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A Mathematical model Trend of Pension Funds' Dynamic Asset Allocation

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This presentation expresses my individual opinion and does not show my belonging company's views.

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--> Strategy for the Risky Asset's weight

From Early days to Date	Spin effects

From Early days to Date

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- ✓ Pricing Theory
- ✓ Merton's Multi-Period Model by Utility
- ✓ → CRRA Utility leads Myopic solution
- ✓ PDE
  - Bellman Principle, HJB
  - Cox-Haung
- ✓ BSDE

Spin effects

- ✓ Option like strategy
  ←→ PDE Viscosity solution
- ✓ Flexibility
  ←→ Stopping time, Reflected BSDE
- ✓ Solution's sensibility
  ←→ Malliavin calculus

- ✓ Numerical Simulations

I. Introduction (Continued)

- ❑ A bit more about Time Diversification
  - ✓ Empirically seems to be RIGHT.
  - ✓ However, a Geometric Brownian Motion with a CRRA Utility model shows longer time investment needs a less risky asset.

The below are thought to be irrelevant for pension fund investments which surely DO NOT END their investments for a fixed determined time period.

Longer term and Larger potential magnitude of losses.

Longer term and Larger option costs.

Longer term and Larger possibility of within period losses.

- ❑ Pricing Theory
  - ✓ Utility Function
  - ✓ Risk Neutral Measure
  - ✓ Hedging Strategy

II. The Merton Model

- The CRRA Utility function leads One-Period Solution for Multi-Period Problem. (Myopic)

$$\sup_{\varphi_t} E\left[U^{STD}(w_T)\right]$$

Constant Relative Risk Aversion

Subject to: $w_t \geq 0$

$U^{STD}(w_t) = (w_t^{1-\gamma} - 1)/(1-\gamma)$: Utility function. See Fig. 1. Constant Relative Risk Averseness (CRRA type).

γ : Risk Averseness. Positive. In case this is 1, the utility function becomes logarithmic function.

w_t : Asset value at time= t . Consists of the risk free asset and the risky asset.

φ_t : Ratio of the risky asset among the portfolio.

S: The risky asset value. $dS_t = S_t \mu^S dt + S_t \sigma^S dB_t$.

μ^S and σ^S are drift and volatility respectively. r^f is a risk free interest rate.

(Brownian motion B_t is on a complete filtered probability space $(\Omega, F, (F_t), P)$ with initial value 0 almost surely. Filtration F_t is all time t available information for the pension fund. Setting a finite time T, $(F_t)_{0 \leq t \leq T}$ satisfies the usual conditions and the augmented sigma-field generated by B_t up to time t. In general expression, the process of portfolio X is a controlled state process valued in R and satisfying:

$dX_t = X_t[\varphi_t(\mu^S - r^f) + r^f]dt + X_t \sigma^S dB_t$.

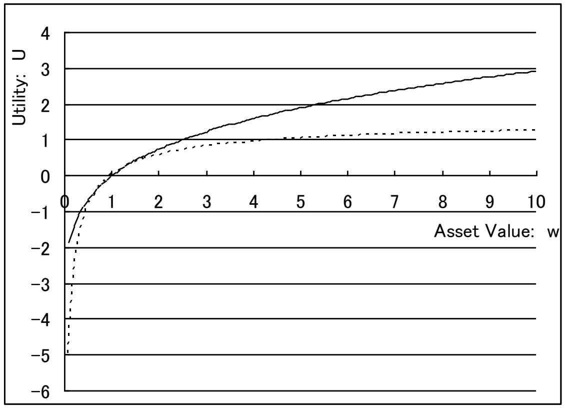


Fig 1 Utility γ=0.8 (real line), 1.6 (dot line)

(The shape of the utility function. w is Asset Value.

U is utility function $(w^{1-\gamma} - 1)/(1-\gamma)$. Two lines are for γ =0.8 case (real line) and for 1.6 case (dot line).)

$$\varphi_t = (\mu^S - r^f)/(\sigma^S)^2/\gamma$$ (=constant)

III. Mathematical Progress

- ❑ Generally, Dynamic Programing Problem can be solved by Bellman Principle and/or by Martingale Method.

✓ Bellman Principle / HJB / Verification Theorem → PDEs (Markovian)

cf. More wider kind of process, Martingale → BSDEs

✓ Martingale Method / Cox-Haung

Hamilton-Jacobi-Bellman Equation

The HJB solution shows that, the problem below is, setting $V^{STD}(w_t, t)$ as a value function,

$$\sup_{\varphi_t} E\left[U^{STD}(w_T)\right].$$

Subject to: $V^{STD}(w_0, 0) = e^{-r^f T} E[w_T]$

This is solved by the below first order condition of HJB.

$$DV^{STD}(w,t) = V^{STD}{}_t + V^{STD}{}_w w_t{}^2\left[\varphi_t(\mu^S - r^f) + r^f\right] + V^{STD}{}_{ww}(\sigma^S)^2 w^2 \varphi^2 / 2 = 0$$

$$V^{STD}(w,T) = U^{STD}(w_T), \quad D: \text{Partial differential operator}$$

The solution of the Merton model is that φ_t is constant. This means that at any time to keep the risky asset weight constant and is called myopic. Merton induced the solution by Stochastic control method. Generally speaking, the Martingale method, like [13]Cox and Haung (1987), [36]Karatzas et al. (1987), [65] Ocone and Karatzas (1991) etc., and the method to use Bellman Principle and Verification theorem(of HJB) are there. The former makes use of the logarithmic utility function case solution which [61] Morita(1997)is discussing. This paper use the later method to see the relationship with Backward Stochastic Differential Equations (BSDEs).

III. Mathematical Progress (Continued)

- ❑ BSDEs find Path from Terminal Value.

- ✓ By Time Driver and Hedging Strategy

- ✓ Risk asset process to set as X_t.
  Value Function to set as Y_t.

$$dX_t = X_t[\varphi_t(\mu^S - r^f) + r^f]dt + X_t\sigma^S dB_t$$

$$dY_t = Y_s + \int_s^t \varphi_\tau X_\tau \bullet dX_\tau / X_\tau$$

$$dY_t = r^f Y_t dt + \varphi_t X_t \bullet [(\mu^S - r^f)dt + \sigma^S dB_t]$$

Have a Solution!

In this subsection, we see how BSDE is related to HJB. A general equation sets for BSDE is described in (3.1). We treat Y as a utility function value.

(3.1) $$dY_t = -f(t, Y_t, Z_t)dt + Z_t dB_t; Y_T = \xi$$

f: Generator (Driver)

Y: Variable under a stochastic process

Z: State variable (Hedge strategy)

ξ: Terminal condition of Y

Regarding the existence and uniqueness of the solution of a specific BSDE, [40]Kobylanski (2000), [62] Morlais(2009), [6] Briand and Elie (2012), [31] Hu and Schweizer (2008), and [21] Fromm et al. (2011) proved in case the generator is a quadratic function and under some specific conditions, which are usual in a utility maximum problems.

Under more general conditions, [68] Pardoux and Peng (1990) and [19] El Karoui and Hamadene (2003) had shown the existence and uniqueness of the solution of a specific BSDE. In many cases, the terminal condition, ξ, has a function of X, i.e., $\xi = g(X_T)$ and X is supposed to be as follows.

(3.2) $$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t$$

$X_s = x$ (s expresses the opening time and x is an initial value.)

By combining (3.1) and (3.2), we can get a (decoupled) FBSDE(Forward Backward Stochastic Differential Equation). The existence and uniqueness of its solution is proved by [1] Antonelli(1993). In addition, [68] Pardoux and Peng (1990) and [69] Peng (1993a) indicate the relationship between Partial Differential Equation(PDE) (HJB is one of PDE) and FBSDE using the general expression form of PDE by Feynman-Kac, as see the below (3.3).

(3.3) $$V_t + bV_x + f(t, x, V, \sigma' v_x)dt + tr(\sigma\sigma' V_{xx})/2 = 0$$

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t$$

$$dY_t = -f(t, Y_t, Z_t)dt + Z_t dB_t$$

$$V(T, x) = g(x), \quad X_s = x, \quad Y_T = g(X_T)$$

Regarding Y and Z, [68] Pardoux and Peng (1990) proved that;

$$Y_s^{t,x} = v(t, X_s^{t,x}), \quad Z_s^{t,x} = \sigma'(t, X_s^{t,x}) v_x(t, X_s^{t,x})$$

are the solutions of the FBSDE.

In the multi-period optimal asset allocation problem through maximizing the utility function, once the utility function is given, the value function is set and the problem of (3.3) is finally solved by (2.2) unless no time-dependency for b and σ above.

In addition, in case of the stochastic process is Markovian, [46] Ma, Protter and Yong (1994) showed that the FBSDE is reduced to a nonlinear PDE by Ito formula, meaning easy to see the analytic solution and/or numerical solution (4 Step Method). [43,44] Lepeltier and San Martin (1997, 1998) and [40] Kobylanski (2000) got a solution with less strict premise

IV. The Merton Model Variation

❑ Utility Function with Kreps-Porteus characteristics

Regarding the equilibrium model for asset pricing, the Merton model use only one parameter for risk averseness and time substituent for consumption, i. e., use only γ for two different parameters, and it is hard to explain a time dependent strategy. There are studies that use both γ and Ψ (time substituent consumption parameter), which mean Kreps-Porteus type utility function, and consider Consumption CAPM. For example, Eqstiein-Zin type utility function below is popular. (For instance, [38] Kraft et al.(2011) for continuous time case and [7] Campbel and Viceira(2002) for discrete time model.)

$$\mathrm{U}(w_{\mathrm{t}}, t) = \left[(1-\delta) C_t^{(1-\gamma)/\theta} + \delta (E_t [U(w_{t+1}, t+1)])^{1/\theta} \right]^{\theta/(1-\gamma)}$$

δ : Ratio for consumption, C : Consumpton, $\theta = (1-\gamma)/(1-1/\Psi)$

IV. The Merton Model Variation (Continued)

❑ Utility Function of more directly terminal value related form

✓ Contrarian Strategy

[29] Hojgaard and Vigna(2007) arranged the target function as the below.

This is for not only maximizing end period asset amount but also for controlling the volatility of expected end period asset amount.

$$\sup_{\varphi_t(w_t)} \left(E[w_T] - \alpha \mathrm{Var}[w_T] \right)$$

The analytic solution is as the below.

$$\varphi_t(w_t) = (\mu^S - r^f)/\sigma_S{}^2 \bullet (\exp[-r^f(T-t) + (\mu^S - r^f)^2/(\sigma_S)^2 T]/2\alpha w_t + 1/(w_t/(w_0 e^{r^f})) - 1)$$

This is the case of no cash flow and about to say that the exposure to the risky asset is almost counter-proportional to the market value of the asset. This looks like reasonable because we have a rule of thumb that when the value up, then to sell to fix the gain and when the value down, then buy more at cheap (contrarian). This strategy make the volatility of expected end period's asset smaller.

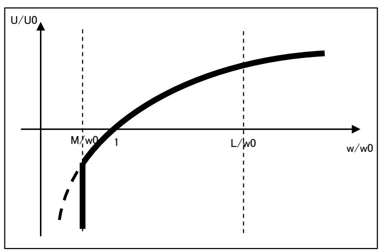
$$\varphi(w_t) = K \bullet \left(\frac{P(t)}{w_t} + \frac{1}{(w_t / R(t))} - 1 \right)$$

Contrarian Solution!

IV. The Merton Model Variation (Continued)

- Kinked Utility Functions
  - ✓ Kinked Utilities express more granularity of risk averseness.

[79] Yamashita(2011a) and [80] Yamashita(2011b) set the utility function as follows. Please see Fig. 2-1. and Fig. 2-2.



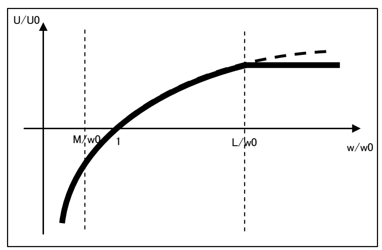


Fig. 2-1. (left): U/U0 is standardized utility function. The vertical axis shows the value of the utility function and the horizontal axis shows standardized asset value w/w0. The utility function has a kink at w/w0=M/w0 and the value becomes minus infinite.

Fig. 2-2. (right): U/U0 is standardized utility function. The vertical axis shows the value of the utility function and the horizontal axis shows standardized asset value w/w0. The utility function has a kink at w/w0=L/w0 and larger asset values do not make any increase of the value of the utility function.

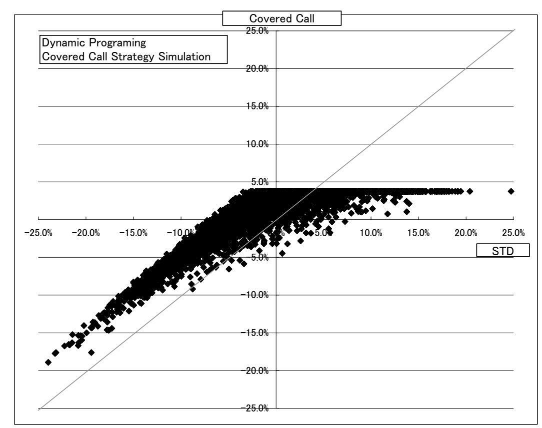
Protective Put type / Covered Call type Solutions and a Simulation Sample

Analytic Solutions

$$w_t = \zeta_t \left(X_t^U + Put\left(\frac{M}{\zeta_t}, T-t\right) \right)$$

$$w_t = \zeta_t \left(X_t^U - Call\left(\frac{L}{\zeta_t}, T-t\right) \right)$$

Covered Call type solution's simulation sample



V. Developments

- ❑ How option like strategy be treated? -> Viscosity Solutions

  - ✓ Some solutions are read as Viscosity Solutions

In terms of the solution for PDEs (including HJB), the analytic solution is hard to find out. There is a concept of a weak form of the solution of PDEs and it does not have continuity etc. In case there is a unique solution of the weak form, it is called a viscosity solution. Generally, [23] Grandall et al. (1992) describes as follows.

$$F(t, w, V, q, p, M) = -q - H(w, p, M)$$

Hamiltonian $H(w, p, M) = \underset{\varphi_t}{Sup}\left[b(w, \varphi)p + (\sigma^S)^2 M / 2\right]$

The “b” shows a drift, q= $\partial_t V$, p= $\partial_w V$, M= $\partial_w{}^2 V$.

Here, for example, the solution of [79] Yamashita (2011) for kinked utility functions are viscosity solutions. For the covered call solution, g satisfies the below.

$$\min[-V_t - H(w, p, M), -g_{ww}] = 0 \quad \text{and} \quad \min[V - g_{ww}, -g_{ww}] = 0$$

In Sobolev problem, [40] Kobylanski (2000), [54] Matoussi and Xu (2008), and [5] Briand et al. (2003) discusses the relationship of related BSDE and viscosity solutions.

V. Developments (Continued)

- BSDEs lead Semi-Analytical Solutions

  ✓ Example 1

  Find Martingale Expression from Utility Function.

  Use Martingale Measure from Utility maximization problems.

  ✓ Example 2

  Find Generator from Utility Function.

  Use BSDEs from Utility maximization problems.

Since, in a complete market, the utility maximization problem is identical to the Martingale measure probability density problem, [75] Pliska(1986), [13] Cox and Huang(1989) and [36] Karatzas et al. (1987) showed the existence and uniqueness of the solution of the problem. In case of an incomplete market, [28] He and Pearson (1991), [34] Karatzas et al. (1991) and [37] Kramkov and Schachermayer (1999) discussing the existence and uniqueness of the solution of the problem. In this section, we discuss two works about how BSDE is made use of for the utility maximization problem.

The fist work is by [50,51] Mania and Tevzadze (2003, 2008). With several assumptions, the utility maximization problem is converted to a FBSDE and using martingale methodology, they showed that the portfolio X (it initial value: X0) is expressed as the below.

$$X_t^{\varphi^*} = X_0^{\varphi^*} - \int_0^t [\psi_x(u, X_u^{\varphi^*}) + \lambda(u) V_x(u, X_u^{\varphi^*})] / V_{xx}(u, X_u^{\varphi^*}) dS_u \,.$$

S: By Martingale M and scaler λ, $S_t = M_t + \int_0^t d < M >_s \lambda_s$

$\int_0^t \psi(s,x) dM_S$: V’s Martingale part

In case the utility function is CRRA, the above can be the bellow.

Optimal solution: $X_t^* = x \mathcal{E}_t[\gamma(\psi_t / V_t + \lambda_t) S_t]$

$$\varphi_t^* = x\gamma(\psi_t / V_t + \lambda_t) \mathcal{E}_t[\gamma(\psi_t / V_t + \lambda_t) S_t]$$

$\mathcal{E}$: Dolean-Dade Exponential. (The expression is changed using this paper’s notation.)

Secondly, I will discuss works by [32] Hu et al. (2005), [77] Sekine (2006), and [73] Pham (2010). They solved the BSDE problem, which related to the utility maximization problem of CRRA utility function case or the exponential function case, by defining BSDE’s generator. In case of CRRA utility function, the value function: $V(x)$ is described as $V(x) = x^{1-\gamma} e^{Y_0}$, and Y_t can be expressed as the following.

$$Y_t = 0 - \int_t^T Z_s dW_s - \int_t^T f(s, Z_s) ds$$

$$f(t,z) = \gamma(1-\gamma) dist^2((z+\theta_t)/\gamma, C_1)/2 - (1-\gamma)|z+\theta_t|^2 / (2\gamma) - |z|^2 / 2.$$

$\theta_t = b_t / \sigma_t$, $dist_C(a) = \min_{b \in C} |a-b|$, C : convex trading strategy closed set

Regarding the optimal solution φ^*_t , the below is true. $\varphi^*_t \in \Pi_C((Z_t + \theta_t)/\gamma)$

$(\Pi_C(a) = \{b \in C : |a-b| = dist_C(a)\})$

(The expression is changed using this paper’s notation.)

V. Developments (Continued)

❑ Reflected BSDEs and Stopping Time

✓ BSDEs can treat

Obstacle problems:
Upper and/or Lower limits.

Stopping Time

In a multi-period problems, there happened to have a opportunity just to stop the risky asset investment and can achieve the target maximization etc. This is called the stopping time problem.

For example, to achieve a specific target amount of asset could be obtained by investing into risk free asset after the specific timing by some occurrence so far. The protective put option case and the covered call option case, there could happen this stopping time issue. As a matter of fact, Reflected BSDE can describe this stopping time problem. Reflected BSDE can also treat the snell envelope problem, with barriers cases, and Dynkin games etc.

More specifically, in case the lower/upper limit of Y is set L/U, by the process K_t^L / K_t^M respectively, the problem's BSDE can be described as followings.

$$dY_t = -f(t, Y_t, Z_t)dt + Z_t dB_t .$$

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dB_s + (K_T^L - K_t^L) - (K_T^M - K_t^M) \;, 0 \le t \le T$$

$$L_t \ge Y_t \ge M_t \,, \quad 0 \le t \le T \,, \quad \int_0^T (Y_t - L_t)dK_t^L = 0 \,, \quad \int_0^T (Y_t - M_t)dK_t^U = 0$$

In case of stopping time problem, the Reflected BSDE is described as the below.

$$Y_t = \underset{\tau \in T_{t,T}}{Sup}\, E[\int_0^T f(s)ds + M_\tau 1_{\tau<T} + \xi 1_{\tau=T} \,|F_t \,], \; 0 \le t \le T$$

τ : stopping time

E[ ] : Snell envelope

Details are discussed in [47] Ma et. al. (2008), [25] Hamadene and Hassani (2005), [26] Hamadene and Lepeltier (2000), [27] Hamadene and Popier (2008), [18] El Karoui et al. (1997), [10] Cvitanic and Karatzas (1996), and [42] Lepeltier et al. (2005) etc. In [24] Hamadene and Jeanblanc (2007), stopping and starting problem is treated.

V. Developments (Continued)

- ❑ Malliavin calculus gives us Derivatives of EASIER numerical calculation.

  - ✓ (S)PDEs is not so satisfactory for higher dimension cases.
  - ✓ Malliavin calculus gives us a more easier way.

PDEs

When to find sensitivity / hedging strategies, two ways to explore.

- Information is in Distribution of risky asset returns.
- Information is in terminal value payoff function.

Typically, distribution is not well-known and derivatives of payoff function can be treated easier by Malliavin calculus.

BSDEs

In the papers of [83] Zhang (2001) and [48] Ma and Zhang (2002), the relationship among BSDE and Malliavin calculus is shown. The calculus will be not discussed here (please see [52] Malliavin(2006), [64] Nualart (1995) or [66] Oksendal(1997)) but some discussion here. Based on [8] Cetin (2006), regarding SDE $dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t$, we can express as follows regarding Malliavin derivative operator D.

$$D_s X_t = \sigma\,(s, X_u) + \int_s^t \nabla_x b(u, X_u) D_s X_u du + \int_s^t \nabla_x b(u, X_u) D_s X_u dB_u$$

$\nabla_x f(t, x)$: gradient of f by x

In addition, based on [33] Imkeller (2008), we can obtain the followings.

$$dY_t = -f(\Omega, t, Y_t, Z_t)dt + Z_t dB_t$$

$$Y_s = Y_t + \int_t^s Z_r dB_r - \int_t^s f(\Omega, r, Y_r, Z_r)dr$$

$$D_u Y_s = Z_u + \int_u^s D_u Z_r dB_r - \int_u^s [\partial_y f(\Omega, r, Y_r, Z_r) D_u Y_r + \partial_z f(\Omega, r, Y_r, Z_r) D_u Z_r + D_u f(\Omega, r, Y_r, Z_r)]dr$$

VI. Numerical Simulations

- ✓ Stopping time prevent backward simulations.
- ✓ In a typical situation, the BSDE has a generator with Z dependency.

Since analytic solutions could be found in very limited cases, numerical simulations have been also discussed often. There is a computer capacity issue that even most recent super computers take time for massive years especially for multi-asset cases.

In such circumstances, [63] Munk (2003) used Markov Chain approximation and showed validity of the numerical simulation result of the Merton model, which is a time continuous solution. They caught up with the way to decide the grids Other methodologies for the issue includes [67] Pages et al. (2004)'s quantization algorithm. More generally, Monte Carlo simulation methodologies for the stochastic control problem are expressed and discussed in [15] Detemple et al. (2003) and [9] Cvitanic et al. (2003). In case Dynamic Programing is to be used, [4] Brandt et al. (2005) and [45] Longstaff and Schwartz (2001) researched in that way. In addition, Malliavin calculus is discussed in [78] Takahashi and Yoshida (2004).

On the other hand, the utility maximization problem by BSDE is hard to solve by numerical simulations because of the exact backwardness. However, simulations of Reflected BSDE is easier to understand than the analytic solutions of [50,51] Mania and Tevzadze (2003, 2008) and [32] Hu et al. (2005). Still in that case, f's dependency of Z and need for predetermined boundary conditions are obstacle.

Generally speaking, BSDE numerical simulations are more focused on these days. The simulation by [72] Peng and Xu (2011) is a good example but not looked like investment problem setting. Other simulation discussions includes [76] Porcher et al. (2008), [14] Delarue and Menozzi (2006), [3] Bouchard and Touzi (2005), [12] Chaumont, Imkeller and Muller (2005), [22] Gobet et al. (2005), [70,71] Peng (2003b,2004), [83] Zhang (2001), [48] Ma and Zhang (2002), [55] Memin et al. (2008), [47] Ma et al. (2008), [2] Bally and Pages (2000), [49] Ma and Zhang (2005), [53] Martin and Torres (2007) , [16] Douglas et al. (1996). Some are making use of American Option tactics of [45] Longstaff and Schwartz (2001) methodology.

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