Asymptotic Behavior of Central Order Statistics Under Monotone Normalization and Several Applications to Reinsurance

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Notes

Abstract

Smirnov (1949), derived four limit types of distributions for linearly normalized central order statistics, under the weak convergence. In this paper we investigate the possible limit distributions of the k-th upper order statistics with central rank using monotone regular norming sequences and obtain thirteen possible types. Besides considering the upper order statistics with central rank of sample with deterministic size, we also investigate the asymptotic behavior of randomly indexed upper order statistics using regular norming time-space changes.

Notes

Introduction and Preliminaries

Let $X_1, X_2, ..., X_n$ be a sample of iid rvs with common continuous df F. In our study below all distribution functions are supposed to be left-continuous. We denote $X_{1,n} = \max_{i\in\{1,2,...,n\}}\{X_i\}$, $X_{2,n} = \max_{i\in\{1,2,...,n\}\setminus L}\{X_i\}$, for L the index of the first maximum,$\cdots$, $X_{n,n} = \min_{i\in\{1,2,...,n\}}\{X_i\}$. The sequence $X_{n,n} \le ... \le X_{k,n} \le ... \le X_{1,n}$ is the ordered sample. The random variables $X_{n,n}, ..., X_{k,n}, ..., X_{1,n}$ are called upper order statistics and $X_{k,n}$ is the k-th upper order statistic (u.o.s).

Notes

In [10], Smirnov among others, considered the case of sequence (k_n) with the central order property $\frac{k_n}{n} \to \theta \in (0,1)$, $n \to \infty$. With appropriate restrictions for the (k_n) he obtains $F_{k_n,n}(u_n) := P\left(X_{k_n,n} < u_n\right) \longrightarrow \Phi(\tau)$, as $n \to \infty$, where $\{u_n\}$ is properly chosen increasing sequence.

Furthermore, in the case of central order statistics, Smirnov [10] has proved that the condition $\sqrt{n}\frac{F(a_n x+b_n)-\theta}{(\theta(1-\theta))^{1/2}} \longrightarrow \tau(x)$ is equivalent to the asymptotic relation $F_{k_n,n}(a_n x + b_n) \xrightarrow{w} \Phi(\tau(x))$.

Let (k_n) be a sequence of integers such that $\frac{k_n}{n} \to \theta \in (0,1)$ and $X_{k_n,n}$ the k_n-th u.o.s. We denote by GMA the group of max-automorphisms on $\mathbb{R}$, i.e. strictly increasing, continuous mappings. Also let $C(f)$ be the set of all continuity points of f.

Notes

Theorem 1.

Let H be a non-degenerate df, $\frac{k_n}{n} \to \theta \in (0,1)$ and $\{G_n\}$ be a sequence of norming mappings in GMA. Then

$$F_{k_n,n}\left(G_n(x)\right) = P\left({G_n}^{-1}(X_{k_n,n}) < x\right) \xrightarrow[n]{w} H(x) \qquad (1)$$

if and only if

$$\sqrt{n}\frac{\theta - \overline{F}\left(G_n(x)\right)}{\sqrt{\theta(1-\theta)}} \xrightarrow[n]{w} \tau(x) \qquad (2)$$

where $\tau(x)$ is a non-decreasing function uniquely determined by the equation

$$H(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\tau(x)} e^{-\frac{x^2}{2}}\, dx = \Phi(\tau(x)). \qquad (3)$$

Notes

The aim of our work is to give a characterization of the class of limit distributions in (1). In such problems the main analytical tool is the convergence to type theorem. Unfortunately it doesn't hold in the model of maxima. This is why we have to set additional conditions on the normalizing sequence, namely we suppose it is regular.

Definition 1.

The sequence $\{G_n\}$ is a regular norming sequence on the set $\mathbb{X} \times \mathbb{T}$ if for $x \in \mathbb{X}$ and $t \in \mathbb{T}$, there exists $\lim_{n\to\infty} {G_{[nt]}}^{-1} \circ G_n(x) = g_t(x) \in GMA$ and the correspondence $t \to g_t$ is continuous 1-1 mapping. Here $[x]$ is the integer part of x.

For our purposes, $\mathbb{X} = (l_H, r_H)$, $\mathbb{T} = (0, \infty)$. Clearly, for $t = 1$, ${G_n}^{-1} \circ G_n(x) = x = g_1(x) \in GMA$.

Notes

Lemma 1.

Let the sequence $\{G_n\} \subset GMA$ be a regular norming sequence on $(l_H, r_H) \times (0, \infty)$. Then the limit mapping $g_t(x)$ satisfies the functional equation

$$g_t \circ g_s(x) = g_{t \cdot s}(x), \quad s, t \in (0, \infty) . \tag{4}$$

Definition 2.

A distribution function F belongs to the normal domain of θ - attraction of H if there exists a norming sequence $\{G_n\} \subset GMA$ and the second order condition $\left(\frac{k_n}{n} - \theta\right)\sqrt{n} \to 0$ holds, such that $F_{k_n,n}(G_n(x)) \xrightarrow[n]{w} H(x)$.

Remark.

Further on we will abbreviate the normal domain of θ - attraction of H with $\theta - NDA(H)$.

Notes

Theorem 2.

If a df H has a $\theta - NDA$ with respect to a regular norming sequence $\{G_n\} \subset GMA$, then its corresponding function τ satisfies the following functional equation:

$$\sqrt{t} \cdot \tau(x) = \tau\left(g_t(x)\right), \forall t \in (0, \infty), x \in C(\tau) . \tag{5}$$

Denote $\mathcal{D} := \{x \mid \tau(x) \neq 0, \tau(x) \neq \pm\infty\}$.
Let $x \notin \mathcal{D}$. For $\tau(x) = \Phi^{-1}(H(x)) = \infty$, $H(x) = 1$, hence $x \geq r_H$. For $\tau(x) = -\infty$, $H(x) = 0$, hence $x \leq l_H$. And for all x such that $\tau(x) = 0$, $H(x) = \frac{1}{2}$. Defining the median of H by $m_H = \sup\left\{x \mid H(x) < \frac{1}{2}\right\}$, we reach uniqueness of the median. One may notice that $\tau(m_H+) = 0$.

Proposition 1.

$\tau(x)$ is continuous and strictly increasing on $\mathcal{D}$.

Notes

Let us now consider the set $S \cap \mathcal{D}^c$. Here S is the open interval (l_H, r_H) and $\mathcal{D}^c$ consists of all points $x \leq l_H$, $x \geq r_H$ and all x with $\tau(x) = 0$. Hence, the set $S \cap \mathcal{D}^c$ consists either of only one point m_H or it is a connected interval $I \subseteq S$ whose left end point is m_H and on which $\tau \equiv 0$.

Proposition 2.

$I = (m_H, r_H)$.

Corollary 1.

Let $I = (m_H, r_H)$. Then $\tau(x) = \begin{cases} -\infty, & x \leq l_H = m_H \\ 0, & x \in (l_H = m_H, r_H) \\ +\infty, & x \geq r_H. \end{cases}$.

This case will be referred to as a singular case. The corresponding limit distribution H is the two jump distribution, with jumps of height 1/2 in l_H and r_H, and is constant on $(l_H = m_H, r_H)$.

Notes

Further on, we ask for the explicit form of τ in the remaining case, $I = \{m_H\}$, which we assume to hold in all of the following analysis. As a consequence of proposition 1, τ (respectively H) might jump only at l_H, m_H and r_H. The following three cases are the only possible cases:

i) $S=(l_H=m_H, r_H)$, i.e. H jumps at l_H by $1/2$ and H might have a jump at r_H;

ii) $S=(l_H, m_H=r_H)$, i.e. H jumps at r_H by $1/2$ and H might have a jump at l_H;

iii) $S=(l_H, r_H)$ and $l_H < m_H < r_H$, i.e. H might have jumps at r_H, l_H and m_H.

Proposition 3.

The three points l_H, m_H and r_H are the only possible fixed points of g_t.

Notes

Solution of the functional equations (4) and (5)

Lemma 2.

Let $g_t : (l_H, r_H) \to (l_H, r_H)\,,\ \{g_t \mid t > 0\}$ be c.o.g in GMA, satisfying $g_t \circ g_s = g_{t \cdot s}$, $\forall t, s > 0$. We suppose that

i) g_t is expanding for $x \in (l_H, m_H)$, $t \in (0, 1)$ and for $x \in (m_H, r_H)$, $t \in (1, \infty)$;

ii) g_t is contracting for $x \in (m_H, r_H)$, $t \in (0, 1)$ and for $x \in (l_H, m_H)$, $t \in (1, \infty)$.

Then there exist continuous and strictly increasing mappings $h : (l_H, m_H) \to (-\infty, \infty)$ and $l : (m_H, r_H) \to (-\infty, \infty)$ such that for $t > 0$:

$$g_t(x) = \begin{cases} h^{-1}\left(h(x) - \log t\right), & x \in (l_H, m_H) \\ l^{-1}\left(l(x) + \log t\right), & x \in (m_H, r_H) \end{cases} .$$

Notes

Let $S_1 = (l_H, m_H)\,,\ S_2 = (m_H, r_H)$.
Using the previous analysis we can formulate the following

Lemma 3.

Let $\{g_t \mid t > 0\}$ be the c.o.g from Lemma 2. Suppose τ satisfies the functional equation $\sqrt{t} \cdot \tau(x) = \tau(g_t(x))$, for $t > 0$ and $x \in S \cap \mathcal{D}$, given that $\tau(x) > 0$ on $S_2 \cap \mathcal{D}$ and $\tau(x) < 0$ on $S_1 \cap \mathcal{D}$. Then:

$$\tau(x) = \begin{cases} -c_1 e^{-h(x)/2}, \ c_1 > 0, \ \textit{on } S_1 \cap \mathcal{D} \\ c_2 e^{l(x)/2}, \ \ c_2 > 0, \ \textit{on } S_2 \cap \mathcal{D} \end{cases} .$$

Notes

Using all the previous results, after obtaining the explicit form of $\tau(x)$, we can now state the characterization theorem for the limit distribution H.

Theorem 3.

The non-degenerate df H in the limit relation (1) may take one of the following four explicit forms:

1. $H_1(x) = \begin{cases} 0, & x < m_H = l_H \\ \frac{1}{2}, & x = m_H = l_H \\ \Phi(\tau_2(x)), & x \in (m_H, r_H) \\ 1, & x \geq r_H \end{cases}$ for

$$\tau(x) = \begin{cases} -\infty, & x < m_H = l_H \\ 0, & x = m_H = l_H \\ \tau_2(x), & x \in (m_H, r_H) \\ \infty, & x \geq r_H \end{cases}$$

Notes

2. $H_2(x) = \begin{cases} 0, & x \leq l_H \\ \Phi(\tau_1(x)), & x \in (l_H, m_H) \\ \frac{1}{2}, & x = m_H = r_H \\ 1, & x > m_H = r_H \end{cases}$ for

$$\tau(x) = \begin{cases} -\infty, & x \leq l_H \\ \tau_1(x), & x \in (l_H, m_H) \\ 0, & x = m_H = r_H \\ \infty, & x > m_H = r_H \end{cases}$$

3. $H_3(x) = \begin{cases} 0, & x \leq l_H \\ \Phi(\tau_1(x)), & x \in (l_H, m_H) \\ \Phi(\tau_2(x)), & x \in (m_H, r_H) \\ 1, & x \geq r_H \end{cases}$ for

$$\tau(x) = \begin{cases} -\infty, & x \leq l_H \\ \tau_1(x), & x \in (l_H, m_H) \\ \tau_2(x), & x \in (m_H, r_H) \\ \infty, & x \geq r_H \end{cases}$$

Notes

4. $H_4(x) = \begin{cases} 0, & x < l_H \\ \frac{1}{2}, & x \in [l_H, r_H) \\ 1, & x \geq r_H. \end{cases}$ for

$$\tau(x) = \begin{cases} -\infty, & x \leq l_H = m_H \\ 0, & x \in (l_H = m_H, r_H) \\ +\infty, & x \geq r_H \end{cases} .$$

Remark.

Let H, G be non-degenerate dfs. Recall, in the extreme value theory we say that $H \in type(G)$ if there exists a mapping $\varphi \in GMA$ such that $H = G \circ \varphi$. In the theorem 3 above, we speak of four different explicit forms of H and not of types. If we count the types, in the sense above, we have to acknowledge that there are 13 possible types. Namely, H_1 gives rise to two different types: one with one jump, and another one with two jumps; the same holds for H_2; H_3 gives rise to 8 types with mostly three jumps. The singular H_4 determines only one type, the two jump distribution.

Notes

Some introductory notions on point processes

On a given probability space $(\Omega, \mathcal{A}, P)$, sufficiently rich, let a point process $\mathcal{N} = \{(T_k, X_k) : k \geq 1\}$ be defined in the following way:
i) The random arrival process $\{T_k\}$ consists of increasing time points $0 < T_1 < T_2 < ... < T_n \to \infty$. We suppose the inter-arrival times $Y_k := T_k - T_{k-1}$, $k \geq 1$, $T_0 = 0$ are independent rv's, $T_n = \sum_{k=1}^{n} Y_k$. The corresponding counting process $N(t) := \max\{k : T_k \leq t\}$ is right continuous.
ii) The random state process $\{X_k\}$ is built by positive iid rv's X_k with continuous df F_X.
iii) Both sequences $\{T_k\}$ and $\{X_k\}$ are independent.
In this initial model, at every fixed moment $t > 0$, we are supplied a sample $\{X_1, X_2, ..., X_{N(t)}\}$ of random size $N(t)$. Our interest is focused on the upper order statistics (u.o.s.) of this sample

$$X_{N(t):N(t)} < X_{N(t)-1:N(t)} < ... < X_{k:N(t)} < ... < X_{1:N(t)}$$

Notes

Definition 3.
We call $Y_k : \Omega \times (0, \infty) \to (0, \infty)$, $Y_k(t) := X_{k:N(t)}$ the k-th u.o.s. process.

The asymptotic behavior of $Y_k(t)$ for $t \to \infty$ and k fixed, under linear norming, is considered e.g. in Embrechts, Kluppelberg, Mikosch [11], Chapter 4.3. For $k = 1$ the process $Y_1(t) = \bigvee_{k=1}^{N(t)} X_k$ is an extremal process, investigated e.g. in Balkema, Pancheva [3] and Pancheva [6]. Here "$\bigvee$" denotes the maximum operation.

In this paper we discuss the asymptotic behavior of the k-th u.o.s. process $Y_k(t)$ using monotone normalization. We assume that there exists a sequence of time-space changes on $(0, \infty) \times (0, \infty)$, $\zeta_n(t, x) = (\tau_n(t), u_n(x))$, $\tau_n(t)$ and $u_n(x)$ continuous and strictly increasing, such that the normalized k-th u.o.s. process converges in law to a non-degenerate random process $Y^{(k)}(t)$, i.e.

$$Y_n^{(k)}(t) := u_n^{-1} \circ Y_k \circ \tau_n(t) = u_n^{-1}\left(X_{k:N(\tau_n(t))}\right) \xrightarrow{d} Y^{(k)}(t). \quad (6)$$

Notes

Further on we suppose that the norming sequence $\{\zeta_n\}$ is *regular* in the following sense: $\forall s > 0$ there exist mappings $\mathcal{U}(s, x)$ and $\mathcal{T}(s, t)$, strictly increasing and continuous in x, respectively in t, such that for $n \to \infty$

$$\lim_{n\to\infty} u_{[ns]}^{-1} \circ u_n(x) = \mathcal{U}(s, x) \quad (7)$$

$$\lim_{n\to\infty} \tau_{[ns]}^{-1} \circ \tau_n(x) = \mathcal{T}(s, x). \quad (8)$$

Moreover, the mappings $s \to \mathcal{U}(s, \cdot) =: \mathcal{U}_s(\cdot)$ and $s \to \mathcal{T}(s, \cdot) =: \mathcal{T}_s(\cdot)$ are one-to-one.
The process $Y_n^{(k)}(t)$ is associated with the point process

$$\mathcal{N}_n = \left\{\left(T_{k,n} := \tau_n^{-1}(T_k), X_{k,n} := u_n^{-1}(X_k)\right) : k \geq 1\right\}, n \geq 1.$$

Notes

Let
$N_n(t) = \max\{k : T_{k,n} \leq t\} = \max\{k : T_k \leq \tau_n(t)\} = N(\tau_n(t))$
be the counting process of $\mathcal{N}_n$. Consider the u.o.s. of the n-th sample series
$\{X_{1,n}, X_{2,n}, ..., X_{N(t),n}\}$, namely

$$X_{N_n(t):N_n(t),n} < ... < X_{k:N_n(t),n} < ... < X_{1:N_n(t),n}$$

where $X_{k:N_n(t),n} = u_n^{-1}\left(X_{k:N(\tau_n(t))}\right) = Y_n{}^{(k)}(t)$ is the k-th u.o.s. in the n-th sample series of size $N_n(t)$. In this way, with $n \to \infty$, the sample size $N_n(t)$ increases whereas the value of the state points $X_{k,n}$ decreases.

For the limit process $Y^{(k)}(t)$ in (6) we ask the following questions:
Q_1: When does it exist?
Q_2: Which class does it belong to?
The answers depend essentially on the character of the rank k.

Notes

Fixed rank case

Let us return to the initial model. Denote by $X_{k:n}$ the k-th u.o.s. of the sample $\{X_1, X_2, ..., X_n\}$ with a continuous df F_X. The asymptotic behavior of the normalized $X_{k:n}$ is stated in the following

Proposition 4.

Suppose $F_X \in \max-\mathcal{DA}$ of a max-stable df G w.r.t. a regular norming sequence $\{u_n(\cdot)\}$. Then for fixed k and $n \to \infty$

$$P\left(X_{k:n} < u_n(x)\right) \xrightarrow{w} H(x) = \overline{\Gamma}_k\left(-\log G(x)\right),$$

where $\Gamma_k(x) = \frac{1}{(k-1)!}\int_0^x t^{k-1}e^{-t}dt$ is the Gamma df.

Notes

Before analyzing the limit class of the processes obtained in (6), we need to agree on the asymptotic behavior of the counting process $N_n(t)$. Let us assume that $\forall t > 0$

$$\frac{N_n(t)}{n} \xrightarrow{d} \Lambda_t. \quad (9)$$

Here Λ_t with df F_Λ is random time change.

Definition 4.

Under random time change we understand a strictly increasing and continuous random function $\Lambda : (0, \infty) \to (0, \infty)$, $\Lambda(0) = 0$ and $\Lambda(t) \to \infty$ for $t \to \infty$.

Notes

Now we are ready to formulate our first result on randomly indexed u.o.s.

Theorem 4.

Suppose $F_X \in \max - \mathcal{DA}(G)$ w.r.t. a regular norming sequence $\{u_n\}$. Assume that the counting process $N_n(t)$ satisfies (9). Then for fixed k and $n \to \infty$

$$P\left(X_{k:N_n(t)} < u_n(x)\right) \xrightarrow{w} \int_0^\infty H\left(\mathcal{U}(s,x)\right) dF_{\Lambda(t)}(s) = EH \circ \mathcal{U}(\Lambda_t, x)$$

where $H(x) = \overline{\Gamma}_k(-\log G(x))$ is the limit distribution from Proposition 4 and $\mathcal{U}(s,x) = \lim_{n\to\infty} u_{[ns]}^{-1} \circ u_n(x)$.

Notes

Corollary to Theorem 4.

Let $k = 1$, then

$$P\left(\bigvee_{j=1}^{N_n(t)} X_j < u_n(x)\right) \xrightarrow{w} EG \circ \mathcal{U}(\Lambda_t, x) = EG^{\Lambda_t}(x).$$

Theorem 4 gives answer to our first question Q_1: If $F^n\left(u_n(x)\right) \xrightarrow{w} G(x)$, where $\{u_n\}$ is regular, and if $\frac{N_n(t)}{n} \xrightarrow{d} \Lambda_t$, Λ_t random time change, then there exists a random process $Y^{(k)}(t)$ with df $P\left(Y^{(k)}(t) < x\right) =: g(t,x) = E\overline{\Gamma}_k\left(-\log G\left(\mathcal{U}(\Lambda_t, x)\right)\right)$, such that
$Y_n^{(k)}(t) = u_n^{-1}\left(X_{k:N_n(t)}\right) \xrightarrow{d} Y^{(k)}(t)$.

Notes

Remark.

In the asymptotic results for randomly indexed samples with size N_n, known in the literature, the authors usually suppose convergence in probability $\frac{N_n}{n} \xrightarrow{P} \Lambda$, Λ positive rv, e.g. Galambos [8], Theorem 6.2.1, Bilingsley [12], Theorem 17.2. In our model, we assume the sequences $\{N_n\}$ and $\{X_n\}$ are independent and Λ is random time change. Thus, it is enough to suppose a convergence in distribution $\frac{N_n(t)}{n} \xrightarrow{d} \Lambda_t$.

Next, using the regularity of the norming sequence $\{\tau_n\}$ (see (8)), we give an answer to question Q_2. Let $\stackrel{fdd}{=}$ denote equivalence of the finite dimensional distributions.

Theorem 5.

The limit process $Y^{(k)}$ in (6) is self-similar w.r.t. the continuous one-parameter group $\{\eta_s(t,x) = (\mathcal{T}_s(t), \mathcal{U}_s(x)) : s > 0\}$ of time-space changes, i.e.

$$Y^{(k)}(t) \stackrel{fdd}{=} \mathcal{U}_s^{-1} \circ Y^{(k)} \circ \mathcal{T}_s(t),\ \forall s > 0$$

Notes

Increasing rank case

In this section we consider $Y_n^{(k_n)}(t) = u_n^{-1} \circ X_{k_n:N_n(t)}$, the central u.o.s. process, where the rank $k = k_n$ increases with n in such a way that

$$\frac{k_n}{n} \to \theta \in (0,1)\,. \tag{10}$$

Further we assume that $\frac{N_n(t)}{n} \xrightarrow{d} \Lambda_t$, Λ_t random time change.
We ask for the asymptotic behavior of $Y_n^{(k_n)}(t)$ as $n \to \infty$. Let us preliminary consider a sample with non-random size l_n, $l_n \to \infty$ as $n \to \infty$, namely $\{X_1, X_2, ..., X_{l_n}\}$. We form the c.o.s. $X_{k_n:l_n}$ with the property $\frac{k_n}{l_n} \to \lambda \in (0,1)$. As norming mappings we again use the regular sequence $\{u_n(\cdot)\}$ of space changes $u_n : (0,\infty) \to (0,\infty)$. Denote $p_n(\cdot) = P(X_i \geq u_n(\cdot))$. Just analogously to Theorem 2.5.2 in Leadbetter, Lindgren, Rootzen [9], where $u_n(x)$ are linear, one can prove the following statement:

Notes

Proposition 5.

Suppose $k_n \to \infty$, $k_n < l_n$, $\frac{k_n}{l_n} \to \lambda \in (0,1)$, $p_n(x) := 1 - F(x)$ and $l_n(1-p_n(\cdot))p_n(\cdot) \to \infty$, as $n \to \infty$. Let

$$\frac{k_n - l_n p_n(x)}{\sqrt{l_n p_n(x)(1-p_n(x))}} \xrightarrow{w} \tau_\lambda(x). \tag{11}$$

Then for $n \to \infty$

$$P\left(X_{k_n:l_n} < u_n(x)\right) \xrightarrow{w} \Phi\left(\tau_\lambda(x)\right) \tag{12}$$

where $\Phi(x)$ is the standard normal df.

Let us return to the randomly indexed c.o.s. $X_{k_n:N_n(t)}$ satisfying (9) and (10). On the basis of the previous Proposition 5, we now state the main result:

Notes

Theorem 6.

Suppose $k_n \to \infty$ and (10) is satisfied. Assume there exists a regular norming sequence $\{u_n(\cdot)\}$ such that $n\overline{F}\left(u_n(x)\right) F\left(u_n(x)\right) \to \infty$ and

$$\sqrt{n}\frac{\theta - \overline{F}\left(u_n(x)\right)}{\sqrt{\theta\left(1-\theta\right)}} \xrightarrow{w} \tau(\theta, x), \tag{13}$$

for $n \to \infty$. If additionally (9) holds true, then $\forall t > 0$

$$u_n^{-1} \circ X_{k_n:N_n(t)} \xrightarrow{d} Y_0(t)$$

where the limit process has df

$$g_0\left(t,x\right) = \int_\theta^\infty \Phi \circ \tau\left(\tfrac{\theta}{z}, \mathcal{U}_z(x)\right) dF_{\Lambda(t)}(z)$$

$$= E[\Phi \circ \tau\left(\theta {\Lambda_t}^{-1}, \mathcal{U}_{\Lambda(t)}(x)\right) I\left\{\Lambda_t > \theta\right\}]$$

and $\mathcal{U}(\cdot,\cdot)$ is defined in (7).

Notes

Applications to reinsurance

1. The largest claims reinsurance, guarantees that the reinsurer will cover the largest k claims.

The reinsurers treat is $R_{LC}(t) = \sum_{i=N(t)-k+1}^{N(t)} X_{N(t)-i+1}$, and the insurer keeps the risk $I_{LC}(t) = \sum_{i=1}^{N(t)-k} X_{N(t)-i+1}$. The number of claims covered k might be fixed or growing sufficiently slow with t.

2. The ECOMOR reinsurance, can be seen as an excess of loss reinsurance with a random deductible, determined by the k-th largest claim in the portfolio.

The reinsurers treat is $R_{RExL}(t) = \sum_{i=1}^{N(t)} (X_{(N(t)-i+1)} - X_k)_+$, while the risk left for the insurer is

$I_{RExL}(t) = \sum_{i=1}^{N(t)} (X_{(N(t)-i+1)} \wedge X_{(N(t)-k+1)})$.

Notes

Thank You for your attention!

Notes

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