Life insurance cash flows with policyholder behaviour

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Abstract

The problem of valuation of life insurance payments with policyholder behaviour is studied. First a simple survival model is considered, and it is shown how cash flows without policyholder behaviour can be modified to include surrender and free policy behaviour by calculation of simple integrals. In the second part, a more general disability model with recovery is studied. Here, cash flows are determined by solving a modified Kolmogorov differential equation. This method has been suggested recently in Buchardt et al. [2]. We conclude the paper with numerical examples illustrating the impact of modelling policyholder behaviour.

Keywords: Kolmogorov differential equations; surrender; free policy; Solvency 2

1 Introduction

In a classic multi-state life insurance setup, we consider how to include the modelling of policyholder behaviour when calculating the expected cash flows.

In this paper, the policyholder behaviour consists of two policyholder options. First, the surrender option, where the policyholder may surrender the contract cancelling all future payments and instead receiving a single payment corresponding to the value of the contract on a technical basis. Second, the free policy option[2], where the policyholder may cancel the future premiums, and have the benefits reduced according to the technical

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[2]the free policy option is sometimes referred to as a "paid-up policy" in the literature.

basis. Policyholder modelling has a significant influence on future cash flows. If the technical basis differs considerably from the market basis, policyholder behaviour can also have a substantial impact on the market value of the contract.

The policyholder behaviour is modelled as random transitions in a Markov model, and rationality behind surrender and free policy modelling is thus disregarded. In contrast, one can consider surrender and free policy exercises as rational, where they purely occur if it is benificial for the policyholder with some objective measure. For an introduction to policyholder modelling, see [7] and references therein. Attempts to couple the two approaches have been made for surrender behaviour, where surrender occurs randomly, but where the probability is somewhat controlled by rational factors, e.g. [4] and [1]. From a Solvency II point of view, the modelling of policyholder behaviour is required, see Section 3.5 in [3].

In the first part of the paper a simple survival model is considered. We calculate cash flows without policyholder behaviour as integral expressions. Then we extend the model by including first surrender behaviour and then both surrender and free policy behaviour. We see that these extensions can be obtained via simple modifications of the cash flows without policyholder behaviour. This can be viewed as a formula for extending current cash flows without policyholder behaviour. However, this modification of the cash flows is only correct for the survival model, and not for e.g. a disability model. If the method is applied to cash flows from a disability model, it could be viewed as an approximation to a more correct way of modelling policyholder behaviour. Also, we show that the cash flows with policyholder behaviour can be derived from cash flows with surrender behaviour. This method can be used in the case where one has access to cash flows with surrender behaviour but not free policy behaviour. In practice, many life insurance companies work with cash flows without policyholder behaviour, hence, the proposed method may be viewed as a simple alternative to full, combined modelling of policyholder behaviour and insurance risk. The quality of these formulae as an approximation is not assessed in this paper; This issue is studied numerically in [5], where they examine ways to simplify the calculations when modelling policyholder behaviour.

In the second part, we consider the more correct way of modelling policyholder behaviour in a multi-state life insurance setup. This model is presented in [2] for the general semi-Markov setup, and here we present the special case of a Markov process for the disability model with recovery. Within this setup, the transition probabilities are first calculated using Kolmogorov's differential equations, and then the cash flow can be determined. When including policyholder behaviour, duration dependence is introduced since the future payments are affected by the time of the free policy conversion. This complicates calculations significantly. We present the main result from [2] that allow us to effectively dismiss the duration dependence and calculate cash flows with policyholder behaviour by simply calculating a slightly modified Kolmogorov differential equation. The complexity of the calculations is therefore not increased significantly by inclusion of policyholder behaviour.

In the third part of the paper a numerical example is studied, which illustrates the importance of including policyholder modelling when valuating cash flows. We see that the structure of the cash flows changes significantly in our example, and the dollar duration measuring interest rate risk is reduced by more than 50%. For hedging of interest rate risk, it is thus essential to consider policyholder behaviour.

2 Life insurance setup

The general setup is the classic multi-state setup in life insurance, consisting of a Markov process, $\mathbf{Z}$, in a finite state space $\mathcal{J} = \{0, 1, \ldots, J\}$ indicating the state of the insured. We associate payments with sojourns in states and transitions between states, and this specifies the life insurance contract. We go through the setup and basic results; for more details, see e.g. [8], [6] or [7].

Assume that $\mathbf{Z}$ is a Markov process in $\mathcal{J}$, and that $Z(0) = 0$. The transition probabilities are defined by

$$p_{ij}(s,t) = P\left(Z(t) = j \middle| Z(s) = i\right),$$

for $i, j \in \mathcal{J}$ and $s \leq t$. Define the transition rates, for $i \neq j$,

$$\mu_{ij}(t) = \lim_{h \searrow 0} \frac{1}{h} p_{ij}(t, t+h),$$
$$\mu_{i.}(t) = \sum_{\substack{j \in \mathcal{J} \\ j \neq i}} \mu_{ij}(t).$$

We assume that these quantities exist. Define also the counting processes $N_{ij}(t)$, for $i, j \in \mathcal{J}$, $i \neq j$, counting the transitions between state i and j. They are defined by

$$N_{ij}(t) = \#\left\{s \in (0,t] \,|\, Z(s) = j, Z(s-) = i\right\},$$

where we have used the notation $f(t-) = \lim_{h \searrow 0} f(t-h)$.

The payments consist of continuous payment rates during sojourns in states, and single payments upon transitions between states. Denote by $b_i(t)$ the payment rate at time t if $Z(t) = i$, and let $b_{ij}(t)$ be the payment upon transition from state i to j at time t. Then, the accumulated payments at time t are denoted $B(t)$, and are given by

$$\mathrm{d}B(t) = \sum_{i \in \mathcal{J}} 1_{\{Z(t)=i\}} b_i(t)\,\mathrm{d}t + \sum_{\substack{i,j \in \mathcal{J} \\ i \neq j}} b_{ij}(t)\,\mathrm{d}N_{ij}(t). \tag{2.1}$$

Positive values of the payment functions $b_i(t)$ and $b_{ij}(t)$ correspond to benefits, while negative values corresponds to premiums. It is also possible to include single payments during sojourns in states, but that is for notational simplicity omitted here.

We assume that the interest rate $r(t)$ is deterministic. Then, the present value at time t of all future payments is denoted $PV(t)$, and it is given by

$$PV(t) = \int_t^\infty e^{-\int_t^s r(\tau)\,\mathrm{d}\tau}\,\mathrm{d}B(s).$$

The formula is interpreted as the sum over all future payments, $\mathrm{d}B(s)$, which are discounted by $e^{-\int_t^s r(\tau)\,\mathrm{d}\tau}$. For an actual valuation, we take the expectation conditional on the current state, $\mathrm{E}\left[PV(t)\middle|\, Z(t)=i\right]$. This expected present value is called the prospective (state-wise) reserve.

Definition 2.1. *The prospective reserve at time t for state $i \in \mathcal{J}$ is denoted $V_i(t)$, and given as*

$$V_i(t) = \mathrm{E}\left[\int_t^\infty e^{-\int_t^s r(\tau)\,\mathrm{d}\tau}\,\mathrm{d}B(s)\middle|\, Z(t)=i\right].$$

The prospective reserve can be calculated using the following classic results.

Proposition 2.2. *The prospective reserve at time t given $Z(t)=i$, $i \in \mathcal{J}$, satisfies,*

$$V_i(t) = \int_t^\infty e^{-\int_t^s r(\tau)\,\mathrm{d}\tau} \sum_{j\in\mathcal{J}} p_{ij}(t,s)\left(b_j(s) + \sum_{\substack{k\in\mathcal{J}\\ k\neq j}} \mu_{jk}(s) b_{jk}(s)\right)\mathrm{d}s.$$

Proposition 2.3. *The prospective reserve at time t given $Z(t)=i$, $i \in \mathcal{J}$, satisfies* Thiele's differential equation*,*

$$\frac{\mathrm{d}}{\mathrm{d}t}V_i(t) = r(t)V_i(t) - b_i(t) - \sum_{j\in\mathcal{J}, j\neq i} \mu_{ij}(t)\left(b_{ij}(t) + V_j(t) - V_i(t)\right),$$

with boundary conditions $V_i(\infty)=0$, for $i \in \mathcal{J}$.

Remark 2.4. If a timepoint $T \geq 0$ exists such that $b_i(t) = b_{ij}(t) = 0$ for $t > T$ and all $i,j \in \mathcal{J}$, then the boundary conditions $V_i(T)=0$ for $i \in \mathcal{J}$ are used with Thiele's differential equation. ◇

It can be convenient to calculate not only the expected present value (the prospective reserve), but also the expected cash flow. From here on, we simply refer to the expected cash flow as the cash flow, and it is a function giving the expected payments at any future time s. The cash flow is, in this setup, independent of the interest rate, and thus the cash flow can be useful for hedging and for an assessment of the interest rate risk associated with the life insurance liabilities.

Definition 2.5. *The cash flow at time t associated with the payment process $(B(t))_{t\geq 0}$, conditional on $Z(t)=i$, $i\in\mathcal{J}$, is the function $s\mapsto A_i(t,s)$, given by*

$$A_i(t,s) = \mathrm{E}\left[B(s)-B(t)\,|\,Z(t)=i\right],$$

for $s\in[t,\infty)$.

A formal calculation yields an expression for the cash flow: From Definition 2.1, we note that

$$\begin{aligned} V_i(t) &= \int_t^\infty e^{-\int_t^s r(\tau)\,\mathrm{d}\tau}\,\mathrm{d}(\mathrm{E}\left[B(s)|\,Z(t)=i\right]) \\ &= \int_t^\infty e^{-\int_t^s r(\tau)\,\mathrm{d}\tau}\,\mathrm{d}(\mathrm{E}\left[B(s)-B(t)|\,Z(t)=i\right]) \\ &= \int_t^\infty e^{-\int_t^s r(\tau)\,\mathrm{d}\tau}\,\mathrm{d}A_i(t,s), \end{aligned}$$

where we have used that $B(t)$ is a constant and doesn't change the dynamics in s. We state the result in a proposition.

Proposition 2.6. *The cash flow $A_i(t,s)$ satisfies,*

$$V_i(t) = \int_t^\infty e^{-\int_t^s r(\tau)\,\mathrm{d}\tau}\,\mathrm{d}A_i(t,s),$$

$$\mathrm{d}A_i(t,s) = \sum_{j\in\mathcal{J}} p_{ij}(t,s)\left(b_j(s)+\sum_{\substack{k\in\mathcal{J}\\ k\neq j}}\mu_{jk}(s)b_{jk}(s)\right)\mathrm{d}s.$$

The second result in Proposition 2.6 follows from the first result and from Proposition 2.2.

In order to actually calculate the cash flow, one must first calculate the transition probabilities $p_{ij}(s,t)$. In sufficiently simple models, so-called hierarchical models, where you can not return to a state after you left it, the transition probabilities can be calculated using only integrals and known functions. These kind of models are considered in Section 3. In general Markov models, closed form expressions for the transition probabilities typically do not exist. Instead, the transition probabilities can be found numerically by solving Kolmogorov's forward and backward differential equations.

Proposition 2.7. *The transition probabilities $p_{ij}(t,s)$, for $i,j\in\mathcal{J}$, are unique solutions to* Kolmogorov's backward differential equation,

$$\frac{\mathrm{d}}{\mathrm{d}t}p_{ij}(t,s) = \mu_{i.}(t)p_{ij}(t,s) - \sum_{\substack{k\in\mathcal{J}\\ k\neq i}}\mu_{ik}(t)p_{kj}(t,s),$$

with boundary conditions $p_{ij}(s,s) = 1_{\{i=j\}}$, *and* Kolmogorov's forward differential equation,

$$\frac{\mathrm{d}}{\mathrm{d}s} p_{ij}(t,s) = -p_{ij}(t,s)\mu_{j\cdot}(s) + \sum_{\substack{k \in \mathcal{J} \\ k \neq j}} p_{ik}(t,s)\mu_{kj}(s),$$

with boundary conditions $p_{ij}(t,t) = 1_{\{i=j\}}$.

Using Kolmogorov's differential equations, the transition probabilities needed in order to calculate the cash flow from Proposition 2.6 can be found. It is worth noting, that for calculating the cash flow, the forward differential equations are the easiest way to obtain the desired transition probabilities.

2.1 Technical basis and market basis

In practice and in our examples, we distinguish between calculations on the so-called technical basis, used to settle premiums, and the market basis, used the calculate the market consistent value of the life insurance liabilities, referred to as the market value. A basis is a set of assumptions used for the calculations of life insurance liabilities, and it typically consists of an interest rate $r(t)$ and a set of transition rates $(\mu_{ij}(t))_{i,j \in \mathcal{J}}$. There can also be different administration costs associated with different bases, however administration costs are not considered in this paper. The Markov model can also be different in different bases, and the policyholder behaviour modelling of this paper is an example of this. Here, policyholder behaviour is not included in the technical basis, but is included in the market basis, so the Markov models differ by the surrender and free policy states.

Throughout the paper we let $\hat{r}(t)$ and $\hat{\mu}_{ij}(t)$ be the first order interest and transition rates, respectively, i.e. the interest and transition rates associated with the technical basis. We let $r(t)$ and $\mu_{ij}(t)$ be the interest and transition rates, respectively, for the market basis. In general, values marked with a ˆ are associated with the technical basis. Thus, $V(t)$ is the prospective reserve for the market basis, and $\hat{V}(t)$ is the prospective reserve for the technical basis.

2.2 The policyholder options

We study life insurance contracts with two options for the policyholder. She can surrender the contract at any time or she can stop the premium payments and convert the policy into a so-called free policy.

If the policyholder surrenders the contract at time t, all future payments are cancelled, and instead the policyholder receives a compensation for the premiums she has paid so far. Usually, the prospective reserve calculated on the technical basis, $\hat{V}_i(t)$, is paid out, but the formula allows it to be any deterministic value. In this paper, we allow for a deductible, and say that the payment upon surrender is $(1-\kappa)\hat{V}_i(t)$. Since any deterministic value can be chosen, in particular, we can choose κ to be time dependent.

If the policyholder stops the premium payments, i.e. exercises the free policy option, all future premiums are cancelled, and the size of the benefits are decreased to account for the missing future premium payments. If the free policy option is exercised at time t, all future benefits are decreased by a factor $\rho(t)$. In order to handle this, we split the payment process in positive and negative payments, corresponding to benefits and premiums, respectively. The benefit and premium cash flows are denoted by A^+ and A^-, respectively, and are given by

$$\mathrm{d}A_i^+(t,s) = \sum_{j\in\mathcal{J}} p_{ij}(t,s)\left(b_j(s)^+ + \sum_{\substack{k\in\mathcal{J}\\ k\neq j}} \mu_{jk}(s)b_{jk}(s)^+\right)\mathrm{d}s,$$

$$\mathrm{d}A_i^-(t,s) = \sum_{j\in\mathcal{J}} p_{ij}(t,s)\left(b_j(s)^- + \sum_{\substack{k\in\mathcal{J}\\ k\neq j}} \mu_{jk}(s)b_{jk}(s)^-\right)\mathrm{d}s,$$

where the notation $f(x)^+ = \max(f(x),0)$ and $f(x)^- = \max(-f(x),0)$ for a function $f(x)$ is used. The prospective reserve can then be decomposed as well, and we have

$$V_i^+(t) = \int_t^\infty e^{-\int_t^s r(\tau)\,\mathrm{d}\tau}\,\mathrm{d}A_i^+(t,s),$$

$$V_i^-(t) = \int_t^\infty e^{-\int_t^s r(\tau)\,\mathrm{d}\tau}\,\mathrm{d}A_i^-(t,s),$$

and $V_i(t) = V_i^+(t) - V_i^-(t)$. The relations also hold on the technical basis, thus $\hat{V}_i(t) = \hat{V}_i^+(t) - \hat{V}_i^-(t)$, where $\hat{V}_i^+(t)$ and $\hat{V}_i^-(t)$ are the values of the future benefits and premiums, respectively, valuated on the technical basis.

If the free policy is exercised at time t, then at future time s, the payment rate while in state i is $\rho(t)b_i(s)^+$ and the payment if a transition from state i to j occurs, is $\rho(t)b_{ij}(s)^+$. Hence, the prospective reserve on the technical basis at time s in state i, given the free policy option is exercised at time $t \leq s$, is

$$\int_s^\infty e^{-\int_s^u \hat{r}(\tau)\,\mathrm{d}\tau}\rho(t)\,\mathrm{d}\hat{A}_i^+(s,u) = \rho(t)\hat{V}_i^+(s),$$

where $\mathrm{d}\hat{A}_i^+(s,u)$ is the cash flow calculated with the first order transition probabilities and rates, determined by $\hat{\mu}_{ij}(t)$.

The factor $\rho(t)$ should be deterministic and is usually chosen according to the equivalence principle on the technical basis: The prospective reserve for the technical basis should not change as a consequence of the exercise of the free policy option. We assume in this paper that the free policy conversion can only occur from state 0. Thus, if the free policy option is exercised at time t, the prospective reserve on the technical basis before the free policy option is exercised, $\hat{V}(t)$, should be equal to the prospective reserve after the exercise, $\rho(t)\hat{V}^{+}(t)$. Thus, we require $\hat{V}(t) = \rho(t)\hat{V}^{+}(t)$, yielding

$$\rho(t) = \frac{\hat{V}(t)}{\hat{V}^{+}(t)}.$$

Here, we omitted the subscript 0 from $\hat{V}_0(t)$, and we do that in general the rest of the paper when there is no ambiguity. We see that ρ is the value on the technical basis of benefits less premiums, divided by the value on the technical basis of the benefits only. We refer to $\rho(t)$ as the free policy factor.

3 The survival model

We consider the survival model and extend it gradually to include policyholder behaviour. First, we include the surrender option, and afterwards, we include the free policy option as well. The survival model consists of two states, 0 *(alive)* and 1 *(dead)*, corresponding to Figure 1.

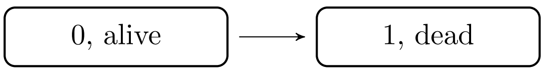

Figure 1: *Survival Markov model*

Assume the insured is x years old at time 0. The payments consist of a benefit rate $b(t)$ and a premium rate $\pi(t)$, and a payment $b_{\text{ad}}(t)$ upon death at time t. Referring to the general setup, we have

$$\begin{aligned} b_0(t) &= b(t) - \pi(t), \\ b_{01}(t) &= b_{\text{ad}}(t). \end{aligned}$$

and also, we denote the mortality intensity $\mu_{01}(t) = \mu_{\text{ad}}(t)$. The prospective reserve on the technical basis at time t in state 0 is given by Proposition 2.2, and we get

$$\hat{V}(t) = \int_t^\infty e^{-\int_t^s \hat{r}(u)\,\mathrm{d}u} {}_{s-t}\hat{p}_{x+t} \left(b(s) - \pi(s) + \hat{\mu}_{\text{ad}}(s) b_{\text{ad}}(s) \right) \mathrm{d}s,$$

We have used the actuarial notation for the survival probability, ${}_{s-t}\hat{p}_{x+t} = \hat{p}_{00}(t,s)$, and it is given by,

$$ {}_t\hat{p}_x = e^{-\int_0^t \hat{\mu}_{\mathrm{ad}}(x+u)\,\mathrm{d}u}. $$

Thus, ${}_t\hat{p}_x$ is the survival probability of an x-year old reaching age $x+t$, calculated on the technical basis.

The market values of benefits and premiums, respectively, are then given by

$$ \begin{aligned} V^+(t) &= \int_t^\infty e^{-\int_t^s r(u)\,\mathrm{d}u} {}_{s-t}p_{x+t} \left(b(s) + \mu_{\mathrm{ad}}(s) b_{\mathrm{ad}}(s)\right) \mathrm{d}s, \\ V^-(t) &= \int_t^\infty e^{-\int_t^s r(u)\,\mathrm{d}u} {}_{s-t}p_{x+t} \pi(s)\,\mathrm{d}s, \end{aligned} $$

and the associated cash flows, conditioning on being alive at time t, are

$$ \begin{aligned} \mathrm{d}A^+(t,s) &= {}_{s-t}p_{x+t} \left(b(s) + \mu_{\mathrm{ad}}(s) b_{\mathrm{ad}}(s)\right) \mathrm{d}s, \\ \mathrm{d}A^-(t,s) &= {}_{s-t}p_{x+t} \pi(s)\,\mathrm{d}s, \end{aligned} \tag{3.1} $$

with $V(t) = V^+(t) - V^-(t)$ and $\mathrm{d}A(t,s) = \mathrm{d}A^+(t,s) - \mathrm{d}A^-(t,s)$ being the total prospective reserve and cash flow, respectively. Here, we have omitted the subscript 0 from the notation $\mathrm{d}\hat{A}_0(t,s)$.

The free policy factor is determined by

$$ \rho(t) = \frac{\hat{V}(t)}{\hat{V}^+(t)}, $$

where $\hat{V}^+(t)$ is the value on the technical basis of the benefits only. If the free policy option is exercised immediately, the market value is

$$ \rho(t) V^+(t) = \frac{\hat{V}(t)}{\hat{V}^+(t)} V^+(t), $$

and in Denmark, this is often referred to as the market value of the guaranteed free policy benefits.

3.1 Survival model with surrender modelling

We continue the example from above and determine the market value including valuation of the surrender option. The Markov model is extended to include a surrender state, corresponding to Figure 2. The surrender modelling is only included in the market basis, and the valuation on the technical basis does not change.

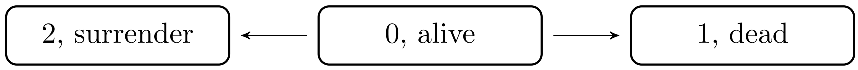

Figure 2: *Survival Markov model with surrender.*

On the market basis, we denote the surrender rate by $\mu_{\text{as}}(t)$. We introduce a quantity ${}_tp_x^{\text{s}}$ which is the probability that an x-year old does not die nor surrender before time $x+t$. It is thus the probability of staying in state 0, and is given by,

$$ {}_{s-t}p_{x+t}^{\text{s}} := p_{00}(t,s) = e^{-\int_t^s (\mu_{\text{ad}}(\tau)+\mu_{\text{as}}(\tau))\,\mathrm{d}\tau} = {}_{s-t}p_{x+t} e^{-\int_t^s \mu_{\text{as}}(\tau)\,\mathrm{d}\tau}. $$

Here, the transition rates μ_{ad} and μ_{as} are for an x-year old at time 0, which for simplicity is suppressed in the notation.

The payment upon surrender at time s is $(1-\kappa)\hat{V}(s)$, and the cash flow valuated at time t is, by Proposition 2.6,

$$ \mathrm{d}A^{\text{s}}(t,s) = {}_{s-t}p_{x+t}^{\text{s}} \Big(b(s) - \pi(s) + \mu_{\text{ad}}(s) b_{\text{ad}}(s) + \mu_{\text{as}}(s)(1-\kappa)\hat{V}(s) \Big)\,\mathrm{d}s. \tag{3.2} $$

We decompose the cash flow in all payments excluding the surrender payments,

$$ \begin{aligned} \mathrm{d}A^{\text{s1}}(t,s) &= {}_{s-t}p_{x+t}^{\text{s}} \left(b(s) - \pi(s) + \mu_{\text{ad}}(s) b_{\text{ad}}(s) \right)\,\mathrm{d}s \\ &= e^{-\int_t^s \mu_{\text{as}}(\tau)\,\mathrm{d}\tau}\,\mathrm{d}A(t,s), \end{aligned} $$

and the surrender payments,

$$ \mathrm{d}A^{\text{s2}}(t,s) = {}_{s-t}p_{x+t}^{\text{s}} \mu_{\text{as}}(s)(1-\kappa)\hat{V}(s)\,\mathrm{d}s. $$

Here, $\mathrm{d}A(t,s)$ is the cash flow from the model in Figure 1, as defined by (3.1). The market value calculated on the market basis including surrender is denoted $V^{\text{s}}(t)$, and is given by

$$ \begin{aligned} V^{\text{s}}(t) &= \int_t^\infty e^{-\int_t^s r(\tau)\,\mathrm{d}\tau} \left(\mathrm{d}A^{\text{s1}}(t,s) + \mathrm{d}A^{\text{s2}}(t,s) \right) \\ &= \int_t^\infty e^{-\int_t^s r(\tau)\,\mathrm{d}\tau} e^{-\int_t^s \mu_{\text{as}}(\tau)\,\mathrm{d}\tau}\,\mathrm{d}A(t,s) \\ &\quad + \int_t^\infty e^{-\int_t^s r(\tau)\,\mathrm{d}\tau} {}_{s-t}p_{x+t}^{\text{s}} \mu_{\text{as}}(s)(1-\kappa)\hat{V}(s)\,\mathrm{d}s. \end{aligned} \tag{3.3} $$

We see that the cash flow and market value including surrender modelling are found using the original cash flow without surrender modelling, $\mathrm{d}A(t,s)$, and multiplying the probability of no surrender $e^{-\int_t^s \mu_{\text{as}}(\tau)\,\mathrm{d}\tau}$. Thus, finding the cash flow and the market value in the survival model with surrender is particularly simple when the existing cash flow is known.

3.2 Survival model with surrender and free policy modelling

We extend the model to include free policy modelling on the market basis, and the Markov model is extended in Figure 3 to include free policy states. The mortality and surrender transition rates in the free policy states are identical to those in the premium paying states, μ_{ad} and μ_{as}.

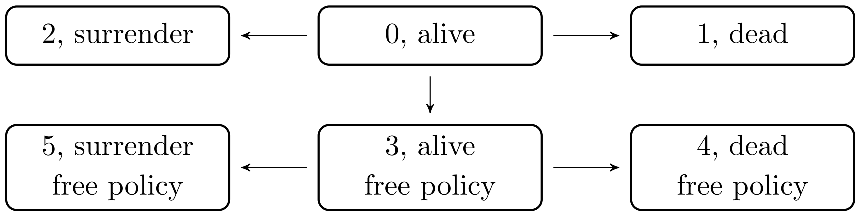

Figure 3: *Survival Markov model with surrender and free policy.*

We introduce a free policy rate $\mu_{\mathrm{af}}(t)$, which is the transition rate of becoming a free policy at time t. We introduce the notation

$$\begin{aligned}
{}_{s-t}p^{\mathrm{fs}}_{x+t} &= e^{-\int_t^s (\mu_{\mathrm{ad}}(\tau)+\mu_{\mathrm{as}}(\tau)+\mu_{\mathrm{af}}(\tau))\,\mathrm{d}\tau} \\
&= e^{-\int_t^s (\mu_{\mathrm{as}}(\tau)+\mu_{\mathrm{af}}(\tau))\,\mathrm{d}\tau}\,{}_{s-t}p_{x+t} \\
&= e^{-\int_t^s \mu_{\mathrm{af}}(\tau)\,\mathrm{d}\tau}\,{}_{s-t}p^{\mathrm{s}}_{x+t},
\end{aligned}$$

which is the probability of staying in state 0, i.e. not becoming a free policy, surrendering nor dying.

If the free policy transition occurs at time t, the future benefits are reduced by a factor $\rho(t)$, and the future premiums are cancelled. Thus, in the free policy state at a later time s, the payment rate is $\rho(t)b(s)$, and the payment upon death is $\rho(t)b_{\mathrm{ad}}(s)$. The surrender payment, if surrender occurs as a free policy, is $\rho(t)(1-\kappa)\hat{V}^+(s)$, where $\rho(t)\hat{V}^+(s)$ is the prospective reserve on the technical basis.

The payment process is dependent on the exact time of the free policy transition, i.e. the payments are dependent on the duration since the free policy transition. It can be shown that the cash flow valuated at time t is given by

$$\begin{aligned}
\mathrm{d}A^{\mathrm{f}}(t,s) = {}& {}_{s-t}p^{\mathrm{fs}}_{x+t}\Big(b(s)-\pi(s)+\mu_{\mathrm{ad}}(s)b_{\mathrm{ad}}(s)+\mu_{\mathrm{as}}(s)(1-\kappa)\hat{V}(s)\Big)\,\mathrm{d}s \\
&+\int_t^s {}_{\tau-t}p^{\mathrm{fs}}_{x+t}\mu_{\mathrm{af}}(\tau)\,{}_{s-\tau}p^{\mathrm{s}}_{x+\tau} \\
&\quad\times\Big(\rho(\tau)b(s)+\mu_{\mathrm{ad}}(s)\rho(\tau)b_{\mathrm{ad}}(s)+\mu_{\mathrm{as}}(s)\rho(\tau)(1-\kappa)\hat{V}^+(s)\Big)\,\mathrm{d}\tau\,\mathrm{d}s.
\end{aligned} \tag{3.4}$$

The result can be obtained as a special case of Proposition 4.1 below, where the disability rate is set to 0, but for completeness, a separate proof is given in Appendix A. The first line is the payments in state 0 and the payments upon death and surrender. The second and third lines contain the payments as a free policy. This expression can be interpreted as the probability of staying in state 0 until time τ, then becoming a free policy at time τ, and then neither dying nor surrendering from time τ to time s. This is multiplied with the payments as a free policy at time s, given the free policy occured at time τ. Finally, we integrate over all possible free policy transition times from s to t.

The cash flow is decomposed into four parts. First, the benefits and premiums, excluding surrender payments, while alive and not a free policy,

$$\begin{aligned} \mathrm{d}A^{\mathrm{f1}}(t,s) &= {}_{s-t}p_{x+t}^{\mathrm{fs}}\left(b(s)-\pi(s)+\mu_{\mathrm{ad}}(s)b_{\mathrm{ad}}(s)\right)\mathrm{d}s \\ &= e^{-\int_t^s(\mu_{\mathrm{as}}(u)+\mu_{\mathrm{af}}(u))\,\mathrm{d}u}\left(\mathrm{d}A^{+}(t,s)-\mathrm{d}A^{-}(t,s)\right). \end{aligned} \tag{3.5}$$

Then, the surrender payments, if the free policy transition has not occured,

$$\begin{aligned} \mathrm{d}A^{\mathrm{f2}}(t,s) &= {}_{s-t}p_{x+t}^{\mathrm{fs}}\mu_{\mathrm{as}}(s)(1-\kappa)\hat{V}(s)\,\mathrm{d}s \\ &= e^{-\int_t^s\mu_{\mathrm{af}}(u)\,\mathrm{d}u}{}_{s-t}p_{x+t}^{\mathrm{s}}\mu_{\mathrm{as}}(s)(1-\kappa)\hat{V}(s)\,\mathrm{d}s. \end{aligned} \tag{3.6}$$

Note that these cash flows correspond to the cash flows in the surrender model, but reduced with the probability of the free policy transition not happening.

The third cash flow is the benefits while a free policy

$$\begin{aligned} \mathrm{d}A^{\mathrm{f3}}(t,s) &= \int_t^s {}_{\tau-t}p_{x+t}^{\mathrm{fs}}\mu_{\mathrm{af}}(\tau)\rho(\tau){}_{s-\tau}p_{x+\tau}^{\mathrm{s}}\,\mathrm{d}\tau\left(b(s)+\mu_{\mathrm{ad}}(s)b_{\mathrm{ad}}(s)\right)\mathrm{d}s \\ &= \int_t^s {}_{\tau-t}p_{x+t}^{\mathrm{fs}}\mu_{\mathrm{af}}(\tau)\rho(\tau)e^{-\int_\tau^s\mu_{\mathrm{as}}(u)\,\mathrm{d}u}\,\mathrm{d}A^{+}(\tau,s)\,\mathrm{d}\tau, \end{aligned} \tag{3.7}$$

and the fourth cash flow is the surrender payments while a free policy,

$$\mathrm{d}A^{\mathrm{f4}}(t,s) = \int_t^s {}_{\tau-t}p_{x+t}^{\mathrm{fs}}\mu_{\mathrm{af}}(\tau)\rho(\tau){}_{s-\tau}p_{x+\tau}^{\mathrm{s}}\,\mathrm{d}\tau\cdot\mu_{\mathrm{as}}(s)(1-\kappa)\hat{V}^{+}(s)\,\mathrm{d}s. \tag{3.8}$$

The third cash flow (3.7) seems complicated, since the cash flows at time s evaluated at time τ, $\mathrm{d}A^{+}(\tau,s)$, is needed for any $\tau\in(t,s)$ and all $s\geq t$. However, a straightforward calculation yields,

$$\begin{aligned} &{}_{\tau-t}p_{x+t}^{\mathrm{fs}}\mu_{\mathrm{af}}(\tau)\rho(\tau)e^{-\int_\tau^s\mu_{\mathrm{as}}(u)\,\mathrm{d}u}\,\mathrm{d}A^{+}(\tau,s) \\ &= e^{-\int_t^\tau(\mu_{\mathrm{as}}(u)+\mu_{\mathrm{af}}(u))\,\mathrm{d}u}\mu_{\mathrm{af}}(\tau)\rho(\tau)e^{-\int_\tau^s\mu_{\mathrm{as}}(u)\,\mathrm{d}u}{}_{\tau-t}p_{x+t}\,\mathrm{d}A^{+}(\tau,s) \\ &= e^{-\int_t^\tau(\mu_{\mathrm{as}}(u)+\mu_{\mathrm{af}}(u))\,\mathrm{d}u}\mu_{\mathrm{af}}(\tau)\rho(\tau)e^{-\int_\tau^s\mu_{\mathrm{as}}(u)\,\mathrm{d}u}\,\mathrm{d}A^{+}(t,s), \end{aligned}$$

which simplifies things, and insertion of this into $\mathrm{d}A^{\mathrm{f3}}$ yields,

$$\mathrm{d}A^{\mathrm{f3}}(t,s) = \left(\int_t^s e^{-\int_t^\tau(\mu_{\mathrm{as}}(u)+\mu_{\mathrm{af}}(u))\,\mathrm{d}u}\mu_{\mathrm{af}}(\tau)\rho(\tau)e^{-\int_\tau^s\mu_{\mathrm{as}}(u)\,\mathrm{d}u}\,\mathrm{d}\tau\right)\mathrm{d}A^{+}(t,s).$$

Define the quantity

$$r^{\rho}(t,s) = \int_t^s e^{-\int_t^{\tau} \mu_{\mathrm{af}}(u)\,\mathrm{d}u} \mu_{\mathrm{af}}(\tau)\rho(\tau)\,\mathrm{d}\tau, \tag{3.9}$$

and note that

$$\begin{aligned}
\mathrm{d}A^{\mathrm{f3}}(t,s) &= r^{\rho}(t,s) e^{-\int_t^s \mu_{\mathrm{as}}(u)\,\mathrm{d}u}\,\mathrm{d}A^{+}(t,s),\\
\mathrm{d}A^{\mathrm{f4}}(t,s) &= r^{\rho}(t,s)\,{}_{s-t}p_{x+t}^{\mathrm{s}} \mu_{\mathrm{as}}(s)(1-\kappa)\hat{V}^{+}(s)\,\mathrm{d}s.
\end{aligned}$$

The market value including surrender and free policy modelling is denoted $V^{\mathrm{f}}(t)$, and it may finally be written as

$$\begin{aligned}
V^{\mathrm{f}}(t) &= \int_t^{\infty} e^{-\int_t^s r(\tau)\,\mathrm{d}\tau} \Big(\mathrm{d}A^{\mathrm{f1}}(t,s) + \mathrm{d}A^{\mathrm{f2}}(t,s) + \mathrm{d}A^{\mathrm{f3}}(t,s) + \mathrm{d}A^{\mathrm{f4}}(t,s) \Big)\\
&= \int_t^{\infty} e^{-\int_t^s r(\tau)\,\mathrm{d}\tau} e^{-\int_t^s (\mu_{\mathrm{as}}(u)+\mu_{\mathrm{af}}(u))\,\mathrm{d}u} \big(\mathrm{d}A^{+}(t,s) - \mathrm{d}A^{-}(t,s) \big)\\
&\quad + \int_t^{\infty} e^{-\int_t^s r(\tau)\,\mathrm{d}\tau}\,{}_{s-t}p_{x+t}^{\mathrm{fs}} \mu_{\mathrm{as}}(s)(1-\kappa)\hat{V}(s)\,\mathrm{d}s\\
&\quad + \int_t^{\infty} e^{-\int_t^s r(\tau)\,\mathrm{d}\tau} r^{\rho}(t,s) e^{-\int_t^s \mu_{\mathrm{as}}(u)\,\mathrm{d}u}\,\mathrm{d}A^{+}(t,s)\\
&\quad + \int_t^{\infty} e^{-\int_t^s r(\tau)\,\mathrm{d}\tau} r^{\rho}(t,s)\,{}_{s-t}p_{x+t}^{\mathrm{s}} \mu_{\mathrm{as}}(s)(1-\kappa)\hat{V}^{+}(s)\,\mathrm{d}s.
\end{aligned} \tag{3.10}$$

The last four lines in (3.10) have the following interpretation.

- The first line is the value of the original cash flow (3.1) without policyholder behaviour, reduced by the probability of not surrendering and not becoming a free policy.
- The second line is the value of the surrender payments, when not a free policy.
- The third line is the benefit payments as a free policy, i.e. the positive payments reduced with the free policy factor $\rho(\tau)$ at the time τ of the free policy transition.
- The fourth line is the surrender payments if surrender occurs after the free policy transition.

The formula gives the market value of future guaranteed payments, including valuation of the surrender and free policy options. In order to calculate the value, the following quantities are needed

- The original cash flows $\mathrm{d}A^{+}(t,s)$ and $\mathrm{d}A^{-}(t,s)$,

- The prospective reserve on the technical basis $\hat{V}^+(s)$ and $\hat{V}^-(s)$, for all future time points $s \geq t$, which allow us to determine the surrender payments and the free policy factor $\rho(s)$.
- The factor $r^\rho(t,s)$, which is a simple integral of the free policy transition rate.

3.3 Free policy modelling when surrender is already modelled

In the previous section, we found the market value including surrender and free policy modelling based on cash flows without any policyholder behaviour modelling. It is also possible to find this market value based on cash flows including surrender modelling. This could be relevant if the existing cash flows already include surrender modelling, and one wishes to modify these cash flows to include free policy modelling. Thus, we assume that the cash flow including surrender behaviour modelling, (3.2), are available, and that it is split in a cash flow associated with the benefits and a cash flow associated with premiums, i.e.

$$\begin{aligned} \mathrm{d}A^{\mathrm{s},+}(t,s) &= {}_{s-t}p^{\mathrm{s}}_{x+t}\Big(b(s) + \mu_{\mathrm{ad}}(s)b_{\mathrm{ad}}(s) + \mu_{\mathrm{as}}(s)(1-\kappa)\hat{V}^+(s)\Big)\,\mathrm{d}s, \\ \mathrm{d}A^{\mathrm{s},-}(t,s) &= {}_{s-t}p^{\mathrm{s}}_{x+t}\Big(\pi(s) + \mu_{\mathrm{as}}(s)(1-\kappa)\hat{V}^-(s)\Big)\,\mathrm{d}s. \end{aligned} \tag{3.11}$$

Note that the payment upon surrender is split between the two cash flows, through the decomposition $\hat{V}(t) = \hat{V}^+(t) - \hat{V}^-(t)$, i.e. the value of the future benefits less the value of the future premiums.

The market value with surrender modelling, but not free policy modelling, $V^{\mathrm{s}}(t)$ from (3.3), is then given by

$$V^{\mathrm{s}}(t) = \int_t^\infty e^{-\int_t^s r(\tau)\,\mathrm{d}\tau}\left(\mathrm{d}A^{\mathrm{s},+}(t,s) - \mathrm{d}A^{\mathrm{s},-}(t,s)\right).$$

We find the cash flow including free policy modelling by modifying the existing cash flows into two cash flows: One, which is reduced by the probability of not becoming a free policy, and a special free policy cash flow. With a few calculations using (3.5), (3.6) and (3.11), we see that

$$\begin{aligned} \mathrm{d}A^{\mathrm{f1}}(t,s) + \mathrm{d}A^{\mathrm{f2}}(t,s) &= {}_{s-t}p^{\mathrm{fs}}_{x+t}\Big(b(s) - \pi(s) + \mu_{\mathrm{ad}}(s)b_{\mathrm{ad}}(s) + \mu_{\mathrm{as}}(s)(1-\kappa)\hat{V}(s)\Big)\,\mathrm{d}s \\ &= e^{-\int_t^s \mu_{\mathrm{af}}(u)\,\mathrm{d}u}\left(\mathrm{d}A^{\mathrm{s},+}(t,s) - \mathrm{d}A^{\mathrm{s},-}(t,s)\right), \end{aligned}$$

and also, by (3.7), (3.8), (3.9) and (3.11),

$$\begin{aligned} &\mathrm{d}A^{\mathrm{f3}}(t,s) + \mathrm{d}A^{\mathrm{f4}}(t,s) \\ &= \int_t^s {}_{\tau-t}p^{\mathrm{fs}}_{x+t}\mu_{\mathrm{af}}(\tau)\rho(\tau)\,{}_{s-\tau}p^{\mathrm{s}}_{x+\tau}\,\mathrm{d}\tau\Big(b(s) + \mu_{\mathrm{ad}}(s)b_{\mathrm{ad}}(s) + \mu_{\mathrm{as}}(s)(1-\kappa)\hat{V}^+(s)\Big)\,\mathrm{d}s \\ &= r^\rho(t,s)\,\mathrm{d}A^{\mathrm{s},+}(t,s). \end{aligned}$$

The total cash flow is then given as

$$\mathrm{d}A^{\mathrm{f}}(t,s) = e^{-\int_t^s \mu_{\mathrm{af}}(u)\,\mathrm{d}u}\left(\mathrm{d}A^{\mathrm{s},+}(t,s) - \mathrm{d}A^{\mathrm{s},-}(t,s)\right) + r^{\rho}(t,s)\,\mathrm{d}A^{\mathrm{s},+}(t,s).$$

This cash flow can be interpreted as a weighted average between the original cash flow, reduced with the probability of not becoming a free policy, and the payments as a free policy. The payments as a free policy are the positive payments multiplied with $r^{\rho}(t,s)$. The quantity $r^{\rho}(t,s)$ is interpreted as the probability of becoming a free policy multiplied with the free policy factor $\rho(\tau)$ at the time τ of the free policy transition.

The market value from before, $V^{\mathrm{f}}(t)$, can then be calculated as

$$\begin{aligned}
V^{\mathrm{f}}(t) &= \int_t^{\infty} e^{-\int_t^s r(\tau)\,\mathrm{d}\tau}\Big(\mathrm{d}A^{\mathrm{f1}}(t,s) + \mathrm{d}A^{\mathrm{f2}}(t,s) + \mathrm{d}A^{\mathrm{f3}}(t,s) + \mathrm{d}A^{\mathrm{f4}}(t,s)\Big) \\
&= \int_t^{\infty} e^{-\int_t^s r(\tau)\,\mathrm{d}\tau} e^{-\int_t^s \mu_{\mathrm{af}}(u)\,\mathrm{d}u}\left(\mathrm{d}A^{\mathrm{s},+}(t,s) - \mathrm{d}A^{\mathrm{s},-}(t,s)\right) \\
&\quad + \int_t^{\infty} e^{-\int_t^s r(\tau)\,\mathrm{d}\tau} r^{\rho}(t,s)\,\mathrm{d}A^{\mathrm{s},+}(t,s).
\end{aligned}$$

If we only include surrender modelling, the needed extra quantities are simple integrals of the surrender rate $\mu_{\mathrm{as}}(t)$. If we in addition include free policy modelling, the free policy factor $\rho(t)$ must also be found, which requires access to future prospective reserves on the technical basis. When these are found, the market value is relatively simple to calculate.

An essential assumption for these calculations is that there are no payments after leaving the active state, i.e. that the prospective reserve is 0 in the dead and surrender states. That is, after the payment upon death or surrender, there are no future payments. If one adds a disability state, similar simple results can only be obtained if the prospective reserve is 0 in the disability state. This is typically not satisfied, and as such the methods of modifying the cash flows presented here are not applicable. However, the method may be used as an approximation to results obtained with more sophisticated policyholder behaviour in a more general model, e.g. a disability model.

4 A general disability Markov model

In this section we consider the survival model extended with a disability state, from which it is possible to recover. We extend the model further by including states for surrender and free policy, and end up with an 8-state model, see Figure 4. By solving certain ordinary differential equations for the relevant transition probabilities and a special free policy quantity, similar to r^{ρ} from (3.9), the cash flow and prospective reserve can be found.

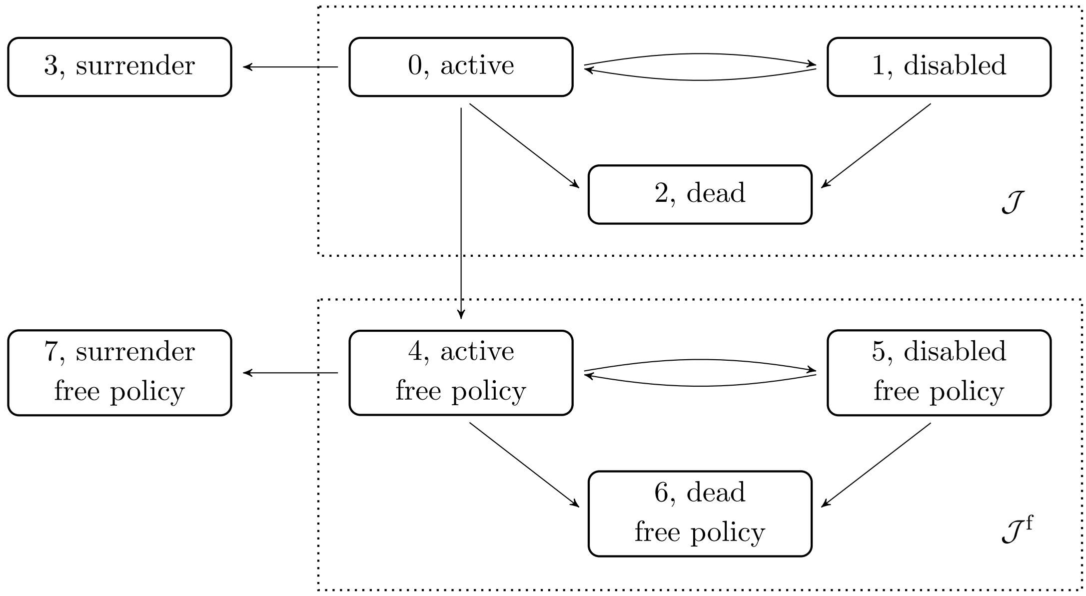

Figure 4: *The 8-state Markov model, with disability, surrender and free policy. The transition rates between states* 0*,* 1 *and* 2 *are identical to the transition rates between states* 4*,* 5 *and* 6*. The two surrender states can be considered one state, and then this model is known as the so-called "7-state model".*

The results can easily be extended to more general Markov models than the disability model, as long as free policy conversion only occurs from the active state 0. A more general setup is studied in [2], which is here specialised to the case of the survival-disability model.

For valuation on the technical basis, the survival-disability Markov model, consisting of states 0, 1 and 2, are used. In this section, the payments are labelled by the state they correspond to instead of the labels used previously. Thus, the payment rate in state 0, *active*, is $b_0(t)$ and in state 1, *disabled*, it is $b_1(t)$. Upon disability there is a payment $b_{01}(t)$, upon death as active there is a payment $b_{02}(t)$, and upon death as disabled there is a payment $b_{12}(t)$. The payment function in state 0, $b_0(t)$, is decomposed in positive payments $b_0(t)^+$, which are benefits, and negative payments, $b_0(t)^-$, which are premiums. Thus,

$$b_0(t) = b_0(t)^+ - b_0(t)^-.$$

We assume that all other payments functions are positive. The notation corresponds to the notation used in (2.1) for the payment functions $b_0(t)$, $b_1(t)$, $b_{01}(t)$, $b_{02}(t)$ and $b_{12}(t)$, and all other payment functions b_i and b_{ij} are zero. The transition rates are also labelled by numbers, e.g. the transition rate from state i to j is $\mu_{ij}(t)$.

Using Proposition 2.6, the cash flow for state 0 under the technical basis is,

$$\begin{aligned} \mathrm{d}\hat{A}(t,s) &= \hat{p}_{00}(t,s)\left(b_0(s)+\hat{\mu}_{02}(s)b_{02}(t)+\hat{\mu}_{01}(s)b_{01}(s)\right) \\ &\quad + \hat{p}_{01}(t,s)\left(b_1(s)+\hat{\mu}_{12}(s)b_{12}(s)\right), \end{aligned}$$

where the notation $\hat{p}$ and $\hat{\mu}$ refers to the transition probabilities and rates on the technical basis. The first line contains payments while in state 0, *active*, and payments during transitions out of state 0. The payments on the second line are payments in state 1, *disabled*, and payments during transitions out of state 1. We decompose the cash flow in positive and negative payments, and define,

$$\begin{aligned} \mathrm{d}\hat{A}^{+}(t,s) &= \hat{p}_{00}(t,s)\left(b_0(s)^{+}+\hat{\mu}_{02}(s)b_{02}(s)+\hat{\mu}_{01}(s)b_{01}(s)\right)\mathrm{d}s \\ &\quad + \hat{p}_{01}(t,s)\left(b_1(s)+\hat{\mu}_{12}(s)b_{12}(s)\right)\mathrm{d}s, \\ \mathrm{d}\hat{A}^{-}(t,s) &= \hat{p}_{00}(t,s)b_0(s)^{-}\,\mathrm{d}s, \end{aligned}$$

such that $\mathrm{d}\hat{A}(t,s) = \mathrm{d}\hat{A}^{+}(t,s) - \mathrm{d}\hat{A}^{-}(t,s)$. The prospective reserve on the technical basis $\hat{V}(t)$ is also decomposed,

$$\begin{aligned} \hat{V}^{+}(t) &= \int_t^{\infty} e^{-\int_t^s \hat{r}(u)\,\mathrm{d}u}\,\mathrm{d}\hat{A}^{+}(t,s), \\ \hat{V}^{-}(t) &= \int_t^{\infty} e^{-\int_t^s \hat{r}(u)\,\mathrm{d}u}\,\mathrm{d}\hat{A}^{-}(t,s), \end{aligned}$$

and we have $\hat{V}(t) = \hat{V}^{+}(t) - \hat{V}^{-}(t)$. Here we again omit the notation 0 for the state in the reserves and cash flows.

For valuation on the market basis, we consider the extended Markov model in Figure 4. We define a duration, $U(t)$, which is the time since the free policy option was exercised (or since surrender),

$$U(t) = \inf\left\{s \geq 0 \,|\, Z(t-s) \in \{0,1,2\}\right\}.$$

If the free policy option is exercised, and the current time is t, the time of the free policy transition is then $t - U(t)$. Upon transition to a free policy, the benefits are reduced by the factor $\rho(t - U(t))$, and the premiums are cancelled. The payments in the free policy states at thus duration dependent, and at time t they are,

$$\begin{aligned} b_4(t,U(t)) &= \rho(t-U(t))b_0(t)^{+}, \\ b_5(t,U(t)) &= \rho(t-U(t))b_1(t), \\ b_{45}(t,U(t)) &= \rho(t-U(t))b_{01}(t), \\ b_{46}(t,U(t)) &= \rho(t-U(t))b_{02}(t), \\ b_{56}(t,U(t)) &= \rho(t-U(t))b_{12}(t). \end{aligned}$$

Upon surrender from state 0, an amount $(1-\kappa)\hat{V}(t)$ is paid out, where $\hat{V}(t)$ is the prospective reserve on the technical basis. If the free policy option is exercised and surrender occurs from state 4, the prospective reserve on the technical basis is the value of the future benefits, reduced by the free policy factor $\rho(t-U(t))$. Thus, the payment upon surrender as a free policy is $(1-\kappa)\rho(t-U(t))\hat{V}^+(t)$. The parameter κ is a surrender strain and is usually 0. We have,

$$
\begin{aligned}
b_{03}(t) &= (1-\kappa)\hat{V}(t), \\
b_{47}(t, U(t)) &= (1-\kappa)\rho(t-U(t))\hat{V}^+(t).
\end{aligned}
$$

The total payment process is then given by,

$$
\begin{aligned}
\mathrm{d}B(t) = &\left(1_{\{Z(t)=0\}} b_0(t) + 1_{\{Z(t)=1\}} b_1(t)\right) \mathrm{d}t \\
&+ b_{01}(t)\,\mathrm{d}N_{01}(t) + b_{02}(t)\,\mathrm{d}N_{02}(t) + b_{12}(t)\,\mathrm{d}N_{12}(t) \\
&+ (1-\kappa)\hat{V}(t)\,\mathrm{d}N_{03}(t) \\
&+ \rho(t-U(t))\Big\{ \left(1_{\{Z(t)=4\}} b_0(t)^+ + 1_{\{Z(t)=5\}} b_1(t)\right) \mathrm{d}t \\
&\quad + b_{01}(t)\,\mathrm{d}N_{45}(t) + b_{02}(t)\,\mathrm{d}N_{46}(t) + b_{12}(t)\,\mathrm{d}N_{56}(t) \\
&\quad + (1-\kappa)\hat{V}^+(t)\,\mathrm{d}N_{47}(t)\Big\}.
\end{aligned}
\tag{4.1}
$$

The first two lines contain the benefits and premiums in the states 0, *alive*, 1, *disabled* and 2, *dead*. Line three contains the payment upon surrender as a premium paying policy, and line six contains the payment upon surrender as a free policy. Lines four and five contain the payments as a free policy.

We find the cash flow, and to this end it is convenient to define the quantity

$$
p_{ij}^{\rho}(t,s) = \mathrm{E}\left[1_{\{Z(s)=j\}}\rho(s-U(s)) \,\middle|\, Z(t)=i\right],
$$

for $i \in \{0,1,2\}$, $j \in \{4,5,6\}$, and $t \le s$. Then, it holds that

$$
p_{ij}^{\rho}(t,s) = \int_t^s p_{i0}(t,\tau)\mu_{04}(\tau)p_{4j}(\tau,s)\rho(\tau)\,\mathrm{d}\tau. \tag{4.2}
$$

For a proof of (4.2), see Appendix B. For $\rho(t)=1$, this quantity is simply the transition probability from state i to j: It is the probability of going form state i to 0 at time τ, and then transitioning to state 4 at time τ, and finally going from state 4 to state j from time τ to s. Since a transition from a state $i \in \{0,1,2\}$ to a state $j \in \{4,5,6\}$ can only occur through a transition from state 0 to 4, this gives the transition probabiliy. When $\rho(t) \neq 1$, the quantity corresponds to the transition probability multiplied by $\rho(t)$ at the time of transition to a free policy.

We now state the cash flow. The proof is found in Appendix C.

Proposition 4.1. *The cash flow in state 0,* $\mathrm{d}A^{\mathrm{f}}(t,s)$*, for payments at time* s *valued at time* t*, is given by*

$$\begin{aligned}\mathrm{d}A^{\mathrm{f}}(t,s) = p_{00}(t,s)&\left(b_0(s) + \mu_{01}(s)b_{01}(s) + \mu_{02}(s)b_{02}(s) + \mu_{03}(s)(1-\kappa)\hat{V}(s)\right)\,\mathrm{d}s\\ &+ p_{01}(t,s)\left(b_1(s) + \mu_{12}(s)b_{12}(s)\right)\,\mathrm{d}s\\ &+ p^{\rho}_{04}(t,s)\left(b_0(s)^{+} + \mu_{45}(s)b_{01}(s) + \mu_{46}(s)b_{02}(s) + \mu_{47}(s)(1-\kappa)\hat{V}^{+}(s)\right)\,\mathrm{d}s\\ &+ p^{\rho}_{05}(t,s)\left(b_1(s) + \mu_{56}(s)b_{12}(s)\right)\,\mathrm{d}s.\end{aligned}$$

Calculation of the cash flow requires $p^{\rho}_{ij}(t,s)$ to be calculated, and with (4.2), this requires the transition probabilities $p_{4j}(\tau,s)$ for all s and τ satisfying $t \le \tau \le s$. However, it turns out that this is not necessary, since since there exists a differential equation for $p^{\rho}_{ij}(t,s)$ similar to Kolmogorov's forward differential equation. Using this, one can calculate all the usual transition probabilities and the $p^{\rho}_{ij}(t,s)$ quantities together. This eliminates the need to calculate $p_{4j}(\tau,s)$ for all τ and s satisfying $t \le \tau \le s$.

Proposition 4.2. *The quantites* $p^{\rho}_{ij}(t,s)$ *satisfy the forward differential equations, for* $i \in \{0,1,2\}$ *and* $j \in \{4,5,6\}$*,*

$$\frac{\mathrm{d}}{\mathrm{d}s}p^{\rho}_{ij}(t,s) = 1_{\{j=4\}}p_{i0}(t,s)\mu_{04}(s)\rho(s) - p^{\rho}_{ij}(t,s)\mu_{j\cdot}(s) + \sum_{\substack{\ell\in\{4,5,6\}\\ \ell\neq j}} p^{\rho}_{i\ell}(t,s)\mu_{\ell j}(s),$$

with boundary conditions $p^{\rho}_{ij}(t,t) = 0$*.*

A more general version of this result is presented in Theorem 4.2 in [2] for the general semi-Markov case, and can also be found for the general Markov case as equation (4.8) in [2]. For completeness, a straightforward proof is given in Appendix D. For the proposition, we recall that $\mu_{j\cdot}(s)$ is the sum of all the transition rates out of state j. Note in particular, that if $j=4$, the last sum is simply the one term $p^{\rho}_{i5}(t,s)\mu_{54}(s)$, and if $j=5$, the last term is $p^{\rho}_{i4}(t,s)\mu_{45}(s)$.

The market value including surrender and free policy modelling is denoted $V^{\mathrm{f}}(t)$ and is given by,

$$V^{\mathrm{f}}(t) = \int_t^{\infty} e^{-\int_t^s r(\tau)\,\mathrm{d}\tau}\,\mathrm{d}A^{\mathrm{f}}(t,s)$$

$$
\begin{aligned}
&= \int_t^\infty e^{-\int_t^s r(\tau)\,\mathrm{d}\tau} p_{00}(t,s)\left(b_0(s)+\mu_{01}(s)b_{01}(s)+\mu_{02}(s)b_{02}(s)\right)\mathrm{d}s \\
&\quad + \int_t^\infty e^{-\int_t^s r(\tau)\,\mathrm{d}\tau} p_{01}(t,s)\left(b_1(s)+\mu_{12}(s)b_{12}(s)\right)\mathrm{d}s \\
&\quad + \int_t^\infty e^{-\int_t^s r(\tau)\,\mathrm{d}\tau} p_{00}(t,s)\mu_{03}(s)(1-\kappa)\hat{V}(s)\,\mathrm{d}s \\
&\quad + \int_t^\infty e^{-\int_t^s r(\tau)\,\mathrm{d}\tau} p^{\rho}_{04}(t,s)\left(b_0(s)^+ +\mu_{45}(s)b_{01}(s)+\mu_{46}(s)b_{02}(s)\right)\mathrm{d}s \\
&\quad + \int_t^\infty e^{-\int_t^s r(\tau)\,\mathrm{d}\tau} p^{\rho}_{05}(t,s)\left(b_1(s)+\mu_{56}(s)b_{12}(s)\right)\mathrm{d}s \\
&\quad + \int_t^\infty e^{-\int_t^s r(\tau)\,\mathrm{d}\tau} p^{\rho}_{04}(t,s)\mu_{47}(s)(1-\kappa)\hat{V}^+(s)\,\mathrm{d}s.
\end{aligned}
$$

The first three lines are the payments when the free policy option is not exercised, and the last three lines are payments as a free policy. The first two lines are the payments without policyholder behaviour which is similar to the first line in (3.10). The third line is the surrender payments when the free policy option is not exercised and this is similar to line two in (3.10). The fourth and fifth line are the payments as a free policy, without the surrender payment, corresponding to the third line in (3.10). The last line is the surrender payments as a free policy which corresponds to the fourth line in (3.10). If the disability state is removed, the formula simplifies to (3.10).

5 Numerical Example

We present a numerical example which illustrates that the modelling of policyholder behaviour may have a considerable effect on the structure of the cash flows. In particular, the interest rate sensitivity (duration) of the cash flow is significantly reduced, which is of importance if one applies duration matching techniques in order to hedge the interest rate risk. The example presented is similar to the numerical example in [2].

For simplicity, we omit disability and only consider a survival model. In the formulae, this can obtained by setting the disability transition rate equal to 0. We consider a 40 year old male with a life annuity at retirement, age 65. A premium is paid of 10,000 annually, and the current savings consist of 100,000. The technical basis consists of

- 2-state survival Markov-model, as given in Figure 1.
- Interest rate of 1.5%
- Mortality rate, the Danish G82M table, $\mu^*(x) = 0.0005 + 0.000075858 \cdot 1.09144^x$.

On the technical basis, the equivalence principle gives a life annuity of size 41,534.

We consider three different market bases, without policyholder behaviour, with surrender modelling, and with both surrender and free policy modelling. This corresponds to the Markov models in Figures 1, 2 and 3, respectively. Common for the three market bases are

- Interest rate from the Danish FSA of 8 May 2013.
- Mortality rate from the Danish FSA benchmark 2011.

For ages less than 65, the surrender and free policy transition rates are given by

$$\mu_{\mathrm{as}}(x) = 0.06 - 0.002 \cdot (x - 40)^+,$$
$$\mu_{\mathrm{af}}(x) = 0.05,$$

respectively, where x is the age, and for $x \geq 65$, the rates are 0. The transition rates loosely resemble the ones used in practice by a large Danish pension fund in the competitive market. The transition rates are shown in Figure 5 together with the transition probabilities, which have been calculated using Kolmogorov's forward differential equations, Proposition 2.7. The probability of surrender and free policy are significant, and already at age 47 the probability of having surrendered or made a free policy conversion is greater than the probability of still being active. We also note that the transition probabilities are smooth except at age 65 where the surrender and free policy transition rates drop to 0.

The left part of Figure 6 contains the premium cash flows. We see that both surrender and free policy modelling greatly reduces the premium cash flow, which is as expected, since future premiums are cancelled upon either surrender or free policy conversion. The surrender and benefit cash flow are also shown in Figure 6 (right): The payments before age 65 are the payments from surrender, and in particular we see that in the basic model where surrender is not modelled, there are no payments before age 65. After age 65, it is no longer possible to surrender, and there is only the life annuity. We see that the introduction of surrender modelling reduces the life annuity part of the cash flow, which is replaced by a significant amount of surrender payments. With free policy modelling, both the benefits and the surrender payments are reduced further, since there is less premiums, and thus smaller surrender payments and benefits. In Figure 7, the total cash flows are shown. They are calculated as the sum of the premium, benefit and surrender cash flows. Both the positive and negative parts are significantly reduced with policyholder behaviour, but since it affects both parts, the change in the market value as measured by the prospective reserve is not of the same magnitude. However, the change in the structure has a significant effect on the interest rate sensitivity, which is seen in the right part of Figure 7. Without policyholder behaviour, the prospective reserve is

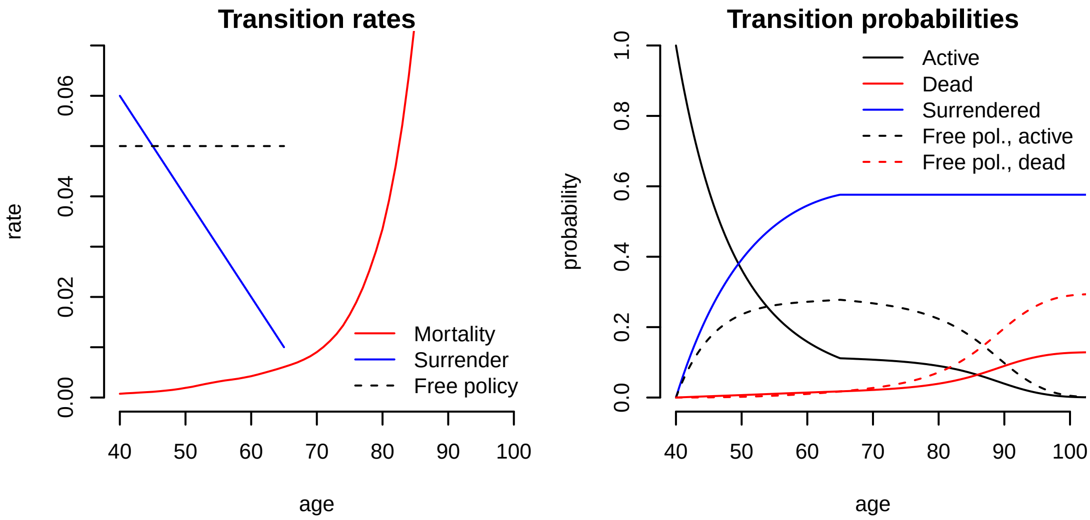

Figure 5: *Transition rates (left) and transition probabilities (right) in the Markov model with surrender and free policy modelling. The surrender rate decreases linearlly and the free policy rate is constant. Policyholder behaviour is seen to be quite significant, and already at age 47 is the probability of either surrender or free policy greater than the probability of still being active.*

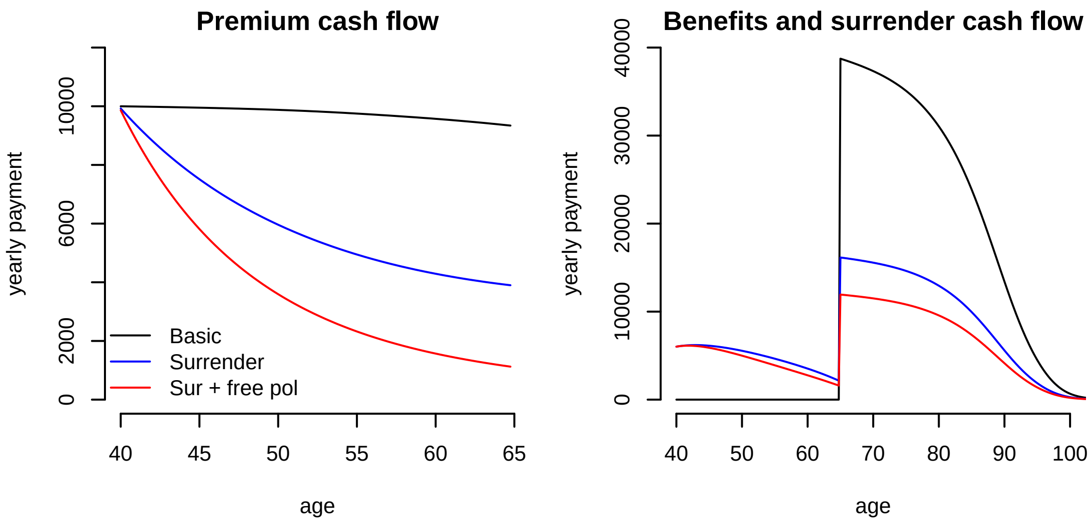

Figure 6: *Cash flows of premiums (left) and benefits (right). The premiums are significantly reduced when including surrender and free policy, as expected. The benefits are also significantly reduced, and we see that the surrender payments appear before age 65.*

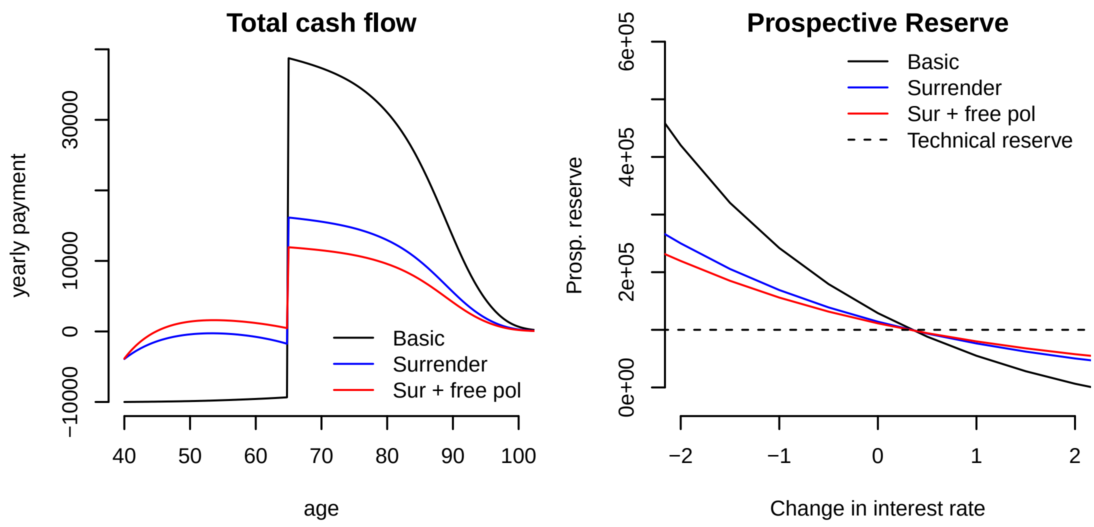

Figure 7: *Total cash flow (left) and prospective reserve plotted against parallel shifts in the market interest rate structure (right). The total cash flows are numerically significantly smaller with policyholder behaviour. To the right the effect of this change is seen on the interest rate sensitivity, which is significantly less with policyholder behaviour.*

more sensitive to changes in the interest rate, and this sensitivity is significantly lowered with policyholder behaviour. We see, that a little above the current market interest rate level, the three lines intersect. Thus, at this level of market interest rates, the cash flow modelling does not have a great effect on the prospective reserve.

	Basic	Surrender	Sur. and free pol.
Prospective reserve	129,919	114,610	111,734
DV01 Total	93,284	46,346	38,087
DV01 Pos. payments	115,550	62,044	46,625
DV01 Premiums	22,266	15,697	8,538

Table 1: *Prospective reserves and dollar durations (DV01), with and without policyholder behaviour. The duration is greatly reduced when policyholder behaviour is included, both for the total cash flows and also for the positive payments and premiums separately.*

In Table 1, the prospective reserve is shown together with the dollar duration (DV01), which measures the change in the prospective reserve for a 100 basis point change in the interest rate structure. The prospective reserve is reduced with policyholder behaviour, as could also be seen from Figure 7. In the example, surrender modelling reduces the dollar duration by approximately 50%, and free policy modelling reduces it by another 18%. Thus, if in practice one applies duration matching techniques in order to hedge the life insurance liabilities, it is essential to take into account both surrender and free policy modelling.

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A Cash flow for Section 3.2

We here prove the formula (3.4) for the model presented in Figure 3. Define first the duration $U(t)$ since entering the free policy state, that is

$$U(t) = \inf \{ s \geq 0 \, | Z(t-s) \in \{0,1,2\} \} \, .$$

Now, the payments in the setup lead to the payment process

$$\begin{aligned} &\mathrm{d}B(t) \\ &= 1_{\{Z(t)=0\}}(b(t) - \pi(t))\,\mathrm{d}t + b_{\mathrm{ad}}(t)\,\mathrm{d}N_{\mathrm{ad}}(t) + (1-\kappa)\hat{V}(t)\,\mathrm{d}N_{\mathrm{as}}(t) \\ &\quad + \rho(t - U(t))\left(1_{\{Z(t)=3\}}b(t)\,\mathrm{d}t + b_{\mathrm{ad}}(t)\,\mathrm{d}N_{\mathrm{af,df}}(t) + (1-\kappa)\hat{V}^{+}(t)\,\mathrm{d}N_{\mathrm{af,sf}}(t)\right). \end{aligned}$$

Here, N_{ad} is the counting process that counts the number of jumps from state *active* to state *dead*. Similarly, N_{as}, $N_{\mathrm{af,df}}$ and $N_{\mathrm{af,sf}}$ counts the number of jumps from state *active* to *surrender*, from state *active, free policy* to *dead, free policy*, and from state *active, free policy* to *surrender, free policy*, respectively.

The cash flow is then given as

$$\begin{aligned} &\int_t^T \mathrm{d}A^{\mathrm{f}}(t,s) \\ &= \mathrm{E}\left[\int_t^T \mathrm{d}B(s) \middle| \, Z(t) = 0\right] \\ &= \mathrm{E}\left[\int_t^T 1_{\{Z(s)=0\}}(b(s) - \pi(s))\,\mathrm{d}s \middle| \, Z(t) = 0\right] \\ &\quad + \mathrm{E}\left[\int_t^T b_{\mathrm{ad}}(s)\,\mathrm{d}N_{\mathrm{ad}}(s) + (1-\kappa)\hat{V}(s)\,\mathrm{d}N_{\mathrm{as}}(s) \middle| \, Z(t) = 0\right] \\ &\quad + \mathrm{E}\left[\int_t^T \rho(s - U(s))1_{\{Z(s)=3\}}b(s)\,\mathrm{d}s \middle| \, Z(t) = 0\right] \\ &\quad + \mathrm{E}\left[\int_t^T \rho(s - U(s))\left(b_{\mathrm{ad}}(s)\,\mathrm{d}N_{\mathrm{af,df}}(s) + (1-\kappa)\hat{V}^{+}(s)\,\mathrm{d}N_{\mathrm{af,sf}}(s)\right) \middle| \, Z(t) = 0\right] \\ &= \int_t^T {}_{s-t}p_{x+t}^{\mathrm{fs}}(b(s) - \pi(s))\,\mathrm{d}s \\ &\quad + \int_t^T {}_{s-t}p_{x+t}^{\mathrm{fs}}\left(b_{\mathrm{ad}}(s)\mu_{\mathrm{ad}}(s) + (1-\kappa)\hat{V}(s)\mu_{\mathrm{as}}(s)\right)\,\mathrm{d}s \\ &\quad + \mathrm{E}\left[\int_t^T \rho(s - U(s))1_{\{Z(s)=3\}}b(s)\,\mathrm{d}s \middle| \, Z(t) = 0\right] \\ &\quad + \mathrm{E}\left[\int_t^T \rho(s - U(s))\left(b_{\mathrm{ad}}(s)\,\mathrm{d}N_{\mathrm{af,df}}(s) + (1-\kappa)\hat{V}^{+}(s)\,\mathrm{d}N_{\mathrm{af,sf}}(s)\right) \middle| \, Z(t) = 0\right]. \end{aligned}$$

For the first expectation, we used that the expectation of an indicator function is a probability. For the second expectation, we recall that the counting process here can be replaced by the predictable compensator.

For the third expectation, we condition on the stochastic variable $s - U(s)$, which is the time of transition. Then, conditional on $Z(t) = 0$, and with the indicator function $1_{\{Z(s)=3\}}$, we know that a transition has occured, thus $s - U(s) \in (t, s)$, which determine the integral limits. Also, the density of the time of the transition from state 0 to state 3 is $\tau \mapsto {}_{\tau-t}p^{\mathrm{fs}}_{x+t}\mu_{\mathrm{af}}(\tau)$. Using these observations, we calculate

$$\begin{aligned}
&\int_t^T \mathrm{E}\left[\rho(s-U(s))1_{\{Z(s)=3\}}b(s)\,\middle|\, Z(t)=0\right]\,\mathrm{d}s \\
&= \int_t^T \int_t^s \mathrm{E}\left[\rho(\tau)1_{\{Z(s)=3\}}b(s)\,\middle|\, Z(t)=0, s-U(s)=\tau\right] \\
&\qquad \times \,\mathrm{d}P\left(s-U(s)\le\tau\,\middle|\, Z(t)=0\right)\,\mathrm{d}s \\
&= \int_t^T \int_t^s \rho(\tau)\,\mathrm{E}\left[1_{\{Z(s)=3\}}\,\middle|\, Z(\tau)=3\right] b(s)\,{}_{\tau-t}p^{\mathrm{fs}}_{x+t}\mu_{\mathrm{af}}(\tau)\,\mathrm{d}\tau\,\mathrm{d}s \\
&= \int_t^T \int_t^s \rho(\tau)\,{}_{s-\tau}p^{\mathrm{s}}_{x+\tau}b(s)\,{}_{\tau-t}p^{\mathrm{fs}}_{x+t}\mu_{\mathrm{af}}(\tau)\,\mathrm{d}\tau\,\mathrm{d}s \\
&= \int_t^T \int_t^s {}_{\tau-t}p^{\mathrm{fs}}_{x+t}\mu_{\mathrm{af}}(\tau)\rho(\tau)\,{}_{s-\tau}p^{\mathrm{s}}_{x+\tau}\,\mathrm{d}\tau\, b(s)\,\mathrm{d}s.
\end{aligned} \tag{A.1}$$

For the fourth expectation, we only consider the first part, since the second part is analogous. Since $U(s)$ is continuous whenever $N_{\mathrm{af,df}}(s)$ (and $N_{\mathrm{af,sf}}(s)$) increase in value, we can replace $U(s)$ by $U(s-)$. Using that $\rho(s-U(s-))$ is predictable, we can integrate with respect to the compensator of the counting process $N_{\mathrm{af,df}}(s)$ instead, so we get

$$\begin{aligned}
&\mathrm{E}\left[\int_t^T \rho(s-U(s))b_{\mathrm{ad}}(s)\,\mathrm{d}N_{\mathrm{af,df}}(s)\,\middle|\, Z(t)=0\right] \\
&= \mathrm{E}\left[\int_t^T \rho(s-U(s-))b_{\mathrm{ad}}(s)\,\mathrm{d}N_{\mathrm{af,df}}(s)\,\middle|\, Z(t)=0\right] \\
&= \mathrm{E}\left[\int_t^T \rho(s-U(s-))b_{\mathrm{ad}}(s)1_{\{Z(s-)=3\}}\mu_{\mathrm{ad}}(s)\,\mathrm{d}s\,\middle|\, Z(t)=0\right] \\
&= \int_t^T \mathrm{E}\left[\rho(s-U(s-))1_{\{Z(s-)=3\}}\,\middle|\, Z(t)=0\right] b_{\mathrm{ad}}(s)\mu_{\mathrm{ad}}(s)\,\mathrm{d}s \\
&= \int_t^T \int_t^s {}_{\tau-t}p^{\mathrm{fs}}_{x+t}\mu_{\mathrm{af}}(\tau)\rho(\tau)\,{}_{s-\tau}p^{\mathrm{s}}_{x+\tau}\,\mathrm{d}\tau\, b_{\mathrm{ad}}(s)\mu_{\mathrm{ad}}(s)\,\mathrm{d}s.
\end{aligned}$$

Since in the second last line, the expression is analogous to the third expectation, the last line was obtained using the same calculations as (A.1). Gathering the results, the cash flow $\mathrm{d}A^{\mathrm{f}}(t,s)$ is obtained.

B Proof of equation (4.2)

Conditioning on the time of transition from state 0 to 4, $s-U(s)$, and using that the density for the transition time is $p_{i0}(t,\tau)\mu_{04}(\tau)$, we find

$$
\begin{aligned}
&p_{ij}^{\rho}(t,s)\\
&= \mathrm{E}\left[1_{\{Z(s)=j\}}\rho(s-U(s))\middle| Z(t)=i\right]\\
&= \int_t^s \mathrm{E}\left[1_{\{Z(s)=j\}}\rho(s-U(s))\middle| Z(t)=i, s-U(s)=\tau\right]\,\mathrm{d}P\left(s-U(s)\le\tau\middle| Z(t)=i\right)\\
&= \int_t^s \mathrm{E}\left[1_{\{Z(s)=j\}}\middle| Z(t)=i, s-U(s)=\tau\right]\rho(\tau)p_{i0}(t,\tau)\mu_{04}(\tau)\,\mathrm{d}\tau\\
&= \int_t^s p_{i0}(t,\tau)\mu_{04}(\tau)\,\mathrm{E}\left[1_{\{Z(s)=j\}}\middle| Z(\tau)=4\right]\rho(\tau)\,\mathrm{d}\tau\\
&= \int_t^s p_{i0}(t,\tau)\mu_{04}(\tau)p_{4j}(\tau,s)\rho(\tau)\,\mathrm{d}\tau.
\end{aligned}
$$

At line five we used that if we know that $s-U(s)=\tau$, then in particular, we know that $Z(\tau)=4$. Since $\mathbf{Z}$ is Markov, we can then drop the condition that $Z(t)=i$ and $s-U(s)=\tau$. Note, that for the proof, it is essential that $i\in\{0,1,2\}$ and $j\in\{4,5,6\}$, since we at the conditioning on line three use that $s-U(s)\in(t,s)$, i.e. a transition to the free policy states occurs in the time interval (t,s). We can do that, since it must hold if $Z(t)=i\in\{0,1,2\}$ and $Z(s)=j\in\{4,5,6\}$.

C Proof of Proposition 4.1

Proof. The cash flow is given as

$$\int_t^T \mathrm{d}A^{\mathrm{f}}(t,s) = \mathrm{E}\left[\int_t^T \mathrm{d}B(s)\middle| Z(t)=0\right],$$

where $B(t)$ is given in (4.1). Inserting $B(t)$ yields,

$$
\begin{aligned}
&\int_t^T \mathrm{d}A^{\mathrm{f}}(t,s)\\
&= \int_t^T \mathrm{E}\left[\left(1_{\{Z(s)=0\}}b_0(s)+1_{\{Z(s)=1\}}b_1(s)\right)\middle| Z(t)=0\right]\mathrm{d}s &\text{(C.1)}\\
&\quad+\int_t^T \mathrm{E}\left[\rho(s-U(s))\left(1_{\{Z(s)=4\}}b_0(s)^{+}+1_{\{Z(s)=5\}}b_1(s)\right)\middle| Z(t)=0\right]\mathrm{d}s &\text{(C.2)}\\
&\quad+\mathrm{E}\Bigg[\int_t^T \Big(b_{01}(s)\,\mathrm{d}N_{01}(s)+b_{02}(s)\,\mathrm{d}N_{02}(s)+b_{12}(s)\,\mathrm{d}N_{12}(s)\\
&\qquad+(1-\kappa)\hat{V}(s)\,\mathrm{d}N_{03}(s)\Big)\Bigg| Z(t)=0\Bigg] &\text{(C.3)}
\end{aligned}
$$

$$+\,\mathrm{E}\left[\int_t^T \rho(s-U(s))\Big(b_{01}(s)\,\mathrm{d}N_{45}(s)+b_{02}(s)\,\mathrm{d}N_{46}(s)+b_{12}(s)\,\mathrm{d}N_{56}(s) + (1-\kappa)\hat{V}^+(s)\,\mathrm{d}N_{47}(s)\Big)\Big| Z(t)=0\right] \tag{C.4}$$

The four expectations (C.1) – (C.4) are calculated separately. The first expectation (C.1) is the expectation of indicator functions, and we replace by the transition probabilities,

$$\int_t^T \left(p_{00}(t,s)b_0(t)+p_{01}(t,s)b_1(t)\right)\,\mathrm{d}s.$$

In the second expectation (C.2), the same calculations as in Section B can be performed to obtain,

$$\int_t^T \left(p^\rho_{04}(t,s)b_0(s)^+ + p^\rho_{05}(t,s)b_1(s)\right)\,\mathrm{d}s.$$

In the third expectation (C.3), we integrate deterministic functions with respect to a counting process. Taking the expectation, we can instead integrate with respect to the predictable compensator, and we get,

$$\begin{aligned}
&\int_t^T \mathrm{E}\Big[1_{\{Z(s-)=0\}}\left(b_{01}(s)\mu_{01}(s)+b_{02}(s)\mu_{02}(s)\right)+1_{\{Z(s-)=1\}}b_{12}(s)\mu_{12}(s)\\
&\quad + 1_{\{Z(s-)=0\}}(1-\kappa)\hat{V}(s)\mu_{03}(s)\,\mathrm{d}s\Big| Z(t)=0\Big]\,\mathrm{d}s\\
&=\int_t^T \Big(p_{00}(t,s)\left(b_{01}(s)\mu_{01}(s)+b_{02}(s)\mu_{02}(s)\right)+p_{01}(t,s)b_{12}(s)\mu_{12}(s)\\
&\quad + p_{00}(t,s)(1-\kappa)\hat{V}(s)\mu_{03}(s)\Big)\,\mathrm{d}s.
\end{aligned}$$

For the fourth expectation (C.4), we start by replacing $U(s)$ with $U(s-)$, since whenever any of $N_{45}(s)$, $N_{46}(s)$, $N_{56}(s)$ or $N_{47}(s)$ are increasing, then $U(s)$ is continuous. Thus, we integrate a predictable process with respect to a counting process, and we can integrate with respect to the predictable compensator instead,

$$\begin{aligned}
&\mathrm{E}\Big[\int_t^T \rho(s-U(s-))\Big(b_{01}(s)\,\mathrm{d}N_{45}(s)+b_{02}(s)\,\mathrm{d}N_{46}(s)\\
&\quad + b_{12}(s)\,\mathrm{d}N_{56}(s)+(1-\kappa)\hat{V}^+(s)\,\mathrm{d}N_{47}(s)\Big)\Big| Z(t)=0\Big]\\
&=\int_t^T \mathrm{E}\Big[\rho(s-U(s-))\Big(1_{\{Z(s-)=4\}}\left(b_{01}(s)\mu_{45}(s)+b_{02}(s)\mu_{46}(s)\right)\\
&\quad + 1_{\{Z(s-)=5\}}b_{12}(s)\mu_{56}(s)+1_{\{Z(s-)=4\}}(1-\kappa)\hat{V}^+(s)\mu_{47}(s)\Big)\Big| Z(t)=0\Big]\,\mathrm{d}s\\
&=\int_t^T \Big(p^\rho_{04}(t,s)\left(b_{01}(s)\mu_{45}(s)+b_{02}(s)\mu_{46}(s)\right)\\
&\quad + p^\rho_{05}(t,s)b_{12}(s)\mu_{56}(s)+p^\rho_{04}(t,s)(1-\kappa)\hat{V}^+(s)\mu_{47}(s)\Big)\,\mathrm{d}s.
\end{aligned}$$

For the last equality, we again used the calculations from Section B. Gathering the four expectations, the result is obtained. □

D Proof of Proposition 4.2

Proof. We differentiate $p^{\rho}_{ij}(t,s)$ for $i \in \{0,1,2\}$ and $j \in \{4,5,6\}$,

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}s} p^{\rho}_{ij}(t,s) \\
&= \frac{\mathrm{d}}{\mathrm{d}s} \int_t^s p_{i0}(t,\tau)\mu_{04}(\tau)p_{4j}(\tau,s)\rho(\tau)\,\mathrm{d}\tau \\
&= p_{i0}(t,s)\mu_{04}(s)p_{4j}(s,s)\rho(s) + \int_t^s p_{i0}(t,\tau)\mu_{04}(\tau)\frac{\mathrm{d}}{\mathrm{d}s}p_{4j}(\tau,s)\rho(\tau)\,\mathrm{d}\tau \\
&= 1_{\{j=4\}} p_{i0}(t,s)\mu_{04}(s)\rho(s) \\
&\quad + \int_t^s p_{i0}(t,\tau)\mu_{04}(\tau)\left(-p_{4j}(\tau,s)\mu_{j\cdot}(s) + \sum_{\substack{\ell\in\{4,5,6\} \\ \ell\neq j}} p_{4\ell}(\tau,s)\mu_{\ell j}(s)\right)\rho(\tau)\,\mathrm{d}\tau \\
&= 1_{\{j=4\}} p_{i0}(t,s)\mu_{04}(s)\rho(s) - p^{\rho}_{ij}(t,s)\mu_{j\cdot}(s) + \sum_{\substack{\ell\in\{4,5,6\} \\ \ell\neq j}} p^{\rho}_{i\ell}(t,s)\mu_{\ell j}(s).
\end{aligned}
$$

For the third equality sign, we used that $p_{4\ell}(\tau,s)\mu_{\ell j}(s) = 0$ for $\ell \notin \{4,5,6\}$. □