Life insurance cash flows

with policyholder behaviour

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Introduction and background

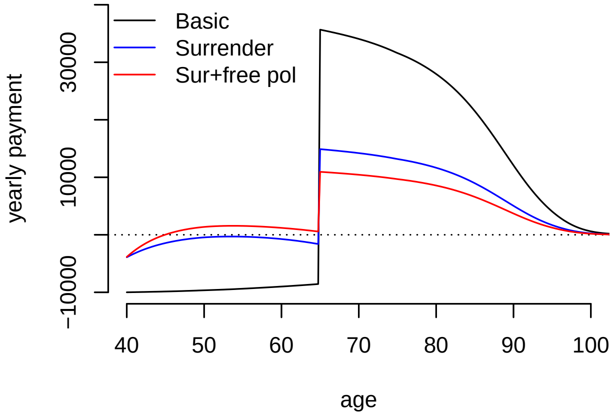
Joint work with Thomas Møller.

- Work originates from practice in PFA Pension
  - Semi-Markov setup
  - A lot of policies requires efficient formulae
  - Want to model policyholder behaviour
- Cash flows are important for hedging interest rate risk
- Requirements from Solvency II to model policyholder behaviour

- This presentation: Markov setup.
- Equivalent results exist for the Semi-Markov setup.
  - Differential eqs. are integro-differential eqs.

Main Result

1. Efficient valuation with free policy modelling
   extra duration eliminated by a modified Kolmogorov forward differential equation.



Agenda

- The (Danish) setup: With-profit products, market values and policyholder behaviour.
- Markov chain life insurance setup
- Cash flows with policyholder behaviour
  - Modification of Kolmogorov's forward diff. eq.
- Numerical illustration

The (Danish) setup: With profit products

1. policyholder pays premium(s)
2. life/pension insurance company guarantees certain benefits

Policies are valued with 2 valuation bases

Technical basis: Safe-side

Determines premiums and guaranteed payments.

- Conservative (low) interest rate r^*
- Safe-side mortality rate, disability rate, etc.

Market basis: Best estimate

Determines balance-sheet value of liabilities (guaranteed payments).

- Market inferred forward interest rate f^r.
- Best estimate mortality rate, disability rate, etc.

Policyholder behaviour

2 policyholder options

Surrender

- cancel all future payments, and
- receives policy value, according to the technical basis

Free policy (eqv. paid-up policy)

- cancel all future premiums, and
- the benefits are reduced, according to the technical basis

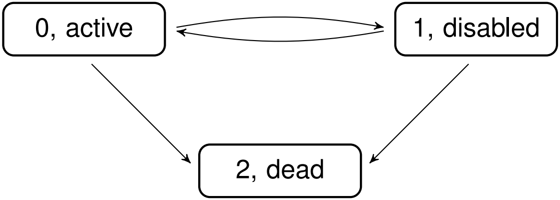
Options are based on the technical basis:
$\Rightarrow$ Introduces risk on the market basis (only).
$\Rightarrow$ Market based valuation should include policyholder behaviour.

The Markov chain Z

- Finite state space $\mathcal{J} = \{0, 1, \ldots, J\}$.
- Stochastic process $Z(t) \in \mathcal{J}$.
- Assume $Z(t)$ is Markov:

$$P(Z(s) = j \mid \mathcal{F}(t)) = P(Z(s) = j \mid Z(t)), \quad t \leq s.$$

Transition probabilities: $p_{ij}(t,s) = P(Z(s) = j \mid Z(t) = i)$.



Life insurance model

- $b_j(t)$, continuous payment rate in state j at time t.
- $b_{ij}(t)$, single payment at transition from state i to j at time t.

Cash flow valuated at t, payments at s,

$$\mathrm{d}A_i(t,s) = \sum_{j\in\mathcal{J}} p_{ij}(t,s)\left(b_j(s) + \sum_{k:k\neq j}\mu_{jk}(s)b_{jk}(s)\right)\mathrm{d}s.$$

Proposition Prospective reserve (discounted cash flow)

$$V_i(t) = \int_t^\infty e^{-\int_t^s r(\tau)\,\mathrm{d}\tau}\,\mathrm{d}A_i(t,s).$$

- Need to calculate the transition probabilities $p_{ij}(t,s)$

Kolmogorov's differential equations

- **Backward**

$$\frac{\mathrm{d}}{\mathrm{d}u} p_{ij}(u,T) = \mu_{i.}(u) p_{ij}(u,T) - \sum_{k \in \mathcal{J}, k \neq i} \mu_{ik}(u) p_{kj}(u,T).$$

  One "solve" yields $p_{ij}(u,T)$ for all u and i.

- **Forward**, $p_{ij}(t,t) = 1_{\{i=j\}}$,

$$\frac{\mathrm{d}}{\mathrm{d}u} p_{ij}(t,u) = -p_{ij}(t,u)\mu_{j.}(u) + \sum_{k \in \mathcal{J}, k \neq j} p_{ik}(t,u)\mu_{kj}(u).$$

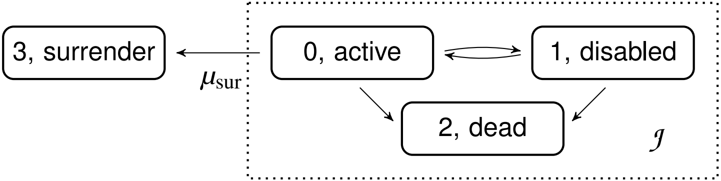
  One "solve" yields $p_{ij}(t,u)$ for all u and j.

Use forward for cash flows

$$\mathrm{d}A_i(t,s) = \sum_{j \in \mathcal{J}} p_{ij}(t,s) \left(b_j(s) + \sum_{k:k \neq j} \mu_{jk}(s) b_{jk}(s) \right) \mathrm{d}s$$

Policyholder behaviour: Surrender modelling

Model policyholder behaviour through random transitions.



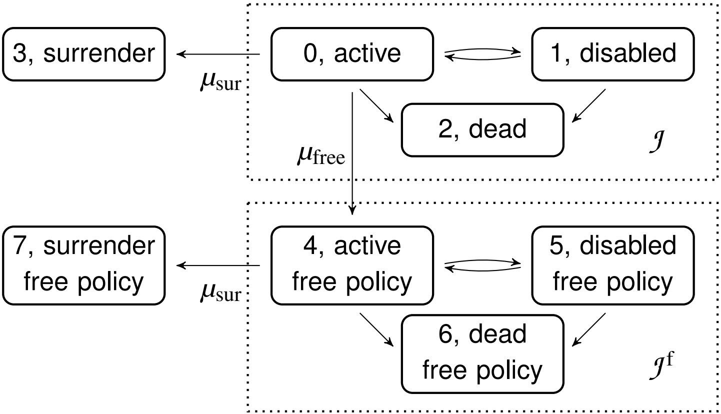
Surrender transition: payment $V_0^*(t)$

Cash flow

$$\mathrm{d}A_i^{\mathrm{s}}(t,s) = \sum_{j\in\mathcal{J}} p_{ij}(t,s)\Big(b_j(s) + \sum_{k:k\neq j} \mu_{jk}(s) b_{jk}(s)\Big)\,\mathrm{d}s$$

$$+\, p_{i0}(t,s)\mu_{\text{sur}}(s)V_0^*(s)\,\mathrm{d}s.$$

State space: Surrender & free policy



Free policy at time t:

- Premiums cancelled
- Future payments reduced by factor $\rho(t)$

Duration $U(t)$: Time since free policy conversion.

Free policy cash flow

Define ρ-modified transition probabilities,

$$p_{ij}^{\rho}(t,s) = \int_t^s p_{i0}(t,\tau)\mu_{\mathrm{free}}(\tau)\rho(\tau)p_{\mathrm{ActFree},j}(\tau,s)\,\mathrm{d}\tau.$$

Proposition The cash flow is, for $i \in \mathcal{J}$,

$$\mathrm{d}A_i^{\mathrm{fs}}(t,s) = \mathrm{d}A_i^{\mathrm{s}}(t,s) + \sum_{j\in\mathcal{J}^{\mathrm{f}}} p_{ij}^{\rho}(t,s)\Bigg(b_j(s)^+ + \sum_{\substack{k\in\mathcal{J}^{\mathrm{f}}\\ k\neq j}} \mu_{jk}(s)b_{jk}(s)^+\Bigg)\,\mathrm{d}s$$
$$+ p_{i,\mathrm{ActFree}}^{\rho}(t,s)\mu_{\mathrm{sur}}(s)V_0^{*,+}(s)\,\mathrm{d}s$$

Expensive to calculate p_{ij}^{ρ}: need a lot of transition probabilities...

p^ρ forward differential equation

Theorem $p^\rho_{ij}(t,s)$ satisfy,

$$\frac{\mathrm{d}}{\mathrm{d}s} p^\rho_{ij}(t,s) = 1_{\{j=\mathrm{ActFree}\}} p_{i0}(t,s)\mu_{\mathrm{free}}(s)\rho(s) - p^\rho_{ij}(t,s)\mu_{j.}(s) + \sum_{\substack{k\in\mathcal{J}^{\mathrm{f}}\\ k\neq j}} p^\rho_{ik}(t,s)\mu_{kj}(s)$$

$$p^\rho_{ij}(t,t) = 0.$$

Compare to Kolmogorov forward diff. eq.

$$\frac{\mathrm{d}}{\mathrm{d}s} p_{ij}(t,s) = -p_{ij}(t,s)\mu_{j.}(s) + \sum_{\substack{k\in\mathcal{J}\\ k\neq j}} p_{ik}(t,s)\mu_{kj}(s),$$

Numerics: cash flows + policyholder behaviour

Example

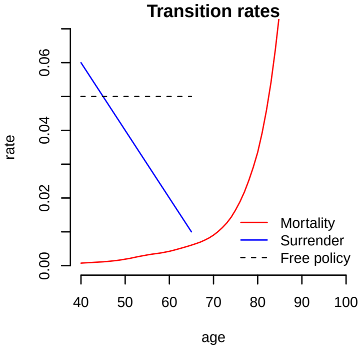
- 40 year old male
- Pension age 65
- Life annuity, size 41,534
- Premium 10,000 per year
- Savings of 100,000

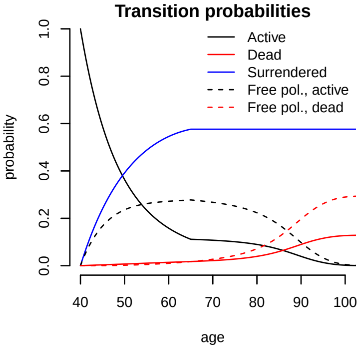
The *technical basis* consists of

- Danish G82M mortality rate
- Interest rate $r^* = 1.5\%$
- 2-state survival semi-Markov model

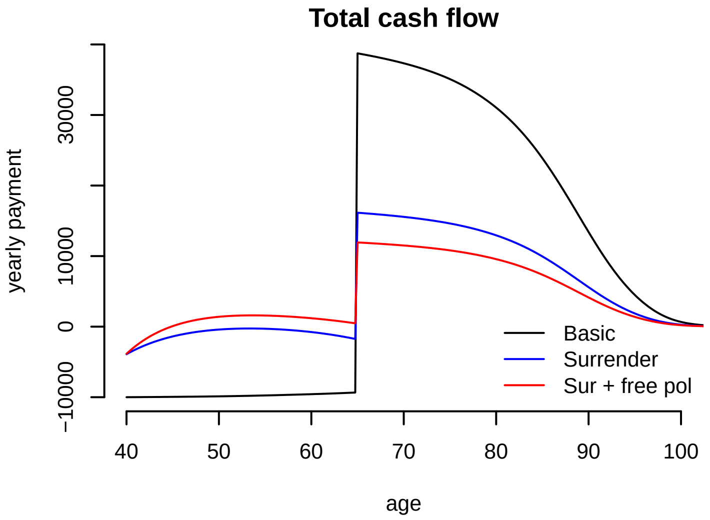
Market bases assumptions

- Danish FSA benchmark mortality
- Surrender rate: $\mu_{\text{sur}}(x) = 0.06 - 0.002 \cdot (x - 40)^+$
- Free policy rate: $\mu_{\text{free}}(x) = 0.05$

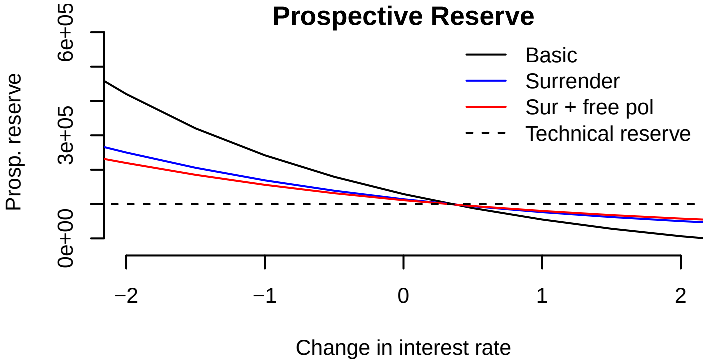




Total cash flow



Interest rate sensitivity of prospective reserves



	Basic	Surrender	Sur. and free pol.
Prospective reserve	129919	114610	111734
DV01 Total	93284	46346	38087

Conclusion

- Reviewed cash flows with the Markov chain life insurance setup
  Kolmogorov's differential equations for transition probabilities
- Cash flows efficiently calculated with policyholder behaviour
  with a modified Kolmogorov forward diff.-eq.
- Policyholder behaviour has a huge effect on cash flows.
  Essential for interest rate sensitivity analysis.
- With policyholder modelling, significantly less interest rate hedging is needed.

References

- K. Buchardt, K. B. Schmidt and T. Møller (2013), *Cash flows and policyholder behaviour in the semi-Markov life insurance setup.* to appear in Scandinavian Actuarial Journal.
- K. Buchardt and T. Møller (2013), *Life insurance cash flows with policyholder behaviour.* Preprint, Department of Mathematical Sciences, University of Copenhagen and PFA Pension.

Thank you!