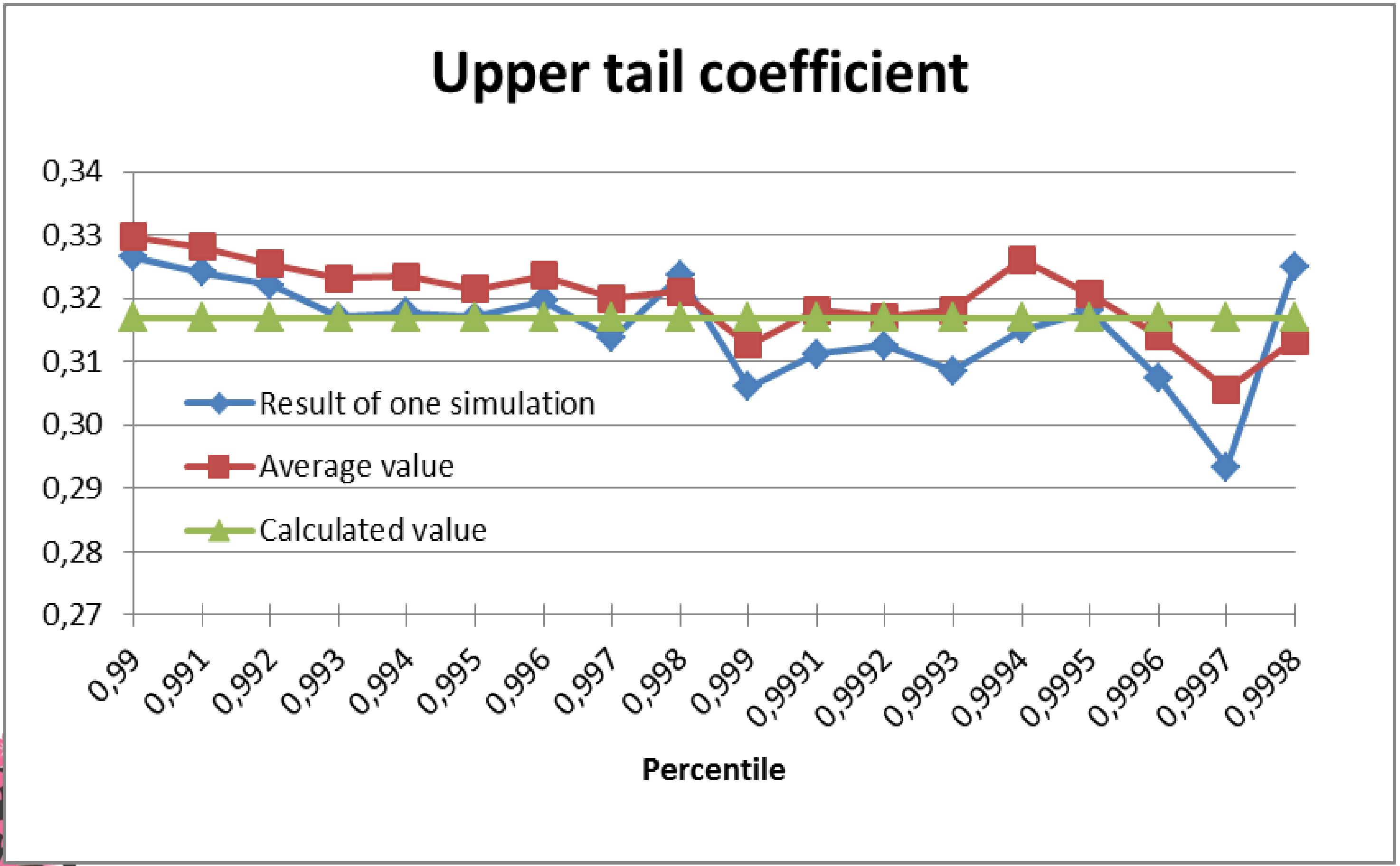
Simulation using both alfas equal 5 and correlation 0.5



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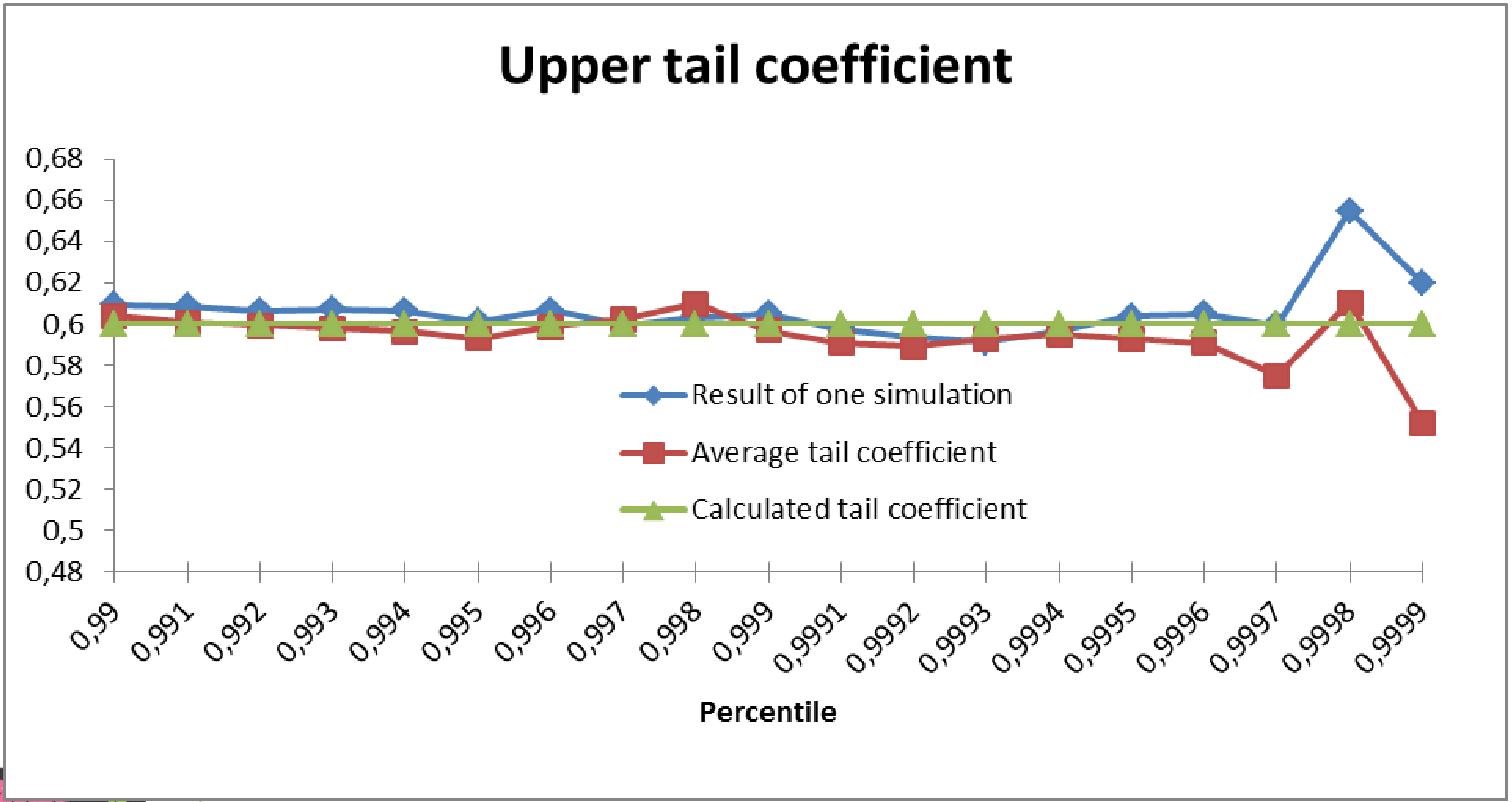
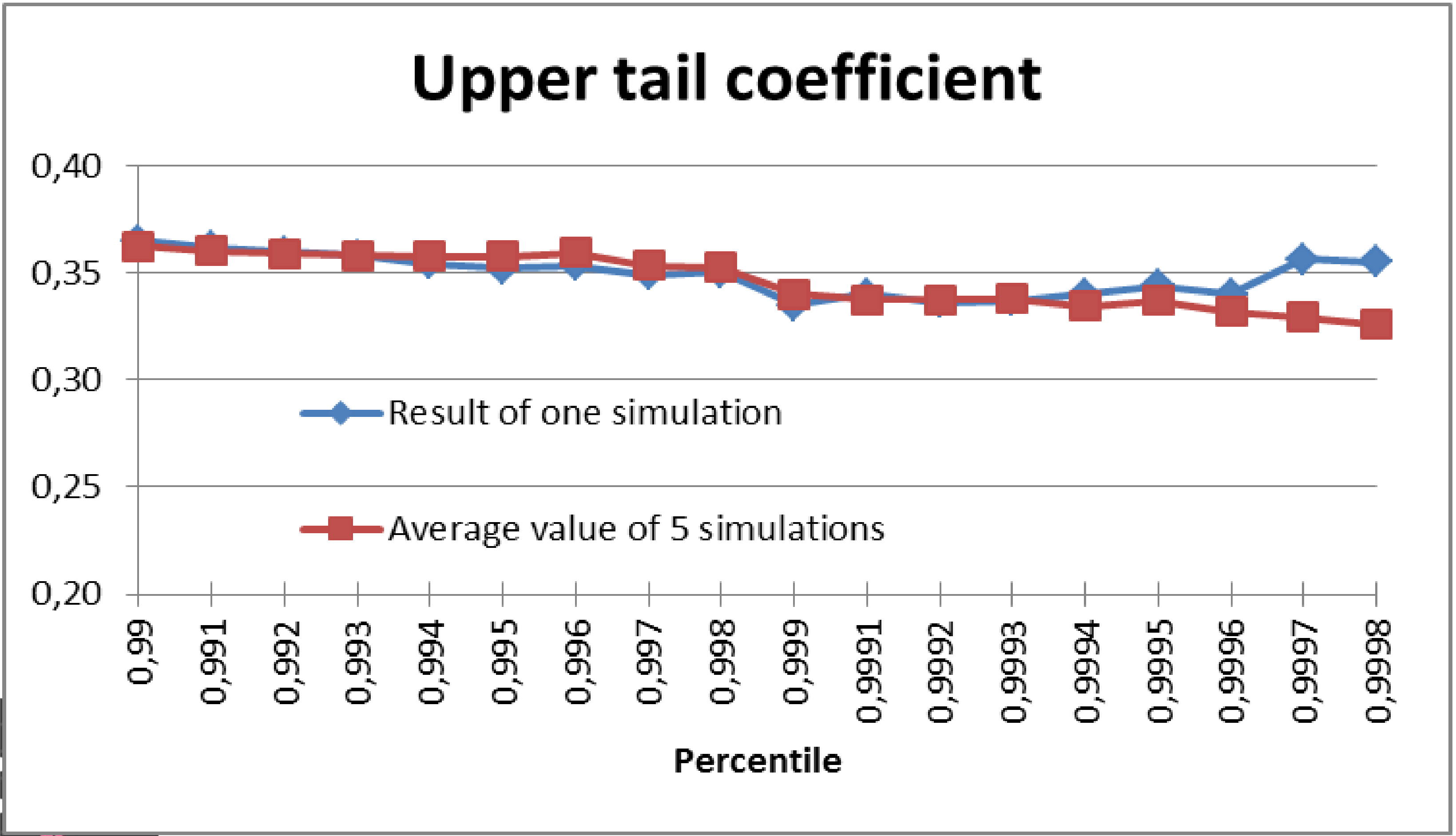
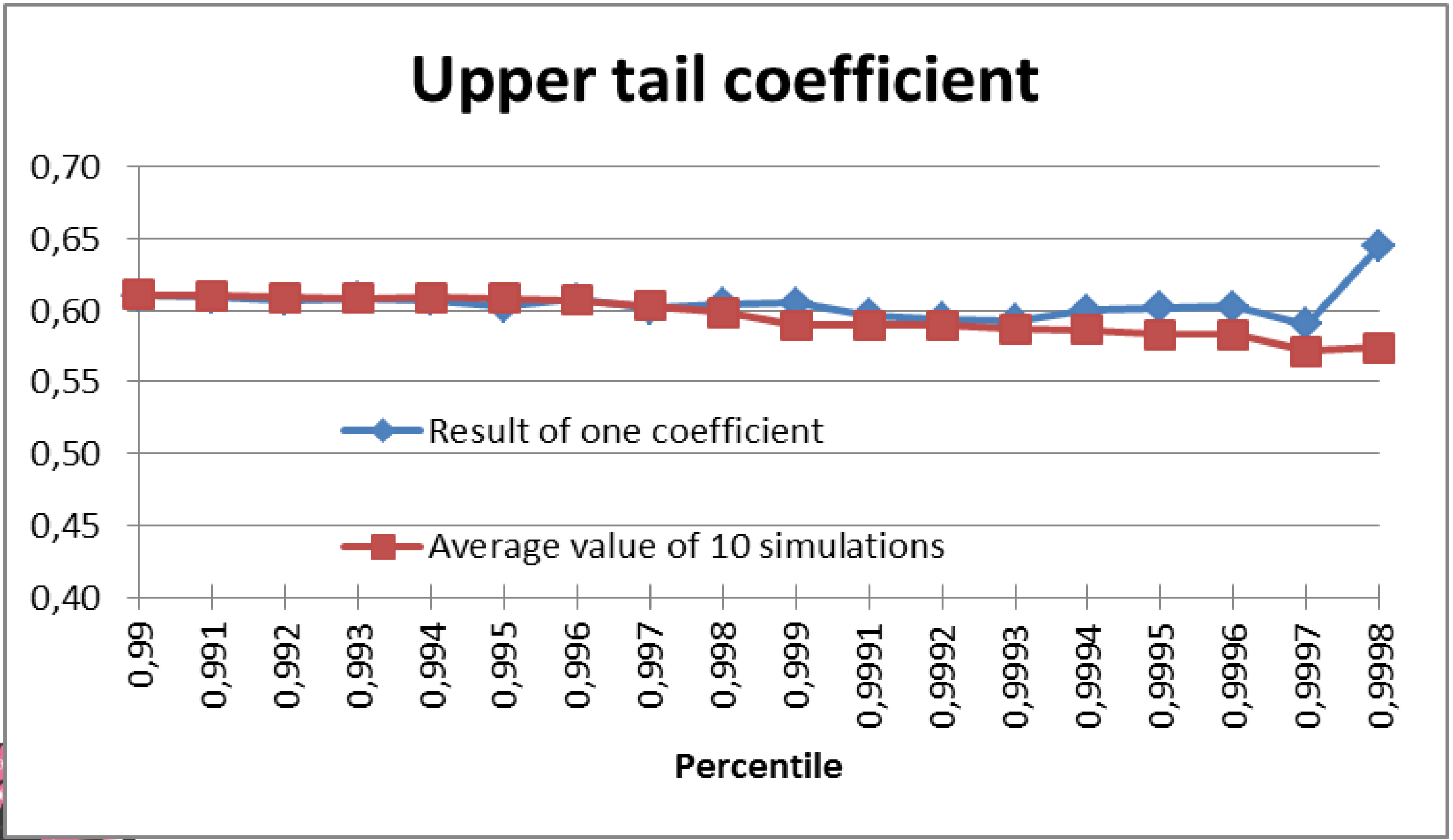
Simulation using both alfas equal 5 and correlation 0.85



Simulation using alfas 20 and 10 and correlation 0.55



Simulation using alfas 20 and 10 and correlation 0.85



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Simulation using alfas -20 and -10 and correlation 0.85

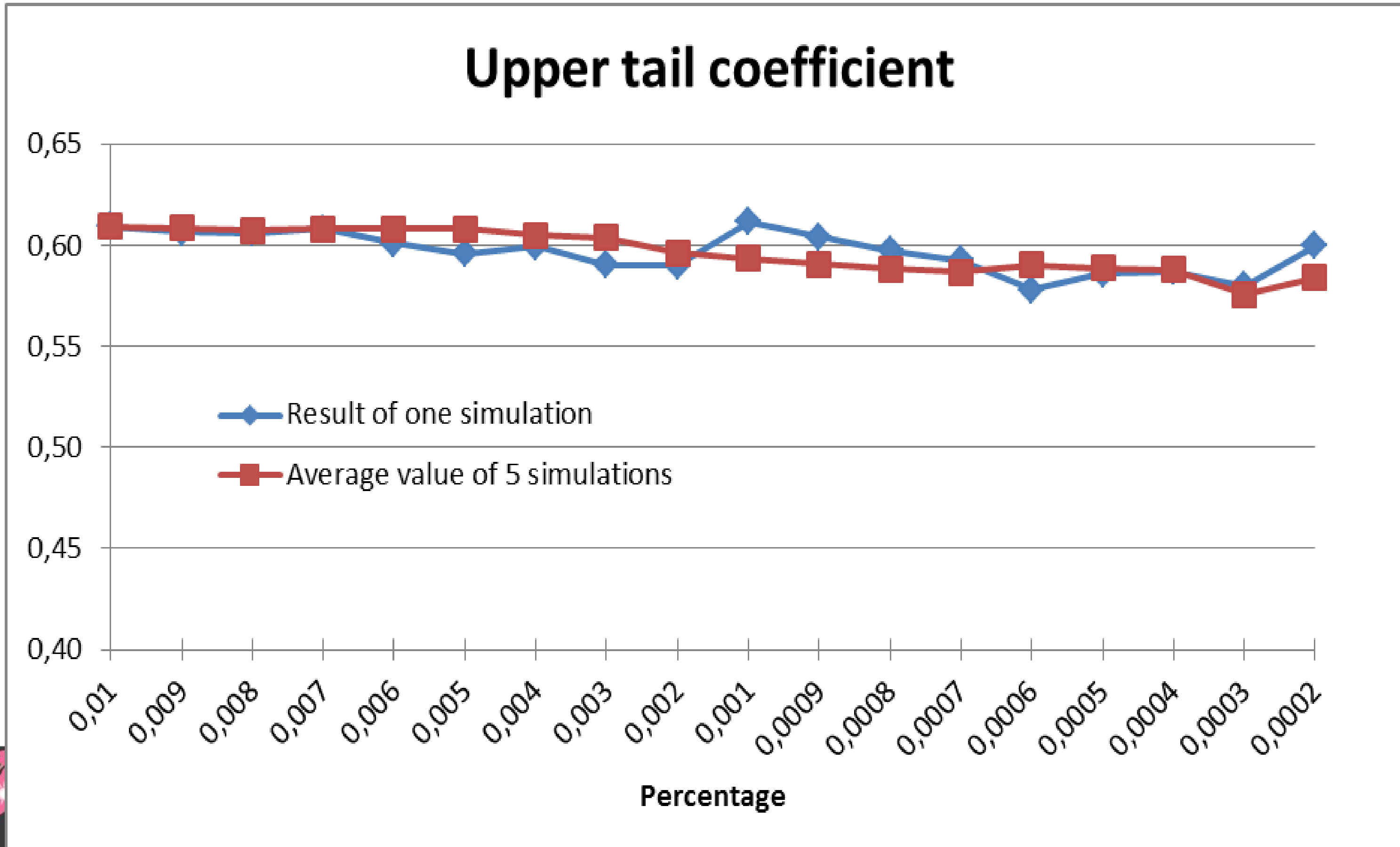

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Table summarizing rezults

No	Alfa 1	Alfa 2	Ro	Percentile	Average tail coefficient	Standard deviation	Coefficient of variation %
1	20	10	0,85	0,999	0,5896	0,0103	1,8
				0,9995	0,5832	0,0182	3,1
2	20	10	0,55	0,999	0,3400	0,0080	2,3
				0,9995	0,3372	0,0149	4,4
3	-20	-10	0,85	0,001	0,5936	0,0155	2,6
				0,0005	0,5888	0,0136	2,3
4	5	5	0,5	0,999	0,3127	0,0071	2,3
				0,9995	0,3137	0,0226	7,2
5	5	5	0,85	0,999	0,5968	0,0074	1,2
				0,9995	0,5929	0,0166	2,8

Simulation using alfas 5 and 0, correlation 0.5 and degrees of freedom 3

n	$\lambda_{up,St,MM}$	$\left\|\lambda_{up,St,MM}-\lambda_{up}\right\|$	$\lambda_{up,St,ML}$	$\left\|\lambda_{up,St,ML}-\lambda_{up}\right\|$
10	0.3675	0.1155	0.3096	0.2123
100	0.3725	0.0548	0.3979	0.0703
1000	0,3712	0.0301	0.3743	0.0204
10 000	0.3734	0.0165	0.3728	0.0070
100 000	0.3729	0.0093	0.3728	0.0023
1 000 000	0.3728	0.0045		

Simulation using alfas 5 and -5, correlation 0.5 and degrees of freedom 3

n	$\lambda_{up,St,MM}$	$\left\|\lambda_{up,St,MM} - \lambda_{up}\right\|$	$\lambda_{up,St,ML}$	$\left\|\lambda_{up,St,ML} - \lambda_{up}\right\|$
10	0.5380	0.4421	0.4051	0.6793
100	0.7349	0.2749	0.7141	0.4265
1000	0,8590	0.1775	0.9217	0.2154
10 000	0.9369	0.0879	0.9594	0.0793
100 000	0.9587	0.0339	0.9734	0.0266
1 000 000	0.9638	0.01047		

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