Quadratic hedging in an incomplete market derived by an influential informed investor

Anne EYRAUD-LOISEL

Université de Lyon, Université Lyon 1, ISFA, Laboratoire SAF

50 avenue Tony Garnier 69007 LYON

Tel +33-4-37-28-74-35

e-mail : anne.eyraud-loisel@univ-lyon1.fr

Abstract

This paper presents a model with an influential and informed investor. A hedging problem is studied from the point of view of a non informed agent in an influenced and informed market. Her lack of information makes the market incomplete. Obtained results, by means of Malliavin calculus and Clark-Ocone Formula, as well as Filtering Theory, are the expressions and a comparison between the strategy of the non informed trader, and the strategy of the informed agent. An expression of the residual risk a non informed trader keeps by detaining an option in this influenced and informed market is derived using a quadratic approach of hedging in incomplete market. Finally, the analysis leads to a measure of the lack of information that makes the market incomplete. The financial interpretation is explained throughout the theoretical analysis, together with an example of such influenced informed model.

1 Introduction

This article deals with a hedging problem in an incomplete market, where the incompleteness of the market is due to the presence of an influential informed investor. This influential informed agent is supposed to possess an additional information, and is also supposed to influence the market. This is a natural extension of the work of Eyraud-Loisel [1], where the informed agent is supposed to be a small investor, who does not influence asset prices. Eyraud-Loisel (2009) [2] has already studied the point of view of the influential informed agent, who wants to hedge against a given contingent claim. Influential and informed investors problems have already been modeled, for example by Grorud and Pontier in [3], in an optimization context, with convex constraints. The point of view here is a large trader model, as introduced Cuoco and Cvitanic in [4], where the large trader is supposed to be asymmetrically informed.

The chosen model is the same as in [2], where the market prices process is influenced by the portfolio of an informed agent. It is more realistic to consider large traders in the market, who may influence the evolution of asset prices, either by their large investment depth, or by their notoriety, when a charter phenomenon appears. Then asset prices may be influenced by certain big agents in the market, and it is quite natural to suppose that such large agents may have more easily access to additional information on the market. This is the reason why it is interesting to develop a model with an influential informed investor. The additional information is supposed to be a strong initial information, modeled by an initial enlargement of the Brownian filtration, as developed for insider trader modeling in Grorud and Pontier [5], or Amendinger [6]. This influenced market has been already studied in [2] from the influential informed agent's point of view, by solving the FBSDE appearing when modeling the hedging problem of the large trader, under the enlarged filtration induced by the additional information. The present article is devoted to the study of the non-informed agent's strategy in this influenced market. Despite the completeness of the influenced informed market, the non-informed market is incomplete : the common agent does not have access to the entire information driving the prices process.

In Section 2, we present the model and notations and quickly recall the results of [2] about the hedging strategy of the influential informed agent, and the completeness of the influenced informed market.

In Section 3 we study the incompleteness of the market from a non informed trader's point of view. As a consequence of the influence of the large informed trader, the filtration generated by prices reflect a part of the additional information, and is not the Brownian filtration anymore. In order to

compare both strategies, of the informed influential agent, and of the small common agent, we use a quadratic hedging approach in this incomplete market. As there is no predictable representation property in this market, we use the local risk minimization approach, introduced by Föllmer and Sonderman [7] in the case where prices are martingales, and developed by Schweizer [8] when prices are only semi-martingales, and Föllmer and Schweizer [9] in terms of minimal martingale measure. This approach shows the existence of a unique risk-minimizing strategy, which may be expressed using the Galtchouk-Kunita-Watanabe decomposition (as developed in Ansel and Stricker [10]). Another possible approach would be the mean-variance hedging method, developed by Gouriéroux, Laurent and Pham [11], and Rheinlander and Schweizer [12], who obtained an expression of the optimal strategy in terms of variance-optimal martingale measure. A very clear review of the subject and of these two quadratic hedging methods in a continuous market is provided in Pham (2000) [13].

We have chosen the local risk-minimization approach developed in Föllmer and Schweizer (1991) [9], as our model fits into their global idea : considering an incomplete market where the incompleteness is due to a lack of information. In our model, when adding the additional information, the market is complete. We compare the hedging strategy of a non informed agent to the strategy of the informed influential agent, in an incomplete market, under Hypothesis ($\mathbf{H_3}$), which differs from classical hedging hypotheses studied in Föllmer and Schweizer [9]. We obtain an expression of the optimal strategy in terms of projection of the strategy of the informed influential agent, with respect of the filtration generated by prices (Filtering Theory is used). Moreover, we give a version of the Clark-Ocone formula in our framework of enlarged filtration, as well as an expression of the informed agent strategy in terms of Malliavin derivatives. We derive also an expression of the residual risk linked to the lack of information of the non informed agent. In the last section, we give an example of information and influence that satisfies all hypotheses of this model, and give the expressions of the strategies and residual risk of the lack of information of the non informed trader derived in the previous section and applied in this case.

2   Model and notations

We consider here a model with an informed and influential agent. The market is modeled by a general probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and prices are supposed to be driven by a multidimensional Brownian motion W, whose natural completed filtration is denoted by $\mathcal{F} = (\mathcal{F}_t)_{t \in [0,T]}$. Wealth process X_t and

investment strategy π_t of the influential agent are supposed to influence both drift and/or volatility of the prices dynamics, such that drift b and volatility σ of the price process are measurable functions of the wealth and investment strategies of the influential agent :

$$P_t = P_0 + \int_0^t b(s, P_s, X_s, \pi_s)ds + \int_0^t < \sigma(s, P_s, X_s, \pi_s), dW_s > .$$

Throughout this analysis, interest rates will be assumed to be constant and equal to r.

We denote by $\mathcal{M}^2(\mathcal{F}, \mathbb{P})$ (resp. $\mathcal{M}^2_{(loc)}(\mathcal{F}, \mathbb{P})$) the space of square-integrable $(\mathcal{F}, \mathbb{P})$-martingales (resp. local martingales) under $\mathbb{P}$, and $\mathcal{M}^2_0(\mathcal{F}, \mathbb{P})$ (resp. $\mathcal{M}^2_{0,(loc)}(\mathcal{F}, \mathbb{P})$) the space of square integrable $(\mathcal{F}, \mathbb{P})$-martingales (resp. local martingales) with initial value 0.

We suppose also that the influential agent has an additional information on the market, so her wealth and portfolio processes are adapted to an initial enlargement of filtration $\mathcal{Y} = (\mathcal{Y}_t)_{t\in[0,T]}$ of the Brownian filtration $\mathcal{F}$. The additional information is modeled by a random variable $L \in \mathcal{F}_{T'}$, where $T' > T$, and the enlarged filtration is

$$\mathcal{Y}_t := \bigcap_{s>t}(\mathcal{F}_s \vee \sigma(L)).$$

The information is supposed to satisfy usual hypothesis (H_3), first introduced by Jeulin [14, 15], and extensively used by Grorud and Pontier [5], Amendinger [6] and Eyraud-Loisel [1] :

Definition 1 *Hypothesis* $(\mathbf{H_3})$
There exists a probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ such that under $\mathbb{Q}$, $\mathcal{F}_t$ and $\sigma(L)$ are independent for all $t < T'$.

Under this hypothesis, and under regularity hypotheses on the coefficients of the model, this influenced market with asymmetrical information is complete. This is a result developed and proved in Eyraud-Loisel [2] : the hedging problem of an agent in this market may be written as a forward-backward stochastic differential equation (FBSDE), to be solved in the enlarged space $\mathcal{Y}$. It has been proved that this enlarged FBSDE, under sufficient conditions, has a unique solution (Theorem 1 of [2]). The point of view studied in the present paper is different. First from the financial point of view : the considered investor has no access to the additional information which is influencing the market, and second from the methodological point of view : used tools are different, as it is an incomplete market study. We consider

in the present work the behavior of the market from the point of view of a normally informed agent, from who the additional information is unknown. The market is incomplete for the non informed trader. This incompleteness is due to his lack of information, as in Föllmer and Schweizer (1991) [9]. The present model is an example of complete market which becomes incomplete from a small non informed investor point of view. The study of such a market uses tools of quadratic hedging in incomplete market, or under incomplete information, and it is the aim of this paper.

3 Quadratic hedging in this incomplete market

3.1 Introduction of a non informed agent

In this model, the market prices are influenced by the insider's wealth and investment process. We consider a non informed agent, who wants to hedge against an option in this market. This non informed agent is supposed to be a small investor, which means that she can not influence asset prices. This can be viewed as a problem of hedging in partial information, or lack of information : the non informed agent can only observe prices. The non informed agent has access to the information represented by the filtration generated by prices :

$$\mathcal{F}_t^P := \sigma(P_s, 0 \leq s \leq t).$$

Define also

$$\mathcal{F} = \bigvee_{t \geq 0} \mathcal{F}_t^P$$

As the coefficients in the diffusion of prices are adapted to the enlarged filtration, due to the influence of the informed agent, even the Brownian filtration is not available to the small trader. Other processes that appear in the BSDE of the hedging problem of the non informed trader, in particular Brownian motion, are not directly observable on the market. Her portfolio strategy and wealth processes have to be $\tilde{\mathcal{F}}$-adapted. As the investment strategies of the informed and non informed traders are adapted to different filtrations, their sets of admissible strategies differ. Even if the insider market is complete, it may not be complete from the non informed trader's point of view. If the non informed trader has a hedging strategy, it may be different from the strategy of the influential informed trader, whose exact replicating strategy is $\mathcal{Y}$-adapted. Completeness of a market (defined as "any square integrable contingent claim is attainable") depends on the considered filtration and probability measure : everything depends on the existence of a martingale representation Theorem. In our context, under Hypothesis (H_3), there

exists a martingale representation Theorem in the enlarged space $\mathcal{Y}$, under probability $\mathbb{Q}$, and in the Brownian filtration $\mathcal{F}$ under probability $\mathbb{P}$, but not under filtration $\tilde{\mathcal{F}}$ (see Jacod and Shiryaev [16] Eyraud-Loisel [1]). The problem becomes a hedging problem in an incomplete market. We will use a quadratic local risk-minimization approach for this study.

We first study in Subsection 3.2 Proposition 1 the Kunita-Watanabe decomposition of the contingent claim in the prices filtration. In Subsection 3.3 a Clark-Ocone formula is obtained in the enlarged space, from which we derive an expression of the integrand of the representation Theorem in this space. Finally we give an expression of the residual quadratic risk linked to the lack of information of the non informed trader, successively under two families of risk-neutral probabilities : the risk-neutral probabilities from the informed trader's point of view in Paragraph 4.1, and the risk-neutral probabilities from the non informed trader's point of view in Paragraph 4.2.

3.2 Hedging strategy and Kunita-Watanabe decomposition

The different considered σ-fields are linked by the following relations :

$$\tilde{\mathcal{F}}_t \subset \mathcal{Y}_t \quad , \text{ and } \mathcal{F}_t \subset \mathcal{Y}_t$$

$$\mathcal{Y}_t = \bigcap_{s>t} \mathcal{F}_s \vee \sigma(L) = \bigcap_{s>t} \tilde{\mathcal{F}}_s \vee \sigma(L).$$

Indeed : $\mathcal{F}_s \subset \tilde{\mathcal{F}}_s$, and so

$$\mathcal{Y}_t = \bigcap_{s>t} \mathcal{F}_s \vee \sigma(L) \subset \bigcap_{s>t} \tilde{\mathcal{F}}_s \vee \sigma(L) \subset \mathcal{Y}_t,$$

which proves the previous equality.
To keep notations as light as possible, we consider only one risky asset here, but all results may be generalized to the d-dimensional case. The dynamics of the risky asset satisfies the following forward equation :

$$P_t = P_0 + \int_0^t b(s, P_s, X_s^L, \pi_s^L) ds + \int_0^t \sigma(s, P_s, X_s^L, \pi_s^L) dW_s \qquad (1)$$

where P_s, X_s^L and π_s^L are respectively the price of the risky asset, and the wealth and portfolio strategy of the informed agent. If this strategy is a hedging strategy of a contingent claim, it can be derived as a solution of a FBSDE in the enlarged space (see [2]).
The non informed trader wants to hedge against a European contingent claim

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$H = h(P_T) \in \mathcal{L}^2(\Omega, \tilde{\mathcal{F}}, \mathbb{Q})$. H has the form $H(W, L)$. This point of view may be used to compare investments strategies of two small agents, one who has access to the additional information and the other who can only observe prices and has just a part of the total information. If the same contingent claim is considered, it may lead to a comparison of the strategies of both agents, the first one being influential and informed and the other one having no influence and no additional information. If not, the general idea is to find a risk-minimizing hedging strategy in this incomplete market, and to measure the residual risk : it will give a measure of the lack of information, and its impact on the hedging problem.

Under $\mathbb{Q}$, probability measure defined in Definition 1 (Hypothesis ($\mathbf{H_3}$)), W_t is a $\mathcal{Y}$-Brownian motion, and P_t is a $(\mathcal{Y}, Q)$-semi-martingale. As P_t is $\tilde{\mathcal{F}}$-adapted, it is also a $(\tilde{\mathcal{F}}, \mathbb{Q})$-semi-martingale. The market is not complete: as H is attainable in $(\mathcal{Y}, \mathbb{Q})$, the replicating strategy is adapted to filtration $\mathcal{Y}$, and so is not admissible for the non informed trader in the smaller space. We develop quadratic hedging arguments in order to find the hedging strategy of the non informed trader that minimize the intrinsic residual risk of the contingent claim, risk linked to the lack of information on this influenced informed market.

Denote by $\mathcal{Q}$ the set of all equivalent martingale measures under filtration $\mathcal{Y}$, i.e. the set of all probability measures equivalent to $\mathbb{Q}$ (and so equivalent to $\mathbb{P}$) under which P_t is a $\mathcal{Y}$-martingale. Financially speaking it is the set of all risk-neutral probability measures from the informed trader's point of view.

Denote also by $\mathcal{Q}_N$ the set of all equivalent martingale measures under filtration $\tilde{\mathcal{F}}$, equivalent to $\mathbb{P}$, i.e. the set of all probability measures equivalent to $\mathbb{P}$ (and so equivalent to $\mathbb{Q}$) under which P_t is a $\tilde{\mathcal{F}}$-martingale. It is the set of all risk-neutral probability measures from the non informed trader's point of view.

We have clearly

$$\mathcal{Q} \subset \mathcal{Q}_N.$$

Indeed, intuitively, from her lack of information, the non informed agent will have a larger range of risk-neutral probability measures. Some of the risk-neutral probability measures will appear to be risk-neutral from the non informed agent's point of view, whereas with the additional information, they will appear not to be risk-neutral. Conversely, any risk-neutral probability measure for the informed trader is always also risk-neutral for the non informed trader: if P is a $\mathcal{Y}$-martingale, as it is $\tilde{\mathcal{F}}$-adapted, it is a $\tilde{\mathcal{F}}$-martingale. We suppose sufficient conditions to have $\mathcal{Q}$ not empty : for example $P^{-A}\sigma$

is invertible and $\mathcal{E}(-P^{-1}\sigma^{-1}(P^{-1}b-r)W)$ integrable are classical arguments implying no arbitrage opportunities (see Delbaen and Schachermayer [17]). Under such conditions, $\mathbb{Q}_0$ defined with its Radon-Nikodym derivatives as the previous Doleans-Dade exponential is in $\mathcal{Q}$, which is not empty.

Recall the definition of orthogonality for two martingales (see Protter [18], and Pham [13]):

Definition 2 *Two martingales L, N are (strongly) orthogonal if their product LN is a local martingale with initial value $L_0 N_0 = 0$.*
If L and $N \in \mathcal{M}^2_{loc}$, this is equivalent to $< L, N >= 0$.

We give the definition of the Kunita-Watanabe decomposition (see Ansel and Stricker [10] or Pham [13]) on a general probability space $(\Omega, \mathcal{F}, \mathbb{P})$:

Definition 3 *Let N be a real-valued $(\mathcal{F}, \mathbb{P})$-local martingale, and M a $\mathbb{R}^d$-valued $(\mathcal{F}, \mathbb{P})$-local martingale. If $N \in \mathcal{M}^2_{(loc)}(\mathcal{F}, \mathbb{P})$ and $M \in \mathcal{M}^2_{(loc)}(\mathcal{F}, \mathbb{P})$, then we have the following projection result, called* **Kunita-Watanabe decomposition** *of the $(\mathcal{F}, \mathbb{P})$-martingale N with respect to M :*

$$N_t = N_0 + \int_0^t \theta_u dM_u + L_t, \quad \mathbb{P} - a.s. \ \ 0 \le t \le T \tag{2}$$

where $\theta \in L^2_{(loc)}(M)$, $L \in \mathcal{M}^2_{0,(loc)}(\mathcal{F}, \mathbb{P})$ and L orthogonal to M in $(\mathcal{F}, \mathbb{P})$.

Remark 1 *Decomposition (2) is unique in the sense that if*

$$N_t = N_0 + \int_0^t \theta_u dM_u + L_t = \tilde{N}_0 + \int_0^t \tilde{\theta}_u dM_u + \tilde{L}_t$$

with (N_0, θ, L) et $(\tilde{N}_0, \tilde{\theta}, \tilde{L})$ satisfying Kunita-Watanabe hypotheses, then $N_0 = \tilde{N}_0$, $\int_0^t \theta_u dM_u = \int_0^t \tilde{\theta}_u dM_u$ and $L_t = \tilde{L}_t$ $\mathbb{P}$-a.s. $\forall 0 \le t \le T$.
Remark also that if M and N are in $\mathcal{M}^2(\mathcal{F}, \mathbb{P})$, then $\theta \in L^2(M)$ and $L \in \mathcal{M}^2_0(\mathcal{F}, \mathbb{P})$.

Let $\tilde{\mathbb{Q}} \in \mathcal{Q}$ be risk-neutral from the informed agent, $\tilde{\mathbb{Q}}$ equivalent to $\mathbb{Q}$, under which P is a $(\mathcal{Y}, \tilde{\mathbb{Q}})$-martingale. As P_t is $\tilde{\mathcal{F}}$-adapted, it is a $(\tilde{\mathcal{F}}, \tilde{\mathbb{Q}})$-martingale.

$H \in \mathcal{L}^2(\Omega, \tilde{\mathcal{F}}, \tilde{\mathbb{Q}})$ is attainable with respect to the largest filtration. The martingale representation Theorem in $(\Omega, \mathcal{Y}, \mathbb{Q})$ gives the existence of $\phi_s^L \in L^2(\Omega \times [0,T], \tilde{\mathbb{Q}} \otimes d\langle P \rangle)$ such that

$$\begin{aligned} H &= E_{\tilde{\mathbb{Q}}}(H|\mathcal{Y}_0) + \int_0^T \phi_s^L dP_s \\ &= E_{\tilde{\mathbb{Q}}}(H|\sigma(L)) + \int_0^T \phi_s^L dP_s. \end{aligned} \tag{3}$$

Remark 2 *This decomposition does not depend on the choice of the risk neutral probability $\tilde{\mathbb{Q}} \in \mathcal{Q}$ from the informed agent's point of view. Indeed, according to Grorud and Pontier [19], if $\tilde{\mathbb{Q}}_1$ and $\tilde{\mathbb{Q}}_2$ are two risk neutral probabilities in $\mathcal{Q}$, then $\tilde{\mathbb{Q}}_1 = f(L)\tilde{\mathbb{Q}}_2$, where $f(L)$ is a nonnegative random variable, with mean 1 under $\tilde{\mathbb{Q}}_2$. One can then write :*

$$
\begin{aligned}
E_{\tilde{\mathbb{Q}}_1}(H|\sigma(L)) &= \frac{E_{\tilde{\mathbb{Q}}_2}(f(L)H|\sigma(L))}{E_{\tilde{\mathbb{Q}}_2}(f(L)|\sigma(L))} \\
&= \frac{f(L)E_{\tilde{\mathbb{Q}}_2}(H|\sigma(L))}{f(L)} \\
&= E_{\tilde{\mathbb{Q}}_2}(H|\sigma(L)).
\end{aligned}
$$

Hence, $E_{\tilde{\mathbb{Q}}}(H|\sigma(L))$ does not depend on the choice of $\tilde{\mathbb{Q}}$ risk neutral probability in $\mathcal{Q}$. Then, ϕ_s^L does not either depend on the considered risk-neutral probability measure in $\mathcal{Q}$.

H represents the contingent claim hedged by the informed agent, and $\phi_s^L = \pi_s^L$ the unique $\mathcal{Y}$-adapted hedging strategy of the informed agent (see Eyraud-Loisel [2]). In this continuous framework, all martingales are locally square integrable. So the unique Kunita-Watanabe decomposition of H w.r.t. the $(\tilde{\mathcal{F}}, \tilde{\mathbb{Q}})$-martingale P, for any $\tilde{\mathbb{Q}} \in \mathcal{Q}_N$, is the following:

$$
H = E_{\tilde{\mathbb{Q}}}(H) + \int_0^T \phi_s^{\tilde{\mathbb{Q}}} dP_s + L_T^{\tilde{\mathbb{Q}}}. \tag{4}
$$

Moreover,

$$
V_t := E_{\tilde{\mathbb{Q}}}(H|\tilde{\mathcal{F}}_t) = E_{\tilde{\mathbb{Q}}}(H) + \int_0^t \phi_s^{\tilde{\mathbb{Q}}} dP_s + L_t^{\tilde{\mathbb{Q}}}, \tag{5}
$$

where $\phi_s^{\tilde{\mathbb{Q}}}$ is $\tilde{\mathcal{F}}_s$-measurable and belongs to $L^2(\Omega \times [0, T], \tilde{\mathbb{Q}} \otimes d\langle P \rangle)$ and where the remainder $(L_t^{\tilde{\mathbb{Q}}})$ is a $(\tilde{\mathcal{F}}, \tilde{\mathbb{Q}})$-local martingale orthogonal to the stable space generated by P, such that $E_{\tilde{\mathbb{Q}}}(L_t^{\tilde{\mathbb{Q}}}) = 0$.
We can derive the expression of the integrand (see [9]):

$$
\phi_s^{\tilde{\mathbb{Q}}} = \frac{d < V, P >_s}{d < P >_s}.
$$

This expression is not easily exploitable in our case. By using filtering Theory, we derive another expression in terms of projection on the space of all $\tilde{\mathcal{F}}$-adapted processes under any risk neutral probability measure $\tilde{\mathbb{Q}} \in \mathcal{Q}$.

Theorem 1 *Under any equivalent martingale measure $\tilde{\mathbb{Q}} \in \mathcal{Q}$, the Kunita-Watanabe decomposition of H w.r.t. $(\tilde{\mathcal{F}}, \tilde{\mathbb{Q}})$ and process P is the following:*

$$H = E_{\tilde{\mathbb{Q}}}[H] + \int_0^T E_{\tilde{\mathbb{Q}}}\left(\phi_s^L | \tilde{\mathcal{F}}_s\right) dP_s + L_T^{\tilde{\mathbb{Q}}} \quad a.s. \ ,$$

or in other terms

$$\phi_s^{\tilde{\mathbb{Q}}} = E_{\tilde{\mathbb{Q}}}\left(\phi_s^L | \tilde{\mathcal{F}}_s\right). \tag{6}$$

Proof : If $E_{\tilde{\mathbb{Q}}}(\phi_s^L | \tilde{\mathcal{F}}_s)$ satisfies hypothesis of Kunita-Watanabe decomposition given in Definition 3, by uniqueness of this decomposition, it is $\phi_s^{\tilde{\mathbb{Q}}}$. First of all, $\left(E_{\tilde{\mathbb{Q}}}(\phi_s^L | \tilde{\mathcal{F}}_s)\right)_{0 \leq s \leq T}$ is $\tilde{\mathcal{F}}$-adapted. Define :

$$L_t = E_{\tilde{\mathbb{Q}}}(H | \tilde{\mathcal{F}}_t) - E_{\tilde{\mathbb{Q}}}(H) - \int_0^t E_{\tilde{\mathbb{Q}}}(\phi_s^L | \tilde{\mathcal{F}}_s) dP_s.$$

It is sufficient to prove that L is a $(\tilde{\mathcal{F}}, \tilde{\mathbb{Q}})$-martingale with mean 0, orthogonal to P. L is indeed a $(\tilde{\mathcal{F}}, \tilde{\mathbb{Q}})$-martingale: P is a $(\tilde{\mathcal{F}}, \tilde{\mathbb{Q}})$-martingale, so $\int_0^t E_{\tilde{\mathbb{Q}}}(\phi_s^L | \tilde{\mathcal{F}}_s) dP_s$ is a $(\tilde{\mathcal{F}}, \tilde{\mathbb{Q}})$-martingale as integral of a $\tilde{\mathcal{F}}$-adapted square integrable process against a $(\tilde{\mathcal{F}}, \tilde{\mathbb{Q}})$-martingale, and $E_{\tilde{\mathbb{Q}}}(H | \tilde{\mathcal{F}}_t)$ is a $(\tilde{\mathcal{F}}, \tilde{\mathbb{Q}})$-martingale. Moreover,

$$E_{\tilde{\mathbb{Q}}}(L_t) = E_{\tilde{\mathbb{Q}}}(H) - E_{\tilde{\mathbb{Q}}}(H) - E_{\tilde{\mathbb{Q}}}\left(\int_0^t E_{\tilde{\mathbb{Q}}}(\phi_s^L | \tilde{\mathcal{F}}_s) dP_s\right) = 0$$

the last term being the expectation of a $(\tilde{\mathcal{F}}, \tilde{\mathbb{Q}})$-martingale null at 0.
It remains to prove that L is orthogonal to the stable space generated by P, stable space generated by P, denoted by $\mathcal{M}(dP)$. Using decomposition (3) of H in $(\mathcal{Y}, \tilde{\mathbb{Q}})$, we obtain:

$$\begin{aligned} L_t &= E_{\tilde{\mathbb{Q}}}\left(E_{\tilde{\mathbb{Q}}}(H | L) + \int_0^T \phi_s^L dP_s \,\middle|\, \tilde{\mathcal{F}}_t\right) - E_{\tilde{\mathbb{Q}}}(H) - \int_0^t E_{\tilde{\mathbb{Q}}}(\phi_s^L | \tilde{\mathcal{F}}_s) dP_s \\ &= E_{\tilde{\mathbb{Q}}}\left[\int_0^T \phi_s^L dP_s | \tilde{\mathcal{F}}_t\right] - \int_0^t E_{\tilde{\mathbb{Q}}}\left[\phi_s^L | \tilde{\mathcal{F}}_s\right] dP_s \\ &\quad + \underbrace{E_{\tilde{\mathbb{Q}}}\left(E_{\tilde{\mathbb{Q}}}(H | \sigma(L)) | \tilde{\mathcal{F}}_t\right) - E_{\tilde{\mathbb{Q}}}(H)}_{N_t}, \end{aligned} \tag{7}$$

where the last term N_t is $\tilde{\mathcal{F}}_t$-measurable.
The following lemma is a classical result of Filtering Theory (see for example the Saint Flour summer school in Probability Theory course by Pardoux in 1989 on non linear filtering theory [20]).

Lemma 1 *(Pardoux (1989) [20])*
Let M be a $(\mathcal{F}, \tilde{\mathbb{Q}})$-martingale, $\theta \in L_{loc}^2(M)$ on the filtered probability space $(\Omega, \mathcal{Y}, \tilde{\mathbb{Q}})$, such that $\mathcal{F}$ satisfied $\mathcal{F}_t \subset \mathcal{Y}_t \; \forall t \in [0, T]$. Then we have:

$$E_{\tilde{\mathbb{Q}}} \left(\int_0^t \theta_s dM_s \,\middle|\, \mathcal{F}_t \right) = \int_0^t E_{\tilde{\mathbb{Q}}} \left(\theta_s | \mathcal{F}_s \right) dM_s.$$

As a consequence of the lemma, we can write

$$E_{\tilde{\mathbb{Q}}} \left(\int_0^t \phi_s^L dP_s | \tilde{\mathcal{F}}_t \right) = \int_0^t E_{\tilde{\mathbb{Q}}}(\phi_s^L | \tilde{\mathcal{F}}_s) dP_s$$

Using the law of iterated expectations, together with the previous lemma, we can simplify the expression of L_t in Equation (7):

$$
\begin{aligned}
L_t &= E_{\tilde{\mathbb{Q}}} \left[\int_0^T \phi_s^L dP_s | \tilde{\mathcal{F}}_t \right] - \int_0^t E_{\tilde{\mathbb{Q}}}(\phi_s^L | \tilde{\mathcal{F}}_s) dP_s + N_t \\
&= E_{\tilde{\mathbb{Q}}} \left[E_{\tilde{\mathbb{Q}}} \left[\int_0^T \phi_s^L dP_s \,\middle|\, \tilde{\mathcal{F}}_T \right] \middle| \tilde{\mathcal{F}}_t \right] - \int_0^t E_{\tilde{\mathbb{Q}}}(\phi_s^L | \tilde{\mathcal{F}}_s) dP_s + N_t \\
&= E_{\tilde{\mathbb{Q}}} \left[\int_0^T E_{\tilde{\mathbb{Q}}} \left(\phi_s^L \middle| \tilde{\mathcal{F}}_s \right) dP_s \,\middle|\, \tilde{\mathcal{F}}_t \right] - \int_0^t E_{\tilde{\mathbb{Q}}}(\phi_s^L | \tilde{\mathcal{F}}_s) dP_s + N_t \\
&= \int_0^t E_{\tilde{\mathbb{Q}}} \left(\phi_s^L \middle| \tilde{\mathcal{F}}_s \right) dP_s - \int_0^t E_{\tilde{\mathbb{Q}}} \left(\phi_s^L \middle| \tilde{\mathcal{F}}_s \right) dP_s + N_t \\
&= N_t. \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad (8)
\end{aligned}
$$

The third equality is obtained because $\int_0^{\cdot} E_{\tilde{\mathbb{Q}}} \left(\phi_s^L \middle| \tilde{\mathcal{F}}_s \right) dP_s$ is a $(\tilde{\mathcal{F}}_t, \tilde{\mathbb{Q}})$-martingale.

Let (θ_s) be a $\tilde{\mathcal{F}}$-adapted bounded process. Equation (8) leads to:

$$E_{\tilde{\mathbb{Q}}} \left(L_t \int_0^t \theta_s dP_s \right) = E_{\tilde{\mathbb{Q}}} \left(N_t \int_0^t \theta_s dP_s \right).$$

Recall that N_t may be written as:

$$N_t = E_{\tilde{\mathbb{Q}}} \left(\underbrace{E_{\tilde{\mathbb{Q}}}(H | \sigma(L))}_{f(L)} \,\middle|\, \tilde{\mathcal{F}}_t \right) - E_{\tilde{\mathbb{Q}}}(H).$$

As $E_{\tilde{\mathbb{Q}}}(H | \sigma(L))$ is measurable with respect to the σ-algebra generated by L, it may be written as $f(L)$, where f is a measurable function. Hence :

$$E_{\tilde{\mathbb{Q}}} \left[N_t \int_0^t \theta_s dP_s \right] = E_{\tilde{\mathbb{Q}}} \left[\left[E_{\tilde{\mathbb{Q}}} \left(f(L) | \tilde{\mathcal{F}}_t \right) - E_{\tilde{\mathbb{Q}}}(H) \right] \int_0^t \theta_s dP_s \right]$$

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$$= E_{\tilde{\mathbb{Q}}} \left(E_{\tilde{\mathbb{Q}}} \left(f(L) \int_0^t \theta_s dP_s \,\middle|\, \tilde{\mathcal{F}}_t \right) \right) - E_{\tilde{\mathbb{Q}}}(H) E_{\tilde{\mathbb{Q}}} \left(\int_0^t \theta_s dP_s \right)$$

$$= E_{\tilde{\mathbb{Q}}} \left(f(L) \int_0^t \theta_s dP_s \right).$$

Literature on initial enlargement of filtration (see Grorud and Pontier [19]) has already established that if there are two risk neutral probability measures $\mathbb{Q}_1$ and $\mathbb{Q}_2$, then there exists z measurable such that $\mathbb{Q}_2 = z(L)\mathbb{Q}_1$, where $z(L)$ is a $\mathcal{Y}_0$-measurable nonnegative random variable, with expectation 1 under $\mathbb{Q}_1$, and conversely any such probability measure is risk neutral. For this reason, the previous computations do not depend on the choice of the risk neutral probability measure $\tilde{\mathbb{Q}} \in \mathcal{Q}$. Then let $\mathbb{Q}^*$ be the probability measure defined as followed:

$$\mathbb{Q}^* = \frac{f(L)}{E_{\tilde{\mathbb{Q}}}(f(L))} \tilde{\mathbb{Q}} \in \mathcal{Q}.$$

As $\mathbb{Q}^*$ is a risk neutral probability measure, from the previous argument, $\int_0^t \theta_s dP_s$ is a $\tilde{\mathcal{F}}$-martingale under $\mathbb{Q}^*$. We get

$$0 = E_{\mathbb{Q}^*} \left(\int_0^t \theta_s dP_s \right) = \frac{1}{E_{\tilde{\mathbb{Q}}}(f(L))} E_{\tilde{\mathbb{Q}}} \left(f(L) \int_0^t \theta_s dP_s \right).$$

We deduce

$$E_{\tilde{\mathbb{Q}}} \left(f(L) \int_0^t \theta_s dP_s \right) = 0$$

and we have finally

$$E_{\tilde{\mathbb{Q}}} \left[N_t \int_0^t \theta_s dP_s \right] = 0, \forall \theta \; \tilde{\mathcal{F}} - \text{measurable}.$$

Under any equivalent martingale measure $\tilde{\mathbb{Q}} \in \mathcal{Q}$, we have the expected orthogonality. This proves that the Kunita-Watanabe decomposition of H has integrand $E_{\tilde{\mathbb{Q}}} \left(\phi_s^L | \tilde{\mathcal{F}}_s \right)$ under any risk-neutral probability measure $\tilde{\mathbb{Q}}$ in $\mathcal{Q}$, which ends the proof of Theorem 1. $\qquad \square$

Remark 3 *As this integrand determines the unique Kunita-Watanabe decomposition, and as it is the same under any risk-neutral probability measure of $\mathcal{Q}$, then the orthogonal rest is the same and the quadratic residual risk is the same : it may be interpreted as the intrinsic risk of the contingent claim H.*

Remark 4 *This also proves that $E_{\tilde{\mathbb{Q}}}(\phi_s^L|\tilde{\mathcal{F}}_s)$ does not depend on the choice of $\tilde{\mathbb{Q}} \in \mathcal{Q}$. This can also be proved by the following way :*
Let $Y_s \in \tilde{\mathcal{F}}_s, Q_2 = z(L)Q_1$, from the previous argument, it follows that

$$E_{Q_1}[Y_s E_{Q_2}[\phi_s^L/\tilde{\mathcal{F}}_s] E_{Q_1}[z(L)/\tilde{\mathcal{F}}_s]] = E_{Q_1}[Y_s \phi^L z(L)].$$

The main consequence of the previous Theorem is then:

Proposition 1 *Prices are the same under two different risk-neutral probability measures.*

Remark 5 *We obtain here a result which is coherent with the result of Grorud and Pontier [19] in a different framework, according to which all risk-neutral probability measures in an incomplete market (where incompleteness is due to additional information) give the same market price.*

Remark 6 *Our study is also coherent with Föllmer and Schweizer's work in [9], but several points differ: their hypothesis of invariance of the decomposition of Doob-Meyer of the prices does not remain valid in our case. They suppose that the decomposition of Doob-Meyer of the prices under filtration $\tilde{\mathcal{F}}$ and probability $\mathbb{Q}$, $P = P_0 + M + A$ where A is $\tilde{\mathcal{F}}$-predictable, and where M is a $(\tilde{\mathcal{F}}, \mathbb{Q})$-martingale, stays the Doob-Meyer Decomposition of prices in the larger filtration, in other words, M remain a $(\mathcal{Y}, \mathbb{Q})$-martingale, although adapted to $\tilde{\mathcal{F}}$ (See Hypotheses (4.1) to (4.3) in [9]). These hypotheses, although weaker, are similar to the so-called Hypothesis ($\mathbf{H}$) in Filtering Theory (used also in progressive enlargement of filtration, in models as those of Jeanblanc [21, 22]). Instead of such hypothesis, our study takes place under Hypothesis ($\mathbf{H_3}$). Furthermore, we distinguish here three different filtrations: the filtration of the Brownian motion, the filtration of prices, and the filtration enlarged with the information. On the other hand, as pointed out before, our study joins perfectly within the framework of incompleteness that they studied: the incompleteness due to a lack of information.*

Finally, our results are coherent with those in previous literature, whereas the model is slightly different. The present study illustrates in a different framework the same kind of incompleteness already studied in this literature: incompleteness due to a lack of information.

3.3 Clark-Ocone Formula

Since $\phi_s^{\tilde{\mathbb{Q}}}$ may be expressed from ϕ_s^L, it is important to study how to obtain ϕ_s^L. We express it as a function of the Malliavin derivative of H, via the use of a Clark-Ocone Formula. We can not use directly the standard formula in our case, under enlarged filtration $\mathcal{Y}$, as $\mathcal{Y}_0 = \sigma(L)$ is not trivial. Nevertheless, thanks to a transformation and the independence of $\sigma(L)$ and $\mathcal{F}$ under $\mathbb{Q}$, we will use the existing Clark-Ocone Formula under filtration $\mathcal{F}$, and obtain an expression of ϕ_s^L.

Theorem 2 *If $H \in L^2(\Omega, \mathcal{Y}, Q)$ and $\forall x, H(., x) \in \mathbb{D}^{1,2}$, we have*

$$\phi_s^L = \left(E_{\mathbb{Q}} \left[D_s(H(., x)) | \mathcal{F}_s \right] |_{x=L} \right) (\sigma_s)^{-1}. \tag{9}$$

Proof In the enlarged space $\mathcal{Y}$, under $(\mathbf{H_3})$, $\sigma(L)$ and $\mathcal{F}_t$ are independent. Moreover, $\mathbb{Q}|_{\mathcal{F}} = \mathbb{P}$ and $\mathbb{Q}|_{\sigma(L)} = \mathbb{P}^L$ (law of L).

H is measurable w.r.t. $\mathcal{Y}_T$, so H may be written as $H = H(W, L)$ (thanks to the independence and because $\mathcal{Y}_T$ is generated by $\mathcal{F}$ and $\sigma(L)$). Then, at fixed $L = x$, $H(W, x)$ is $\mathcal{F}_T$-measurable. So its representation is given by the standard Clark-Ocone Formula:

$$
\begin{aligned}
H(W, x) &= E_{\mathbb{P}} \left(H(W, x) \right) + \int_0^T E_{\mathbb{P}} \left(D_s(H(W, x)) | \mathcal{F}_s \right) dW_s \quad \mathbb{P} \quad a.s. \\
&= E_{\mathbb{Q}} \left(H | L \right) |_{L=x} + \int_0^T E_{\mathbb{Q}} \left(D_s(H(W, x)) | \mathcal{F}_s \right) dW_s.
\end{aligned}
$$

Indeed, derive H w.r.t. W with $L = x$ has a sense under $\mathbb{Q}$ as W and L are independent, and derive $W \mapsto H(W, L)$ under $\mathbb{Q}$ or $\mathbb{P}$ is the same, as W is a $\mathbb{P}$-Brownian motion and also a $\mathbb{Q}$-Brownian motion. Again using independence between $\sigma(L)$ and $\mathcal{F}_t$, an identification leads to:

$$\sigma_s \phi_s^L = E_{\mathbb{Q}} \left[D_s(H(W, x)) | \mathcal{F}_s \right] |_{x=L}. \tag{10}$$

Equation (9) directly comes from Equation (10) and the fact that σ_s^L is invertible, which ends the proof of Theorem 2. $\qquad\square$

4 Residual risk and measure of the lack of information

4.1 Residual risk under a risk neutral probability in $\mathcal{Q}_N$

We derivean expression of the intrinsic residual risk of the contingent claim H under a risk neutral probability in $\mathcal{Q}_N$.

Proposition 2 *Let $\mathbb{Q}^* \in \mathcal{Q}_N$.*
The residual risk $L_t^{\mathbb{Q}^*}$ has the following expression:

$$L_t^{\mathbb{Q}^*} = \int_0^t \left[E_{\mathbb{Q}^*}\left(\phi_s^L | \tilde{\mathcal{F}}_s \right) - \phi_s^{\mathbb{Q}^*} \right] dP_s + E_{\mathbb{Q}^*}\left[E_{\mathbb{Q}^*}(H|\sigma(L))|\tilde{\mathcal{F}}_t \right] - E_{\mathbb{Q}^*}(H).$$
$$(11)$$

A measure of the risk of detaining the contingent claim H in this incomplete market is given by the variance of $L_T^{\mathbb{Q}^*}$ at terminal time:

$$\begin{aligned}
Var_{\mathbb{Q}^*}(L_T^{\mathbb{Q}^*}) &= E_{\mathbb{Q}^*}\left(\int_0^T \left[E_{\mathbb{Q}^*}\left(\phi_s^L | \tilde{\mathcal{F}}_s \right) - \phi_s^{\mathbb{Q}^*} \right] dP_s \times \left(E_{\mathbb{Q}^*}\left[E_{\mathbb{Q}^*}(H|\sigma(L))|\tilde{\mathcal{F}}_T \right] - E_{\mathbb{Q}^*}(H) \right) \right) \\
&\quad + E_{\mathbb{Q}^*}\left(\left(E_{\mathbb{Q}^*}\left[E_{\mathbb{Q}^*}(H|\sigma(L))| \tilde{\mathcal{F}}_T \right] - E_{\mathbb{Q}^*}(H) \right)^2 \right).
\end{aligned}$$
$$(12)$$

Proof Let $\mathbb{Q}^* \in \mathcal{Q}_N$ a risk neutral probability measure for the non informed agent. The expression of the residual risk $L_t^{\mathbb{Q}^*}$ at time t is given by Equations (4) and (5), which are true under any $\mathbb{Q}^* \in \mathcal{Q}_N$:

$$L_t^{\mathbb{Q}^*} = E_{\mathbb{Q}^*}\left[\int_0^T \phi_s^L dP_s | \tilde{\mathcal{F}}_t \right] - \int_0^t \phi_s^{\mathbb{Q}^*} dP_s + E_{\mathbb{Q}^*}\left[E_{\mathbb{Q}^*}(H|\sigma(L))|\tilde{\mathcal{F}}_t \right] - E_{\mathbb{Q}^*}(H).$$

The second part of this expression corresponds to the difference of prices of the contingent claim under $\mathbb{Q}^*$, whether the agent has or does not have the additional information, which is the difference of prices on the informed (complete) market and the non informed (incomplete) market. In the first case, the price will be $E_{\mathbb{Q}^*}(H|\sigma(L))$ projected on the $\tilde{\mathcal{F}}_t$-measurable random variables (which means attainable prices), and in the other case, the price is $E_{\mathbb{Q}^*}(H)$.
The first part may be rewritten thanks to Lemma 1:

$$\begin{aligned}
E_{\mathbb{Q}^*}\left[\int_0^T \phi_s^L dP_s | \tilde{\mathcal{F}}_t \right] &= E_{\mathbb{Q}^*}\left[E_{\mathbb{Q}^*}\left(\int_0^T \phi_s^L dP_s | \tilde{\mathcal{F}}_T \right) | \tilde{\mathcal{F}}_t \right] \\
&= E_{\mathbb{Q}^*}\left[\int_0^T E_{\mathbb{Q}^*}\left(\phi_s^L | \tilde{\mathcal{F}}_s \right) dP_s | \tilde{\mathcal{F}}_t \right] \\
&= \int_0^t E_{\mathbb{Q}^*}\left(\phi_s^L | \tilde{\mathcal{F}}_s \right) dP_s,
\end{aligned}$$

as P is a $(\tilde{\mathcal{F}}, \mathbb{Q}^*)$-martingale.
The residual risk may be rewritten as:

$$L_t^{\mathbb{Q}^*} = \int_0^t \left[E_{\mathbb{Q}^*}\left(\phi_s^L | \tilde{\mathcal{F}}_s \right) - \phi_s^{\mathbb{Q}^*} \right] dP_s + E_{\mathbb{Q}^*}\left[E_{\mathbb{Q}^*}(H|\sigma(L))|\tilde{\mathcal{F}}_t \right] - E_{\mathbb{Q}^*}(H).$$

The sum of these two terms forms the residual risk due to the lack of information of the non informed agent. This risk is the minimal risk under $\mathbb{Q}^*$ which the non informed agent can hope to take by holding the option (see Föllmer and Schweizer [9] and Pham [13]). It is the constituent of the option which is orthogonal to the market prices (see Kunita-Watanabe decomposition).
A measure of this risk, represented by $L_t^{\mathbb{Q}^*}$, martingale with null expectation, is its variance $Var_{\mathbb{Q}^*}(L_T^{\mathbb{Q}^*})$, which represents the quadratic residual risk at terminal time T.
At terminal time, we can write:

$$L_T^{\mathbb{Q}^*} = \int_0^T \left[E_{\mathbb{Q}^*}\left(\phi_s^L | \tilde{\mathcal{F}}_s \right) - \phi_s^{\mathbb{Q}^*} \right] dP_s + E_{\mathbb{Q}^*}\left[E_{\mathbb{Q}^*}(H|\sigma(L))|\tilde{\mathcal{F}}_T \right] - E_{\mathbb{Q}^*}(H).$$

So

$$
\begin{aligned}
Var_{\mathbb{Q}^*}(L_T^{\mathbb{Q}^*}) &= E_{\mathbb{Q}^*}\left(\left(\int_0^T \left[E_{\mathbb{Q}^*}\left(\phi_s^L | \tilde{\mathcal{F}}_s \right) - \phi_s^{\mathbb{Q}^*} \right] dP_s \right)^2 \right) \\
&\quad + 2E_{\mathbb{Q}^*}\left(\int_0^T \left[E_{\mathbb{Q}^*}\left(\phi_s^L | \tilde{\mathcal{F}}_s \right) - \phi_s^{\mathbb{Q}^*} \right] dP_s \times \left(E_{\mathbb{Q}^*}\left[E_{\mathbb{Q}^*}(H|\sigma(L))|\tilde{\mathcal{F}}_T \right] - E_{\mathbb{Q}^*}(H) \right) \right) \\
&\quad + E_{\mathbb{Q}^*}\left(\left(E_{\mathbb{Q}^*}\left[E_{\mathbb{Q}^*}(H|\sigma(L))|\tilde{\mathcal{F}}_T \right] - E_{\mathbb{Q}^*}(H) \right)^2 \right).
\end{aligned}
$$

As $L_T^{\mathbb{Q}^*}$ is orthogonal to P, this expression may be slightly simplified, and the variance may be rewritten as:

$$
\begin{aligned}
Var_{\mathbb{Q}^*}(L_T^{\mathbb{Q}^*}) &= E_{\mathbb{Q}^*}\left(\int_0^T \left[E_{\mathbb{Q}^*}\left(\phi_s^L | \tilde{\mathcal{F}}_s \right) - \phi_s^{\mathbb{Q}^*} \right] dP_s \times \left(E_{\mathbb{Q}^*}\left[E_{\mathbb{Q}^*}(H|\sigma(L))|\tilde{\mathcal{F}}_T \right] - E_{\mathbb{Q}^*}(H) \right) \right) \\
&\quad + E_{\mathbb{Q}^*}\left(\left(E_{\mathbb{Q}^*}\left[E_{\mathbb{Q}^*}(H|\sigma(L))|\tilde{\mathcal{F}}_T \right] - E_{\mathbb{Q}^*}(H) \right)^2 \right).
\end{aligned}
$$

This is the general expression of the quadratic residual risk under a probability measure in $\mathcal{Q}_N$. $\qquad\square$
In the case of a measure in $\mathcal{Q}_N$ but not in $\mathcal{Q}$, we can not simplify better the previous expression. Nevertheless, with orthogonality arguments we obtain a simpler expression.

Proposition 3 *A measure of the residual risk under a risk neutral probability measure $\mathbb{Q}^* \in \mathcal{Q}_N \backslash \mathcal{Q}$ is given by:*

$$Var_{\mathbb{Q}^*}\left(L_T^{\mathbb{Q}^*} \right) = E_{\mathbb{Q}^*}\left((H - E_{\mathbb{Q}^*}(H))^2 \right) - E_{\mathbb{Q}^*}\left(\left(\int_0^T \phi_s^{\mathbb{Q}^*} dP_s \right)^2 \right). \quad (13)$$

Proof Let $\mathbb{Q}^* \in \mathcal{Q}_N$ be a probability measure that does not belong to $\mathcal{Q}$. Then the Kunita-Watanabe decomposition may be written as follows:

$$H - E_{\mathbb{Q}^*}(H) = \int_0^T \phi_s^{\mathbb{Q}^*} dP_s + L_T^{\mathbb{Q}^*},$$

P being a $(\tilde{\mathcal{F}}, \mathbb{Q}^*)$-martingale, but not a $(\mathcal{Y}, \mathbb{Q}^*)$-martingale (otherwise $\mathbb{Q}^*$ would belong to $\mathcal{Q}$). There is no martingale representation Theorem.

As $L_T^{\mathbb{Q}^*}$ is orthogonal to the price process, we get Equation (13) and the expected expression of the quadratic residual risk. $\square$

4.2 Residual risk under a risk neutral probability in $\mathcal{Q}$

In the case of a measure in $\mathcal{Q} \subset \mathcal{Q}_N$, Equations (11) and (13) of Proposition 2 may be simplified. We obtain the following result:

Proposition 4 *A measure of the residual risk under a risk neutral probability measure $\mathbb{Q}^*$ in $\mathcal{Q}$ is:*

$$Var(L_T^{\mathbb{Q}^*}) = E_{\mathbb{Q}^*}\left(E_{\mathbb{Q}^*}\left[E_{\mathbb{Q}^*}(H|\sigma(L))|\tilde{\mathcal{F}}_T \right]^2 \right) - E_{\mathbb{Q}^*}(H)^2, \qquad (14)$$

Proof In the case of a risk neutral probability $\mathbb{Q}^*$ in $\mathcal{Q}$, the first term of $L_T^{\mathbb{Q}^*}$ in Equation (11) nullifies, in application of Proposition 1. The expression becomes simpler, and we deduce Equation (14), which gives a measure of the residual risk under a risk neutral probability $\mathbb{Q}^* \in \mathcal{Q}$. $\square$

Remark 7 *$Var(L_T^{\mathbb{Q}^*})$ measures the revelation of the information in the prices, because the not informed agent does not possess the whole information, but she "sees" all the same a part of the information showing through in market prices.*

A minimal risk exists, which is taken over all possible equivalent martingale measures in $\mathcal{Q}$ and it is the following:

$$\inf_{Q^* \in \mathcal{Q}} \left\{ E_{\mathbb{Q}^*}\left(E_{\mathbb{Q}^*}\left[E_{\mathbb{Q}^*}(H|\sigma(L))|\tilde{\mathcal{F}}_T \right]^2 \right) - E_{\mathbb{Q}^*}(H)^2 \right\}.$$

It can be approached by a minimizing sequence.

Remark 8 *We can notice that measuring the residual risk under a risk-neutral probability is arbitrary, because choosing this equivalent martingale measure is already an arbitrary choice, and also because at first, the model took into account a historic probability, which we do not take into account in this measure of risk. Moreover the estimated risk is the risk of this model under this measure, and not an intrinsic risk.*

4.3   Existence of a minimal martingale measure

Let us write the Doob-Meyer decomposition of the $(\tilde{\mathcal{F}}, \mathbb{Q})$-semi-martingale of prices P :

$$P_t = P_0 + M_t + A_t,$$

where M is a $(\tilde{\mathcal{F}}, \mathbb{Q})$-martingale of null expectation (as a uniformly integrable local martingale, from the no-arbitrage conditions, see Lépingle and Mémin [23] Theorems II-2 and III-7), and A is a $\tilde{\mathcal{F}}$-predictable finite-variations process, whose increments are square integrable under $\mathbb{Q}$ and satisfy $A_0 = 0$.
By Girsanov Theorem, and as the existence of an equivalent martingale measure is ensured, A may be written as follows:

$$A_t = \int_0^t \lambda_u d < M >_u \, , \ 0 \le t \le T,$$

where λ is a $\tilde{\mathcal{F}}$-predictable process.
Define

$$K_t = \int_0^t \lambda'_u d < M >_u \lambda_u \, , \ 0 \le t \le T$$

and denote by $\hat{Z}$ the process solution of the following SDE:

$$d\hat{Z}_t = -\hat{Z}_t \lambda_t dM_t \, , \ 0, \le t \le T, \ \ \hat{Z}_0 = 1.$$

Then:

$$\hat{Z}_t = \exp\left(-\int_0^t \lambda_u dM_u - \frac{1}{2} K_t \right), \ 0 \le t \le T. \tag{15}$$

$\hat{Z}$ is a positive uniformly integrable $(\tilde{\mathcal{F}}, \mathbb{Q})$-local martingale, so it is a $(\tilde{\mathcal{F}}, \mathbb{Q})$-martingale (as soon as some Novikov-type hypotheses on λ are satisfied). This process defines a probability measure $\hat{\mathbb{Q}}$ equivalent to $\mathbb{Q}$ by

$$\frac{d\hat{\mathbb{Q}}}{d\mathbb{Q}} = \hat{Z}_T \in L^2(\mathbb{Q}). \tag{16}$$

As P is a continuous process and $\hat{Z} \in \mathcal{M}^2(\mathbb{Q})$, Proposition 4.3 of Pham [13] may apply, and we establish the following result in our case:

Proposition 5 *Probability measure $\hat{\mathbb{Q}}$ is an equivalent martingale measure, called* minimal martingale measure. *It satisfies the following property: any square integrable $(\tilde{\mathcal{F}}, \mathbb{Q})$-martingale orthogonal to M under $\mathbb{Q}$ remains a $(\tilde{\mathcal{F}}, \hat{\mathbb{Q}})$-martingale:*

$$L \in \mathcal{M}^2(\tilde{\mathcal{F}}, \mathbb{Q}), L \perp_{\mathbb{Q}} M \Rightarrow L \text{ is a martingale under } (\tilde{\mathcal{F}}, \hat{\mathbb{Q}}).$$

Also following Pham [13] (Paragraph 4.3 Theorem 4.2), we can mention the following result, which gives an expression of the strategy minimizing the local risk, in the case where K is uniformly bounded:

Proposition 6 *If K is uniformly bounded, there exists a strategy minimizing the local risk, given by:*

$$
\begin{aligned}
V_t^* &= E_{\hat{\mathbb{Q}}}\left(H|\tilde{\mathcal{F}}_t\right) \\
\phi_t^* &= \phi_t^H,
\end{aligned}
$$

where ϕ_t^H is the integrand of the Föllmer-Schweizer decomposition of H under $\mathbb{Q}$, which is:

$$
H = H_0 + \int_0^T \phi_t^H dP_t + L_t^H
$$
$$
H_0 \in \mathbb{R}, \ \phi^H \in L^2(M) \cap L^2(A), \ L^H \in \mathcal{M}(\tilde{\mathcal{F}}, \mathbb{Q}) \perp M.
$$

Proof It is the Kunita-Watanabe decomposition of the $(\tilde{\mathcal{F}}, \hat{\mathbb{Q}})$-martingale V^* against the $(\tilde{\mathcal{F}}, \hat{\mathbb{Q}})$-martingale P. Indeed, L^H is a $(\tilde{\mathcal{F}}, \mathbb{Q})$-martingale orthogonal to M, and so remains a martingale under $\hat{\mathbb{Q}}$. $\qquad\square$

Remark 9 *By uniqueness of the Kunita-Watanabe decomposition under $\hat{\mathbb{Q}}$, we have $H_0 = E_{\hat{\mathbb{Q}}}(H)$, which is the right term of the Föllmer-Schweizer decomposition.*

5   Example

Let us give an example satisfying all hypotheses of the previous model. This example was first introduced in the last section of [2]. Suppose that the price process is driven by the following dynamics (stochastic volatility model) :

$$
dP_t = b'(P_t, X_t, \pi_t)P_t dt + \sigma_t'(\eta)P_t dW_t, \tag{17}
$$

where

$$
\sigma_t'(\eta) = \sigma^0 \mathcal{I}_{[0,\eta[}(t) + \sigma^1 \mathcal{I}_{[\eta,T]}(t) \ , \ \sigma^0, \sigma^1 \neq 0. \tag{18}
$$

The volatility of this model is piecewise constant, taking two possible values σ^0 and σ^1 fixed by the model, η is a random variable satisfying Hypothesis ($\mathbf{H_3}$), taking its values in $[0, T+\varepsilon]$.

The information of the insider trader is the following strong initial information: $L = \eta$, $\mathcal{F}_{T+\varepsilon}$ − measurable . The drift parameter is chosen as the following:

$$
b'(X_t, P_t, \pi_t) = b_0 + \frac{b_1}{(1+P_t)(1+\pi_t^2)} \ , \ b_0, b_1 \in \mathbb{R} \text{ fixed.}
$$

Interest rate r is supposed to be constant.

Drift b' is bounded, and may vary between two thresholds b_0 and $b_0 + a$. Two cases may appear, depending on the sign of b_1. If $b_1 < 0$, the influence is a positive influence: the bigger is the investment portfolio, the higher is the drift of the prices. This is moderated by the level of prices: the higher are the prices, the lower is the influence. If $b_1 > 0$, it is the converse principle: when the level of the portfolio increases, the drift of the prices decreases, and the influence is stronger when the level of prices is high. Remark that the case $b_1 = 0$ is the case treated in the previous section, without influence. Depending on the sign of b_1, representing the amplitude of the influence, this influence will have either a leverage effect or a return effect on the drift of the price process around the value b_0. The influence is from the insider's portfolio on the price process, which remains bounded according to the hypotheses. We can also notice that

$$\sigma_s'^{-1}(b_s' - r_s) = \sigma_s'^{-1}(b_0 - r + \frac{b_1}{(1 + P_t)(1 + \pi_t^2)})$$

is bounded, as well as σ'. So there exists a risk-neutral probability measure $\tilde{\mathbb{Q}}$ under which $dP_t = \sigma_t' P_t d\tilde{W}_t$ is a positive uniformly integrable martingale.

Remark 10 *We don't have here constraints on the signs of b_0 or b_1, whereas it is often the case in previous influence models developed in the literature, such as in the model introduced by Cuoco and Cvitanic (1998) [4], and treated deeply in Grorud and Pontier (2005) [3] (their influence form is slightly different from the one treated in this work). This may be explained by the fact that we consider a hedging problem, whereas they considered an optimization problem, and therefore we do not need the convexity of the parameters here.*

For the present model, considering the hedging of a European call option with maturity T and strike K, the parameters are the following :

$$
\begin{aligned}
f(s, P_s, X_s, \pi_s) &= X_s r + \left(b_0 - r + \frac{b_1}{(1 + P_t)(1 + \pi_t^2)} \right) \pi_s \ , \\
g(P_T) &= (P_T - K)_+ \ .
\end{aligned}
$$

This leads to the following FBSDE modeling the hedging problem of the informed agent :

$$
\begin{cases}
P_t = P_0 + \int_0^t \left(b_0 + \frac{b_1}{(1 + P_t)(1 + \pi_t^2)} \right) P_s ds + \int_0^t \sigma_s(\eta) P_s dW_s \\
X_t = (P_T - K)_+ - \int_t^T (X_s r + (b_0 + a h(\pi_s)) \pi_s - r\pi_s) ds - \int_t^T \sigma_s(\eta) \pi_s dW_s.
\end{cases}
$$

According to [2], hypotheses of Theorem **??** are satisfied, and the influential informed agent has a unique hedging strategy in this market. As explained before, the informed market is complete, whereas the non informed market is incomplete.

From Subsection 3.3, we derive an expression of the strategy of the informed agent:

$$\sigma_s^L \phi_s^L = E_{\mathbb{Q}}[D_s(H(W,x))|\mathcal{F}_s]|_{x=L}.$$

In the case of a European call option $H(W,x) = (P_T^x - K)_+$, where P_T^x is the price at terminal time when the volatility jump appeared at time x,

$$\sigma_s^L \phi_s^L = E_{\mathbb{Q}}[\mathbb{1}_{P_T^x > K} D_s P_T^x | \mathcal{F}_s]_{x=L}.$$

We obtain an expression of the optimal hedging strategy of a non informed agent in the market, thanks to results of Section 3 (Equation (9)):

$$\phi_s^{\mathbb{Q}^*} = E_{\mathbb{Q}^*}\left[\left(\sigma_s^L\right)^{-1} E_{\mathbb{Q}}[\mathbb{1}_{P_T^x > K} D_s P_T^x | \mathcal{F}_s]_{x=L} \,\big|\, \tilde{\mathcal{F}}_s\right].$$

We derive also the expression of the quadratic residual risk from Equation (14):

$$Var(L_T^{\mathbb{Q}^*}) = E_{\mathbb{Q}^*}\left(E_{\mathbb{Q}^*}\left[E_{\mathbb{Q}^*}((P_T^L - K)_+|L)\big| \tilde{\mathcal{F}}_T\right]^2\right) - E_{\mathbb{Q}^*}((P_T^L - K)_+)^2.$$

Remark 11 *We have in this example*

$$\sigma_t(L)^2 = \frac{1}{P_t^2}\frac{d<P>_t}{dt}.$$

So $\sigma_t(L)^2$ is $\tilde{\mathcal{F}}$-measurable, hence observable.

6   Conclusion

As a consequence of the last remark, imagine a weak informed agent, who only knows the law of L (such information is commonly called weak information). Then, as soon as the agent observes the jump in the volatility process, she has the same information as the strong informed agent. This would mean intuitively that in this example, the weak information would be enough to complete the market, because it would be sufficient to hedge against this contingent claim. A possible continuation of this model would be the study of BSDE or FBSDE with other types of additional information

(weak, progressive), and thus under other types of enlargement of filtration. Other techniques were proposed to study the presence of asymmetry of information, as Malliavin calculus (Imkeller [24], Nualart et al. [25]), either models of weak information (Baudoin [26]) or models of punters (Corcuera, Imkeller, Kohatsu-Higa and Nualart [27]).

Several studies have been developed on discretization schemes for BSDE and FBSDE, from Chevance (1997) [28], to Delarue (2002) [29], Gobet, Lemor and Warin (2005) [30], or more recently Bouchard and Elie (2008) [31], or Bouchard, Elie and Touzi (2009) [32]. Even if these new schemes may open a way to simulate and use more extensively such study, and even if these new works and their future extensions give interesting tracks for efficient schemes for FBSDEs, there is still a difficulty that leaves in the use of such equations. The main difficulty is to express explicitly the solutions, especially when the filtration is not easy to express as an enlarged filtration of the Brownian filtration. Since then, the risk minimization and the quadratic hedging approaches allows to have all the same an expression of the hedging strategy, and of the minimal risk a non informed trader is taking by detaining the contingent claim.

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