Mathematical model Trend Survey of Pension Fund's Dynamic Asset Allocation*

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Abstract

This paper surveys dynamic asset allocation models which use stochastic optimal control problem methods. The Merton model and its variation models, the use of Hamilton Jacobi Bellman Equation (HJB), Backward Stochastic Differential Equation (BSDE), Malliavin calculus, and numerical simulation are discussed.

* This paper is describing an individual opinion and is not expressing a belonging company's opinion.

Contents

1. Introduction

Popular among academic circles and investment practitioners is the modelling of problems to maximize the expected utility of end-of-period wealth by allocating wealth between a risky security and a riskless security over some investment horizon. Merton's CRRA utility maximization investment strategy problem ([50-60] Merton(1969, 1971, 1973a, 1973b, 1992)) is a typical example.

Currently, especially among investment practitioners, the pension fund manager is one of the most interested practitioners to think about this multi-period allocation strategy. Pension funds had static asset allocation strategies for a long time. This is because financial markets have given good returns in a sense of a long term even though sometimes markets experienced large draw downs. However, we now realized recent lower economic growth and interest rate, globally and for to some extent a longer term, which means lower expected returns for financial assets. In addition, financial markets started to be very volatile. Those convinced us that a static allocation strategy no longer gives us enough return in a short term / long term, nor easy ways to recover from large draw downs. Now pension fund managers are seeking a dynamic multi-period asset allocation strategy. Here I survey models which are under very idealized conditions, but this will be of use.

This paper treats, how Hamilton Jacobi Bellman equation (HJB), Backward Stochastic Differential Equation (BSDE), and Malliavin calculus are applied to the Merton model. Also some extended models of the Merton model are introduced, in which viscosity solutions of stochastic differential equations are discussed. [30] Honda(2002) summarized the Merton model from Inter-temporal CAPM point of view and here describe those expansion. [81] Yoshida(2003)is a good overview for BSDE use in financial areas but here recent developments are discussed. [78] Takahashi and Yoshida (2004) described Malliavin calculus and asymptotic expansion and here I discuss some of those. This paper starts for the Merton model and HJB in section 2. Next section 3 describe BSDE and Malliavin calculus. In section 4, the expansion works of the Merton model and viscosity solutions are discussed. Numerical simulations are discussed in section 5 and final section 6 will conclude and describe future challenges.

2. The Merton model and HJB

2.1 The Merton model

An investor's objective is to maximize the expected utility of end-of-period (time t=T) wealth w by allocating his wealth w between two assets, a risky security (Risky Asset) and a riskless security (Risk Free Asset), over some investment horizon [0,T], which is called a strategy and expressed by the risky asset weight . Risky Asset's characteristics are determined by its price S under geometric Brownian motion with drift and volatility. Using utility function , the problem is as follows.

(2.1) $$\underset{\varphi_t}{Sup} E\left[U^{STD}(w_T)\right]$$

Subject to: $w_t \geq 0$

$U^{STD}(w_t) = (w_t^{1-\gamma} - 1)/(1-\gamma)$: Utility function. See Fig. 1. Constant Relative Risk Averseness (CRRA type).

γ : Risk Averseness. Positive. In case this is 1, the utility function becomes logarithmic function.

w_t : Asset value at time= t . Consists of the risk free asset and the risky asset.

φ_t : Ratio of the risky asset among the portfolio.

S: The risky asset value. $dS_t = S_t \mu^S dt + S_t \sigma^S dB_t$.

μ^S and σ^S are drift and volatility respectively. r^f is a risk free interest rate. (Brownian motion B_t is on a complete filtered probability space $(\Omega, F, (F_t), P)$ with initial value 0 almost surely. Filtration F_t is all time t available information for the pension fund. Setting a finite time T, $(F_t)_{0 \leq t \leq T}$ satisfies the usual conditions and the augmented sigma-field generated by B_t up to time t. In general expression, the process of portfolio X is a controlled state process valued in R and satisfying:

$$dX_t = X_t[\varphi_t(\mu^S - r^f) + r^f]dt + X_t \sigma^S dB_t .$$

The decision of the risky asset weight φ_t is the control. Generally, the control is set as $\alpha = (\alpha_s)_{0 \leq s \leq T}$, and it is a progressively measurable process valued in the control set A, a subset of R . The Borelian functions b, σ on $[0, \mathrm{T}] \times \mathrm{R} \times A$ satisfy the usual conditions in order to ensure the existence of a strong solution to the above stochastic process. This is typically satisfied when b and σ satisfy a Lipschitz condition on (t,x) uniformly in A, and α satisfies a square integrability condition.

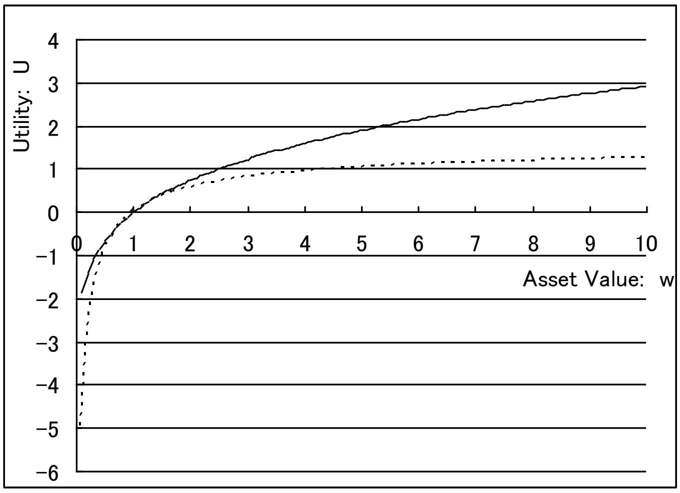


Fig 1 Utility γ=0.8 (real line), 1.6 (dot line)

(The shape of the utility function. w is Asset Value.

U is utility function $(w^{1-\gamma}-1)/(1-\gamma)$. Two lines are for γ =0.8 case (real line) and for 1.6 case (dot line).)

The solution of the Merton model is that φ_t is constant. This means that at any time to keep the risky asset weight constant and is called myopic. Merton induced the solution by Stochastic control method. Generally speaking, the Martingale method, like [13]Cox and Haung (1987), [36]Karatzas et al. (1987), [65] Ocone and Karatzas (1991) etc., and the method to use Bellman Principle and Verification theorem(of HJB) are there. The former makes use of the logarithmic utility function case solution which [61] Morita(1997)is discussing. This paper use the later method to see the relationship with Backward Stochastic Differential Equations (BSDEs).

2.2 HJB

The HJB solution shows that, the problem below is, setting $V^{STD}(w_t,t)$ as a value function,

$$\underset{\varphi_t}{Sup} E\left[U^{STD}(w_T)\right].$$

Subject to: $V^{STD}(w_0,0)=e^{-r^f T}E[w_T]$

This is solved by the below first order condition of HJB.

$$(2.2)\qquad DV^{STD}(w,t)=V^{STD}{}_t+V^{STD}{}_w w_t{}^2\left[\varphi_t(\mu^S-r^f)+r^f\right]+V^{STD}{}_{ww}(\sigma^s)^2w^2\varphi^2/2=0$$

$$V^{STD}(w,T)=U^{STD}(w_T),\quad D\text{: Partial differential operator}$$

This leads that;

$$\varphi_t=(\mu^S-r^f)/(\sigma^S)^2/\gamma\ \text{(=constant)}$$

In case the problem is described by HJB, the problem is solved by the Stochastic Differential Equation (SDE). (See [17] Duffe (1996).)

3. BSDE and Malliavin Calculus

3.1 Value Function and BSDE

Recently BSDE has been paid a lot of attention as a method for solving multi-period / continuous-time asset allocation using SDE. In the problem (2.1), the portfolio value is taken in account by X_t; $dX_t = X_t[\varphi_t(\mu^S - r^f) + r^f]dt + X_t\sigma^S dB_t$ and the value function is discussed by Y_t;

$dY_t = Y_s + \int_s^t \varphi_\tau X_\tau \bullet dX_\tau / X_\tau$, we can see $dY_t = r^f Y_t dt + \varphi_t X_t \bullet [(\mu^S - r^f)dt + \sigma^S dB_t]$

and this shows the BSDE described below in (3.1).

3.2 BSDE and HJB

In this subsection, we see how BSDE is related to HJB. A general equation sets for BSDE is described in (3.1). We treat Y as a utility function value.

(3.1) $$dY_t = -f(t, Y_t, Z_t)dt + Z_t dB_t; Y_T = \xi$$

f: Generator (Driver)

Y: Variable under a stochastic process

Z: State variable (Hedge strategy)

ξ: Terminal condition of Y

Regarding the existence and uniqueness of the solution of a specific BSDE, [40]Kobylanski (2000), [62] Morlais(2009), [6] Briand and Elie (2012), [31] Hu and Schweizer (2008), and [21] Fromm et al. (2011) proved in case the generator is a quadratic function and under some specific conditions, which are usual in a utility maximum problems.

Under more general conditions, [68] Pardoux and Peng (1990) and [19] El Karoui and Hamadene (2003) had shown the existence and uniqueness of the solution of a specific BSDE. In many cases, the terminal condition, ξ, has a function of X, i.e., $\xi = g(X_T)$ and X is supposed to be as follows.

(3.2) $$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t$$

$X_s = x$ (s expresses the opening time and x is an initial value.)

By combining (3.1) and (3.2), we can get a (decoupled) FBSDE(Forward Backward Stochastic Differential Equation). The existence and uniqueness of its solution is proved by [1] Antonelli(1993). In addition, [68] Pardoux and Peng (1990) and [69] Peng (1993a) indicate the relationship between

Partial Differential Equation(PDE) (HJB is one of PDE) and FBSDE using the general expression form of PDE by Feynman-Kac, as see the below (3.3).

(3.3) $$V_t + bV_x + f(t,x,V,\sigma' v_x)dt + tr(\sigma\sigma' V_{xx})/2 = 0$$

$$dX_t = b(t,X_t)dt + \sigma(t,X_t)dB_t$$

$$dY_t = -f(t,Y_t,Z_t)dt + Z_t dB_t$$

$$V(T,x) = g(x)\,,\quad X_s = x\,,\quad Y_T = g(X_T)$$

Regarding Y and Z, [68] Pardoux and Peng (1990) proved that;

$$Y_s^{t,x} = v(t,X_s^{t,x})\,,\quad Z_s^{t,x} = \sigma'(t,X_s^{t,x})v_x(t,X_s^{t,x})$$

are the solutions of the FBSDE.

In the multi-period optimal asset allocation problem through maximizing the utility function, once the utility function is given, the value function is set and the problem of (3.3) is finally solved by (2.2) unless no time-dependency for b and σ above.

In addition, in case of the stochastic process is Markovian, [46] Ma, Protter and Yong (1994) showed that the FBSDE is reduced to a nonlinear PDE by Ito formula, meaning easy to see the analytic solution and/or numerical solution (4 Step Method). [43,44] Lepeltier and San Martin (1997, 1998) and [40] Kobylanski (2000) got a solution with less strict premise of [46] Ma, Protter and Yong(1994). However, [11] Cvitanic and Ma (1996) and [16] Douglas, Ma, and Protter (1996) found that the numerical analysis will see a massive burden for solving numerically as the dimension of the state variables increases.

3.3 BSDE and Utility Maximization Problem

Since, in a complete market, the utility maximization problem is identical to the Martingale measure probability density problem, [75] Pliska(1986), [13] Cox and Huang(1989) and [36] Karatzas et al. (1987) showed the existence and uniqueness of the solution of the problem. In case of an incomplete market, [28] He and Pearson (1991), [34] Karatzas et al. (1991) and [37] Kramkov and Schachermayer (1999) discussing the existence and uniqueness of the solution of the problem. In this section, we discuss two works about how BSDE is made use of for the utility maximization problem.

The fist work is by [50,51] Mania and Tevzadze (2003, 2008). With several assumptions, the utility maximization problem is converted to a FBSDE and using martingale methodology, they showed that the portfolio X (it initial value: X0) is expressed as the below.

$$X_t^{\varphi^*} = X_0^{\varphi^*} - \int_0^t [\psi_x(u, X_u^{\varphi^*}) + \lambda(u) V_x(u, X_u^{\varphi^*})] / V_{xx}(u, X_u^{\varphi^*}) dS_u \ .$$

S: By Martingale M and scaler λ, $S_t = M_t + \int_0^t d < M >_s \lambda_s$

$\int_0^t \psi(s,x) dM_S$: V's Martingale part

In case the utility function is CRRA, the above can be the bellow.

Optimal solution: $X_t^* = x\mathcal{E}_t[\gamma(\psi_t / V_t + \lambda_t) S_t]$

$\varphi_t^* = x\gamma(\psi_t / V_t + \lambda_t)\mathcal{E}_t[\gamma(\psi_t / V_t + \lambda_t) S_t]$

$\mathcal{E}$: Dolean-Dade Exponential. (The expression is changed using this paper's notation.)

Secondly, I will discuss works by [32] Hu et al. (2005), [77] Sekine (2006), and [73] Pham (2010). They solved the BSDE problem, which related to the utility maximization problem of CRRA utility function case or the exponential function case, by defining BSDE's generator. In case of CRRA utility function, the value function: $V(x)$ is described as $V(x) = x^{1-\gamma} e^{Y_0}$, and Y_t can be expressed as the following.

$$Y_t = 0 - \int_t^T Z_s dW_s - \int_t^T f(s, Z_s) ds$$

$$f(t,z) = \gamma(1-\gamma) dist^2((z+\theta_t)/\gamma, C_1)/2 - (1-\gamma)\left|z+\theta_t\right|^2/(2\gamma) - \left|z\right|^2/2 \, .$$

$\theta_t = b_t / \sigma_t$, $dist_C(a) = \min_{b \in C} |a-b|$, C : convex trading strategy closed set

Regarding the optimal solution $\varphi^*{}_t$, the below is true.

$\varphi^*{}_t \in \Pi_C((Z_t + \theta_t)/\gamma) \, (\Pi_C(a) = \{b \in C : |a-b| = dist_C(a)\})$

(The expression is changed using this paper's notation.)

One of the merit of this expression is that after the expression is obtained, we could apply [39] Kunitomo and Takahashi (2004)'s asymptotic expansion.

3.4 Malliavin Calculus and others

In the papers of [83] Zhang (2001) and [48] Ma and Zhang (2002), the relationship among BSDE and Malliavin calculus is shown. The calculus will be not discussed here (please see [52] Malliavin(2006), [64] Nualart (1995) or [66] Oksendal(1997)) but some discussion here. Based on

[8] Cetin (2006), regarding SDE $dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t$, we can express as follows regarding Malliavin derivative operator D.

$$D_s X_t = \sigma\ (s, X_u) + \int_s^t \nabla_x b(u, X_u) D_s X_u du + \int_s^t \nabla_x b(u, X_u) D_s X_u dB_u$$

$\nabla_x f(t,x)$: gradient of f by x

In addition, based on [33] Imkeller (2008), we can obtain the followings.

$$dY_t = -f(\Omega, t, Y_t, Z_t)dt + Z_t dB_t$$

$$Y_s = Y_t + \int_t^s Z_r dB_r - \int_t^s f(\Omega, r, Y_r, Z_r)dr$$

$$D_u Y_s = Z_u + \int_u^s D_u Z_r dB_r - \int_u^s [\partial_y f(\Omega, r, Y_r, Z_r) D_u Y_r + \partial_z f(\Omega, r, Y_r, Z_r) D_u Z_r + D_u f(\Omega, r, Y_r, Z_r)]dr$$

We can search for the structure of the problem by above. Other related discussions are in [82] Yoshida (2010).

4. Expansion of the Merton model

4.1 Expansion Part 1

Regarding the equilibrium model for asset pricing, the Merton model use only one parameter for risk averseness and time substituent for consumption, i. e., use only γ for two different parameters, and it is hard to explain a time dependent strategy. There are studies that use both γ and Ψ (time substituent consumption parameter), which mean Kreps-Porteus type utility function, and consider Consumption CAPM. For example, Eqstiein-Zin type utility function below is popular. (For instance, [38] Kraft et al.(2011) for continuous time case and [7] Campbel and Viceira(2002) for discrete time model.)

$$\mathrm{U}(w_t, t) = \left[(1-\delta)C_t^{(1-\gamma)/\theta} + \delta(E_t[U(w_{t+1}, t+1)])^{1/\theta}\right]^{\theta/(1-\gamma)}$$

δ : Ratio for consumption, C : Consumpton, $\theta = (1-\gamma)/(1-1/\Psi)$

4.2 Expansion Part 2

Regarding (2.1), [29] Hojgaard and Vigna(2007) arranged the target function as the below. This is for not only maximizing end period asset amount but also for controlling the volatility of expected end period asset amount.

$$\underset{\varphi_t(w_t)}{Sup}(E[w_T] - \alpha \mathrm{Var}[w_T])$$

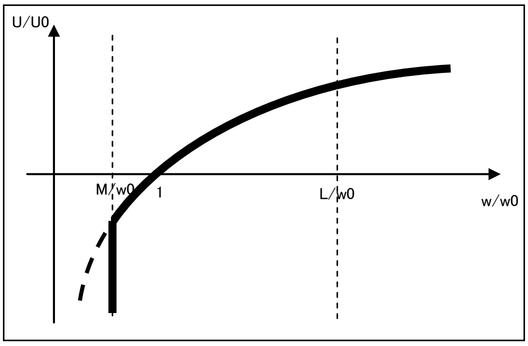
The analytic solution is as the below.

$$\varphi_t(w_t) = (\mu^S - r^f)/\sigma_S^2 \cdot (\exp[-r^f(T-t) + (\mu^S - r^f)^2/(\sigma_S)^2 T]/2\alpha w_t + 1/(w_t/(w_0 e^{r^f})) - 1)$$

This is the case of no cash flow and about to say that the exposure to the risky asset is almost counter-proportional to the market value of the asset. This looks like reasonable because we have a rule of thumb that when the value up, then to sell to fix the gain and when the value down, then buy more at cheap (contrarian). This strategy make the volatility of expected end period's asset smaller.

4.3 Expansion Part 3

[79] Yamashita(2011a) and [80] Yamashita(2011b) set the utility function as follows. Please see Fig. 2-1. and Fig. 2-2.



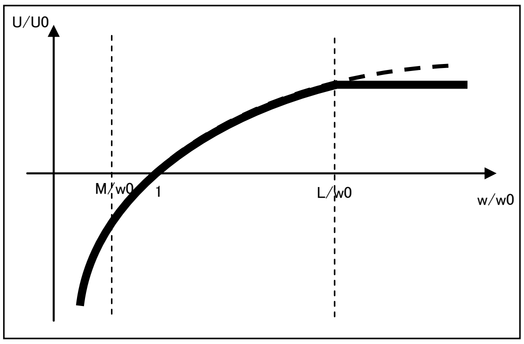


Fig. 2-1. (left): U/U0 is standardized utility function. The vertical axis shows the value of the utility function and the horizontal axis shows standardized asset value w/w0. The utility function has a kink at w/w0=M/w0 and the value becomes minus infinite.

Fig. 2-2. (right): U/U0 is standardized utility function. The vertical axis shows the value of the utility function and the horizontal axis shows standardized asset value w/w0. The utility function has a kink at w/w0=L/w0 and larger asset values do not make any increase of the value of the utility function.

The situation of the Fig. 2-1 is a "protective put option" strategy. Because the utility suddenly has negative infinite value if the pension asset value becomes below "M." In order to avoid investor's wealth value's being below "M," the solution will be to buy a put option of asset with strike price M. In reality, not just a buy-and-hold type strategy but a strategy the strike price will be adjusted by some fixed figure.

The situation of Fig. 2-2 is a "covered call option" strategy. Because there is no incentive to let the wealth increase once the wealth achieves the target amount "L." The asset includes a short position of the call option. To sell a call option (covered call) meaning that the investor wants to achieve the target amount but there is no need to be above level "L." As is the same of the protective put option case, the strike price will be adjusted by a fixed figure because selling call options makes money, and the money should also invested as the same way of itself.

Mathematically, to solve the problems, we have to be careful for non-decreasing but not strictly increasing utility functions. By Legendle-Fenchel transformation (dual utility), we can solve the problems.

4.5 Viscosity Solution of HJB

In terms of the solution for PDEs (including HJB), the analytic solution is hard to find out. There is a concept of a weak form of the solution of PDEs and it does not have continuity etc. In case there is a unique solution of the weak form, it is called a viscosity solution. Generally, [23] Grandall et al. (1992) describes as follows.

$$F(t, w, V, q, p, M) = -q - H(w, p, M)$$

Hamiltonian $H(w, p, M) = \underset{\varphi_t}{Sup}\left[b(w,\varphi)p + (\sigma^S)^2 M / 2\right]$

The "b" shows a drift, q= $\partial_t V$, p= $\partial_w V$, M= $\partial_w{}^2 V$.

Here, for example, the solution of [79] Yamashita (2011) for kinked utility functions are viscosity solutions. For the covered call solution, g satisfies the below.

$$\min[-V_t - H(w, p, M), -g_{ww}] = 0 \quad \text{and} \quad \min[V - g_{ww}, -g_{ww}] = 0$$

In Sobolev problem, [40] Kobylanski (2000), [54] Matoussi and Xu (2008), and [5] Briand et al. (2003) discusses the relationship of related BSDE and viscosity solutions.

4.6 Stopping time and Reflected BSDE

In a multi-period problems, there happened to have a opportunity just to stop the risky asset investment and can achieve the target maximization etc. This is called the stopping time problem. For example, to achieve a specific target amount of asset could be obtained by investing into risk free asset after the specific timing by some occurrence so far. The protective put option case and the covered call option case, there could happen this stopping time issue. As a matter of fact, Reflected BSDE can describe this stopping time problem. Reflected BSDE can also treat the snell envelope problem, with barriers cases, and Dynkin games etc.

More specifically, in case the lower/upper limit of Y is set L/U, by the process K_t^L / K_t^M respectively, the problem's BSDE can be described as followings.

$$dY_t = -f(t, Y_t, Z_t)dt + Z_t dB_t .$$

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dB_s + (K_T^L - K_t^L) - (K_T^M - K_t^M) \text{ ,} 0 \le t \le T$$

$$L_t \ge Y_t \ge M_t , \quad 0 \le t \le T , \quad \int_0^T (Y_t - L_t) dK_t^L = 0 , \quad \int_0^T (Y_t - M_t) dK_t^U = 0$$

In case of stopping time problem, the Reflected BSDE is described as the below.

$$Y_t = \underset{\tau \in \mathrm{T}_{t,T}}{Sup} E[\int_0^T f(s)ds + M_\tau 1_{\tau<T} + \xi 1_{\tau=T} | F_t], \ 0 \le t \le T$$

τ : stopping time

E[ ] : Snell envelope

Details are discussed in [47] Ma et. al. (2008), [25] Hamadene and Hassani (2005), [26] Hamadene and Lepeltier (2000), [27] Hamadene and Popier (2008), [18] El Karoui et al. (1997), [10] Cvitanic and Karatzas (1996), and [42] Lepeltier et al. (2005) etc. In [24] Hamadene and Jeanblanc (2007), stopping and starting problem is treated.

4.7 Others

Other problems discussed in the context of the Merton model expansion are using a HARA type utility function (Hyperbolic absolute risk aversion) or a Power type utility function. Some are using a stochastic process with jump diffusion for the risky asset returns. This type of process could explain fatter downside tail distribution for the risky asset. However it is difficult to obtain a unique solution as Ito's lemma nor Feymann-Kac formula do not work. (Please see [20] Eyraud-Loisel(2005), [74] Platen and Bruti-Liberati (2010) and [41] Kou(2008).)

5. Numerical Simulations

Since analytic solutions could be found in very limited cases, numerical simulations have been also discussed often. There is a computer capacity issue that even most recent super computers take time for massive years especially for multi-asset cases.
In such circumstances, [63] Munk (2003) used Markov Chain approximation and showed validity of the numerical simulation result of the Merton model, which is a time continuous solution. They caught up with the way to decide the grids. Other methodologies for the issue includes [67] Pages et al. (2004)'s quantization algorithm. More generally, Monte Carlo simulation methodologies for the stochastic control problem are expressed and discussed in [15] Detemple et al. (2003) and [9] Cvitanic et al. (2003). In case Dynamic Programing is to be used, [4] Brandt et al. (2005) and [45] Longstaff and Schwartz (2001) researched in that way. In addition, Malliavin calculus is discussed in [78] Takahashi and Yoshida (2004).

On the other hand, the utility maximization problem by BSDE is hard to solve by numerical simulations because of the exact backwardness. However, simulations of Reflected BSDE is easier to understand than the analytic solutions of [50,51] Mania and Tevzadze (2003, 2008) and [32] Hu et al. (2005). Still in that case, f's dependency of Z and need for predetermined boundary conditions are obstacle.

Generally speaking, BSDE numerical simulations are more focused on these days. The simulation by [72] Peng and Xu (2011) is a good example but not looked like investment problem setting. Other simulation discussions includes [76] Porcher et al. (2008), [14] Delarue and Menozzi (2006), [3] Bouchard and Touzi (2005), [12] Chaumont, Imkeller and Muller (2005), [22] Gobet et al. (2005), [70,71] Peng (2003b,2004), [83] Zhang (2001), [48] Ma and Zhang (2002), [55] Memin et al. (2008), [47] Ma et al. (2008), [2] Bally and Pages (2000), [49] Ma and Zhang (2005), [53] Martin and Torres (2007) , [16] Douglas et al. (1996). Some are making use of American Option tactics of [45] Longstaff and Schwartz (2001) methodology.

6. Summary and Future Challenges

I summarized the modelling of problems to maximize the expected utility of end-of-period wealth by allocating wealth between a risky security and a riskless security over some investment horizon. Merton's CRRA utility maximization was the very start and it had been expanded into many variations, i.e., HJB, BSDE, Reflected BSDE and Malliavin calculus. Simulation methodologies are also more lighted for those problems to solve as analytic solutions are difficult to find or understand. Current studies have strong premise / conditions to set. Some trials are Jump diffusion process to implemented or [84] Zhu et al.'s idea that to set the objective not only by the end period status but also the whole period's characteristics of the assets. Others could include utility functions other than CRRA type.

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