The Cramér-Lundberg and the dual risk models: Ruin, dividend problems and duality features

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Renewal Insurance Risk model

In actuarial science and applied probability, *Ruin Theory* uses mathematical models to describe an insurer's vulnerability to insolvency or ruin.

One of such models is the *Renewal Insurance Risk Model*.

$$U(t) = u + P(t) - S(t).$$

Renewal Insurance Risk model

$$U(t) = u + P(t) - S(t).$$

The quantity $U(t)$ is nothing but the insurer's capital balance at a given time t, and the process $U = \{U(t)\}_{t \geq 0}$ describes the cashflow in the portfolio over time.

The function $P(t)$ describes the inflow of capital into business by time t, and $S(t)$ describes the outflow of capital due to payments for claims ocurred in $[0, t]$.

Renewal Insurance Risk model

$$U(t) = u + P(t) - S(t).$$

If $U(t)$ is positive, the company has gained capital, if $U(t)$ is negative it has lost capital. The constant value $U(0) = u \geq 0$ is called initial capital.

It is desirable to start up an insurance business with a sufficiently large initial capital (reserve), which prevents the business from bankruptcy in the first period of it existence.

Renewal Insurance Risk model

$$U(t) = u + P(t) - S(t).$$

The total claim amount process $S(t)$ is defined as

$$S(t) = \sum_{i=1}^{N(t)} X_i, \quad t \geq 0,$$

$\{X_i\}_{i\geq 1}$ is a sequence of individual random non–negative claim sizes.
$\{N(t)\}_{t\geq 0}$ is the claim number process, where $N(t)$ denotes the number of claims ocurred up to time t.

Renewal Insurance Risk model

$$U(t) = u + P(t) - \sum_{i=1}^{N(t)} X_i.$$

We denote the sequence of random interclaim times by $\{W_i\}_{i \geq 1}$. Then the number of claims $N(t)$ has the expression

$$N(t) = \max\{k : \; W_1 + W_2 + \cdots + W_k \leq t\}.$$

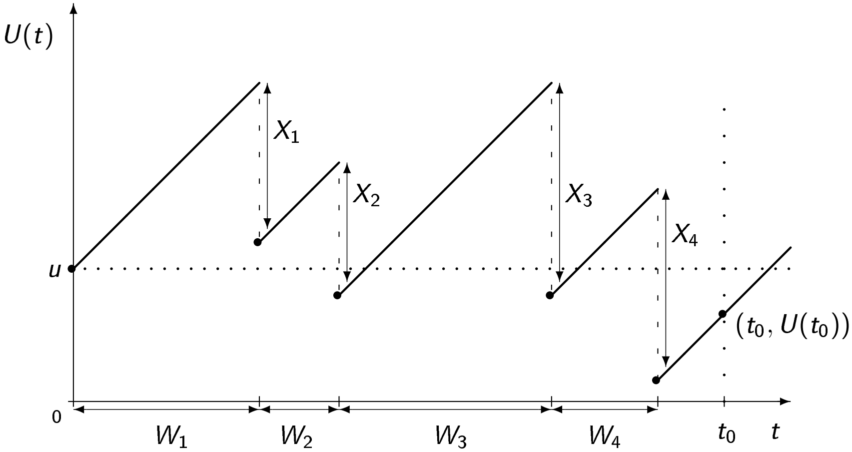
Sparre–Andersen Risk Model

$$U(t) = u + ct - \sum_{i=1}^{N(t)} X_i.$$

The Sparre–Andersen Risk Model is a simplification of the Renewal Insurance Risk model. It assumes

- A deterministic and linear premium income $P(t) = ct$, where $c > 0$ is called the *premium rate*.
- An i.i.d. sequence of claim amounts $\{X_i\}_{i \geq 1}$.
- An i.i.d. sequence of interclaim times $\{W_i\}_{i \geq 1}$, which are also independent of the X_i's.
- The *Net Profit Condition* $cE(W_1) > E(X_1)$.

Sparre–Andersen Risk Model



$$U(t_0) = u + ct_0 - X_1 - X_2 - X_3 - X_4, \quad N(t_0) = 4$$

Figure: The Sparre–Andersen Risk Model

Some quantities of interest in the Sparre–Andersen Model

Time of ruin $T_S = \inf\{t > 0 : \ U(t) < 0\}, \ u \geq 0,$

$$T_S = \infty \ \text{ iff } \ U(t) \geq 0 \ \ \forall t > 0,$$

Ultimate ruin probability $\psi_S(u) = P(T_S < \infty)$,

Survival probability $\phi_S(u) = 1 - \psi_S(u)$,

Time to upcross barrier b $\tau_b = \inf\{t > 0 : U(0) = u, U(t) \geq b\}$,

$R_S(u, b, \delta) = E[e^{-\delta \tau_b} | U(0) = u]$ the Laplace transform of τ_b, for $\delta \geq 0$.
Remark: δ can be interpreted as an interest force.

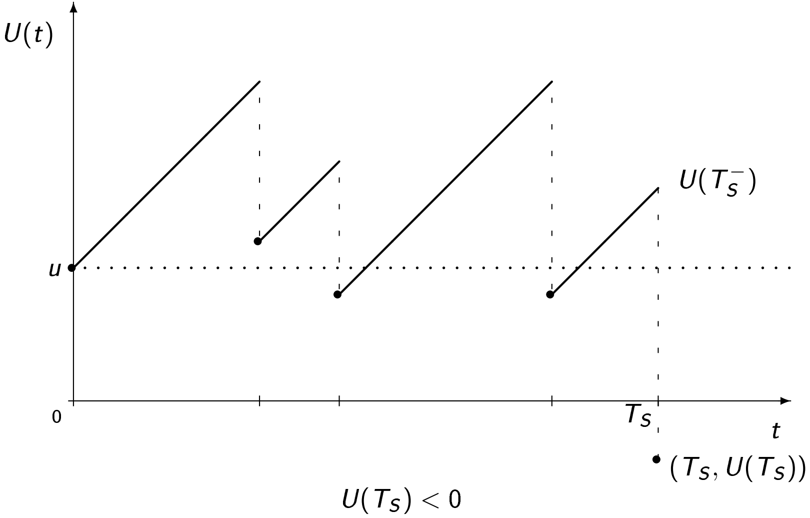
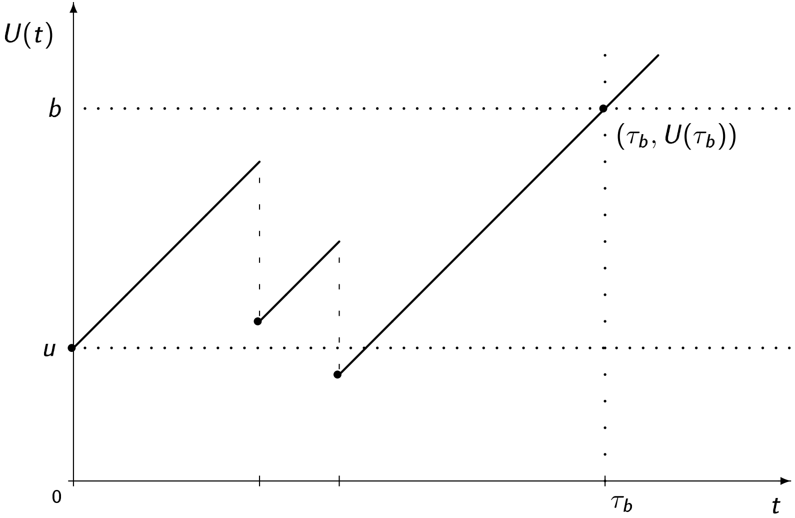


Figure: The time of ruin



$U(\tau_b) = b$

Figure: The time to upcross barrier b

Dual Risk Model

A variation of the Sparre–Andersen model is the so called *Dual Risk Model*

$$U(t) = u - ct + \sum_{i=1}^{N(t)} X_i. \tag{1}$$

In this model the premiums are considered as "costs", and the claims are considered as "gains".

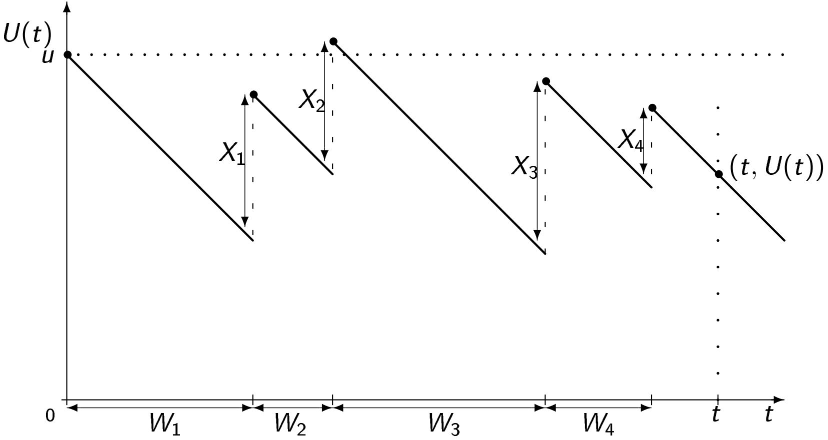
There is a simple but illustrative interpretation, the surplus can be considered as the capital of an economic activity like research and development, where gains are random, at random instants, and costs are certain.

Dual Risk Model

There is an important difference between the Dual and the Sparre–Andersen models: the *Net Profit Condition*

	Model	Net Profit Condition
Sparre–Andersen	$U(t) = u + ct - \sum_{i=1}^{N(t)} X_i$	$cE(W_1) > E(X_1)$
Dual	$U(t) = u - ct + \sum_{i=1}^{N(t)} X_i$	$cE(W_1) < E(X_1)$

Dual Risk Model



$$U(t) = u - ct + \sum_{i=1}^{4} X_i, \quad N(t) = 4$$

Some quantities of interest in the Dual Model

Time of ruin $T_D = \inf\{t > 0 : \ U(t) = 0\}, \ u \geq 0,$

$$T_D = \infty \ \text{ iff } \ U(t) > 0 \ \ \forall t > 0.$$

Ultimate ruin probability $\psi_D(u) = P(T_D < \infty)$.

Survival probability $\phi_D(u) = 1 - \psi_D(u)$.

For a constant $\delta \geq 0$, the Laplace transform of the time of ruin is

$$\psi_D(u, \delta) = E(e^{-\delta T_D} \mathbb{I}(T_D < \infty)).$$

Remark: δ can be interpreted as an interest force.

Duality

HOW DO I GET TO THE OTHER SIDE?
YOU ARE ON THE OTHER SIDE!
The sun shines on both sides
SEAL
www.bobseal.com

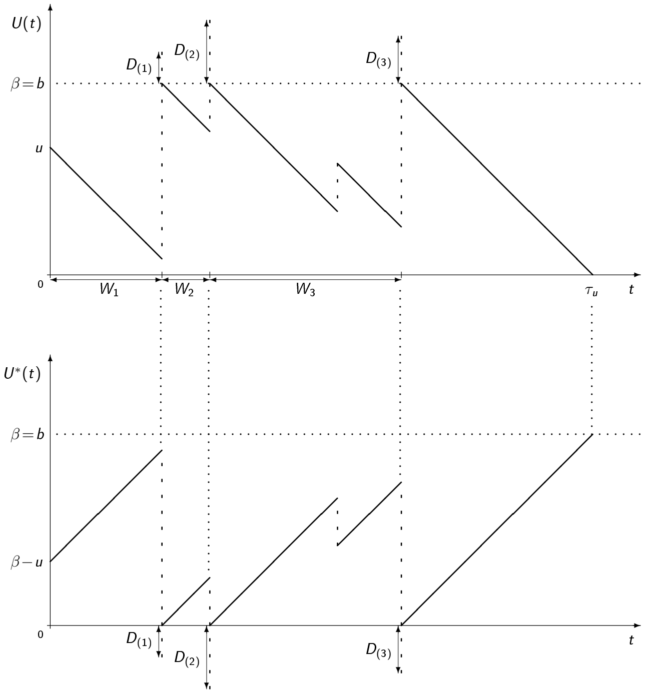


Figure: Sparre–Andersen *vs* Dual model

Results

We have found a connection, or "duality", between two quantities we mentioned before:

- $R_S(u, b, \delta)$, the Laplace transform of the time to upcross barrier b in the Sparre–Andersen Model.
- $\psi_D(u, \delta)$, the Laplace transform of the time of ruin in the Dual Model.

Each of them can be obtained from the another by switching the corresponding *Net Profit Condition*.

Results

This connection appears when we assume the interclaim times W_i follow a Gamma(n, λ) distribution (n positive integer), an hypoexponential distribution or a Phase–Type distribution. It remains open the question: is this the case for every distribution?, or al least for every light tail distribution?

Moreover, are there any other quantities that could show the same kind of duality between the two models?

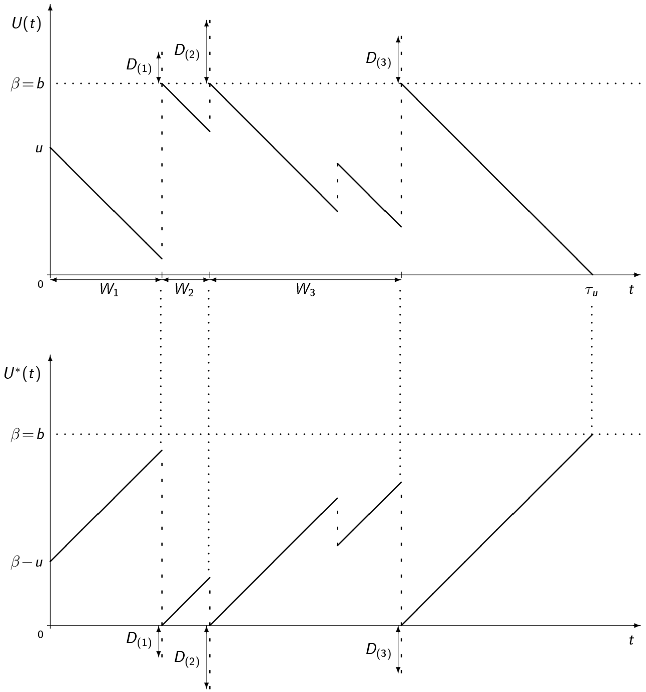


Figure: Sparre–Andersen *vs* Dual model

Conclusions

Under the assumption of some particular interclaim times distributions, it is possible to extrapolate existing results from the Sparre–Andersen Model to the Dual Model, by means of duality interpretations.

References

Afonso, L. B., Cardoso R. M. R. & Egídio dos Reis, A. D. (2013), *Dividend problems in the dual risk model*, Insurance: Mathematics and Economics 53, 906–918.

Bergel, A. I. & Egídio dos Reis, A. D. (2012). *Further developments in the Erlang(n) risk model*, Scandinavian Actuarial Journal, http://dx.doi.org/10.1080/03461238.2013.774112.

Dickson, D. C. M. (2005). *Insurance risk and ruin*, Cambridge University Press, Cambridge.

Li, S. (2008). *The time of recovery and the maximum severity of ruin in a Sparre Andersen model*, North American Actuarial Journal 12(4), 413-424.

References

Li, S. & Garrido, J. (2004), *On ruin for the Erlang(n) risk process*, Insurance: Mathematics and Economics 34(3), 391–408.

Rodríguez–Martínez E., Cardoso R. M. R. & Egídio dos Reis, A. D. (2012), *Some advances on the Erlang(n) dual risk model*, preprint.

Thank you for your attention!