The Modelling of Rare Events:
from methodology to practice and back

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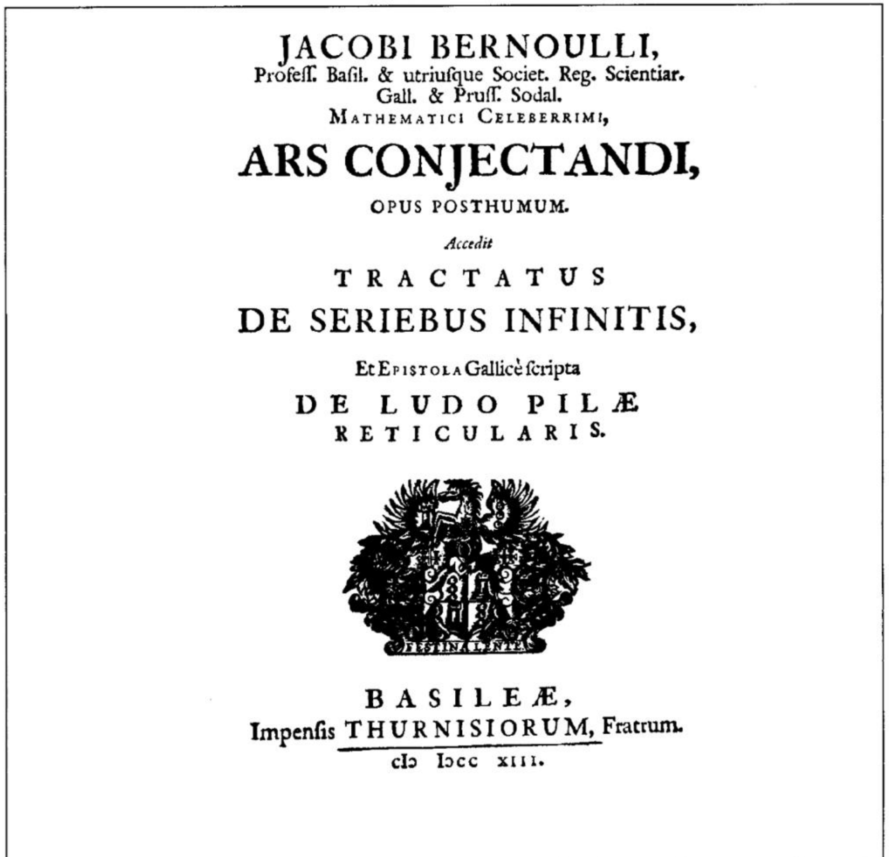
The menu:

- Some EVT history
- Risk measures
- The Pickands – Balkema – de Haan Theorem
- An example: CIs and profile likelihood
- Karamata slow/regular variation
- Rates of convergence in EVT
- An example: micro correlation
- Communication
- Discussion and final example

1713 - 2013



JACOBI BERNOULLI,
Profess. Basil. & utriusque Societ. Reg. Scientiar.
Gall. & Pruss. Sodal.
MATHEMATICI CELEBERRIMI,

ARS CONJECTANDI,

OPUS POSTHUMUM.

Accedit

TRACTATUS
DE SERIEBUS INFINITIS,

Et EPISTOLA Gallicè scripta
DE LUDO PILÆ
RETICULARIS.

FESTINA LENTE

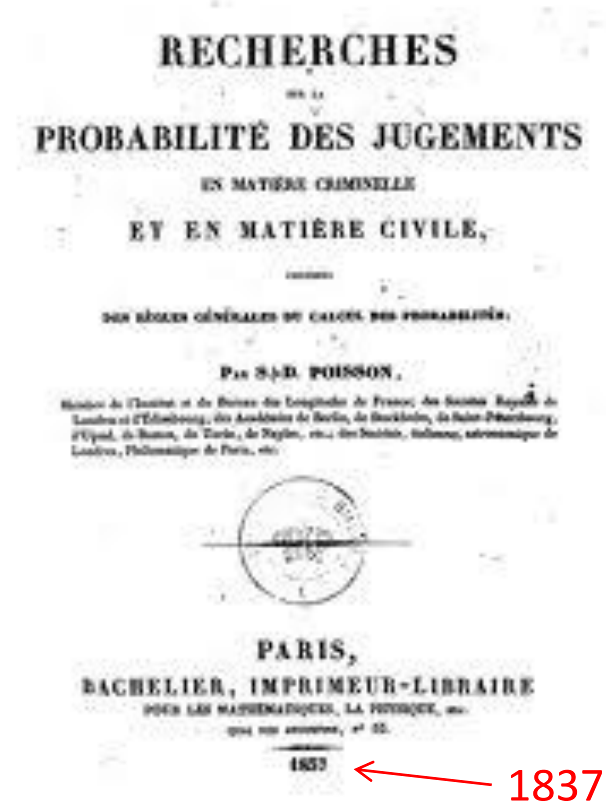
BASILEÆ,
Impensis THURNISIORUM, Fratrum.
cIↄ Iↄcc xiii.

Perhaps the first:

Nicolaus Bernoulli (1687 – 1759) who, in 1709, considered the actuarial problem of calculating the mean duration of life of the last survivor among n men of equal age who all die within t years. He reduced this question to the following:

n points lie at random on a straight line of length t, calculate the mean largest distance from the origin.

Often quoted as the start:



RECHERCHES

PROBABILITÉ DES JUGEMENTS

EN MATIÈRE CRIMINELLE

ET EN MATIÈRE CIVILE,

Par S.-D. POISSON,

PARIS,

BACHELIER, IMPRIMEUR-LIBRAIRE

1837

1837 (p 206)

Simon Denis Poisson (1781 – 1840)

(however Cotes (1714), de Moivre (1718), ...)

However, the real start with relevance to Extreme Value Theory (EVT) was given by:

Ladislaus J. von Bortkiewicz

(1868 – 1931)

The Law of Small Numbers

DAS GESETZ

DER

KLEINEN ZAHLEN

VON

Dr. L. VON BORTKEWITSCH

PRIVATDOZENT IN STRASSBURG

LEIPZIG

DRUCK UND VERLAG VON B. G. TEUBNER

1898

1898

(Prussian army horse-kick data)

Then came numerous developments in the early to mid 20th century with famous names like ...

A.Y. Khinchin

R.A. Fisher

L.H.C. Tippett

M.R. Fréchet

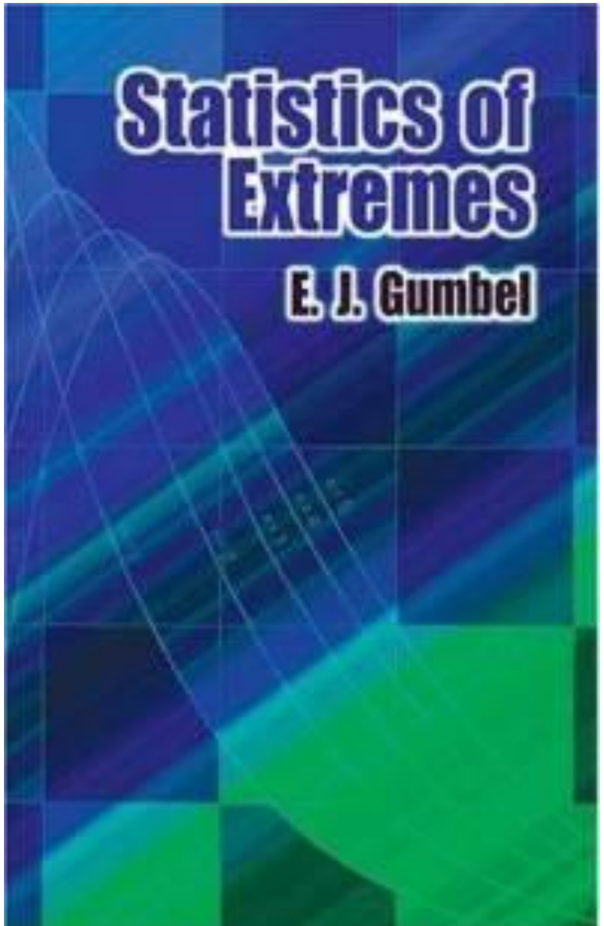
E.H.W. Weibull

E.J. Gumbel

R. von Mises

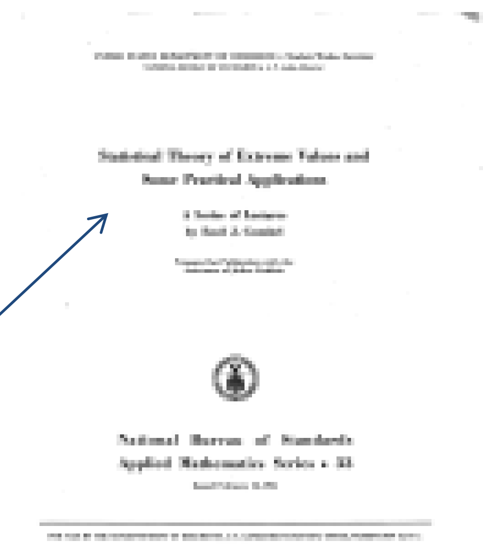
B.V. Gnedenko · · ·

... with as textbook summary:



(1958)

Emil Julius Gumbel
(1891 – 1966)



Statistical Theory of Extreme Values and Some Practical Applications.
National Bureau of Standards, 1954

The later-20th Century, some names:

Laurens de Haan

Sidney I. Resnick

Richard L. Smith

M. Ross Leadbetter

and so many more ...

a second half 20th Century explosion!

When discussing extremes,

Practice is too often frequency oriented ...

- every so often (rare event)
- return period, «once in so many years» event
- Value-at-Risk (VaR) in financial RM (Basel II/III)

... rather than more relevant severity orientation

- what if
- loss size given the occurence of a rare event
- Expected Shortfall E[X I X > VaR] (SST)

This is not just about theory but an attitude! (EVT)

This is well appreciated in the reinsurance world:

- XL-treaties: for a yearly loss variable X one is interested in estimating an attachment point u so that the probability of exceeeding that threshold u is sufficiently small, for instance corresponds to a 1-in-250 year event; mathematically this translates into calculating (estimating) a lower attachment point u in

$$P(X > u) = 1/250$$

but also $F_u(x) = P(X - u \leq x \mid X > u) = ???$

Excess-of-loss

This leads to «The Pickands-Balkema-de Haan Theorem», sometimes quoted as the most important mathematical result for reinsurance!

(Gary Patrick, an American reinsurance actuary)

Asymptotics of Excess Distribution

Theorem. (Pickands–Balkema–de Haan (1974/75)) We can find a positive, measurable function $\beta(u)$ such that

$$\lim_{u \to x_F} \sup_{0 \le x < x_F - u} \left| F_u(x) - G_{\xi,\beta(u)}(x) \right| = 0 \,,$$

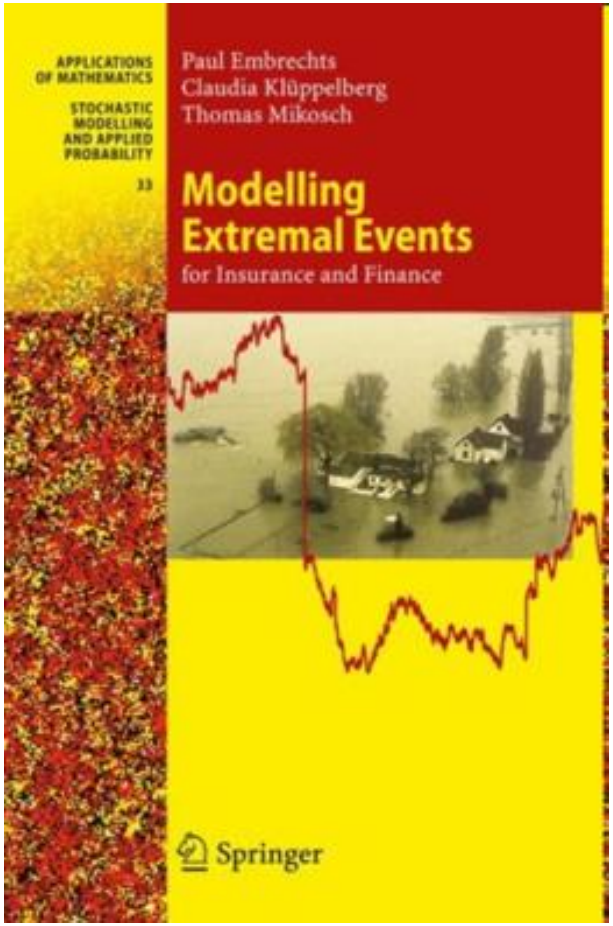
if and only if $F \in \mathrm{MDA}\,(H_\xi)$, $\xi \in \mathbb{R}$.

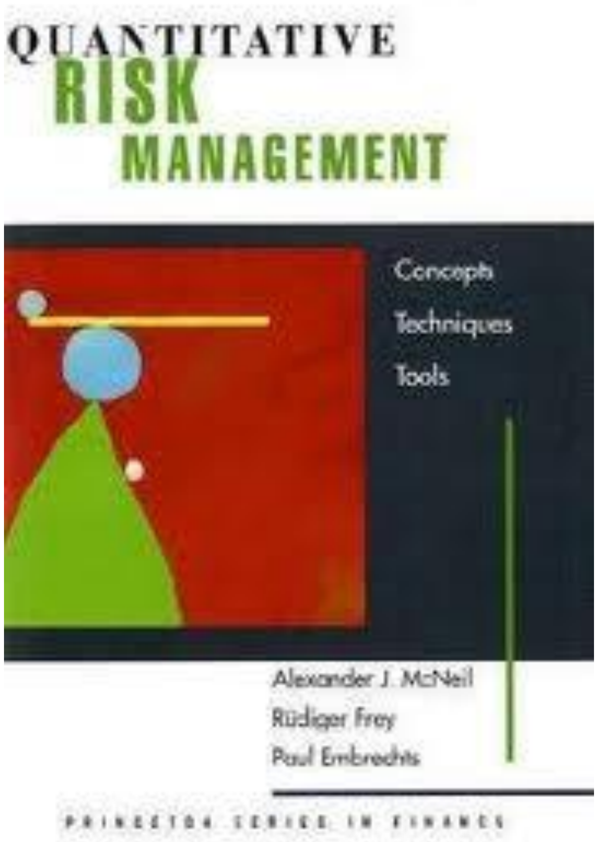
- The GPD is a two parameter distribution with df

$$G_{\xi,\beta}(x) = \begin{cases} 1 - (1 + \xi x/\beta)_+^{-1/\xi} & \xi \neq 0 \,, \\ 1 - \exp(-x/\beta) & \xi = 0 \,, \end{cases}$$

  where $\beta > 0$ and $a_+ = \max(a, 0)$, so the support is $x \ge 0$ when $\xi \ge 0$ and $0 \le x \le -\beta/\xi$ when $\xi < 0$.

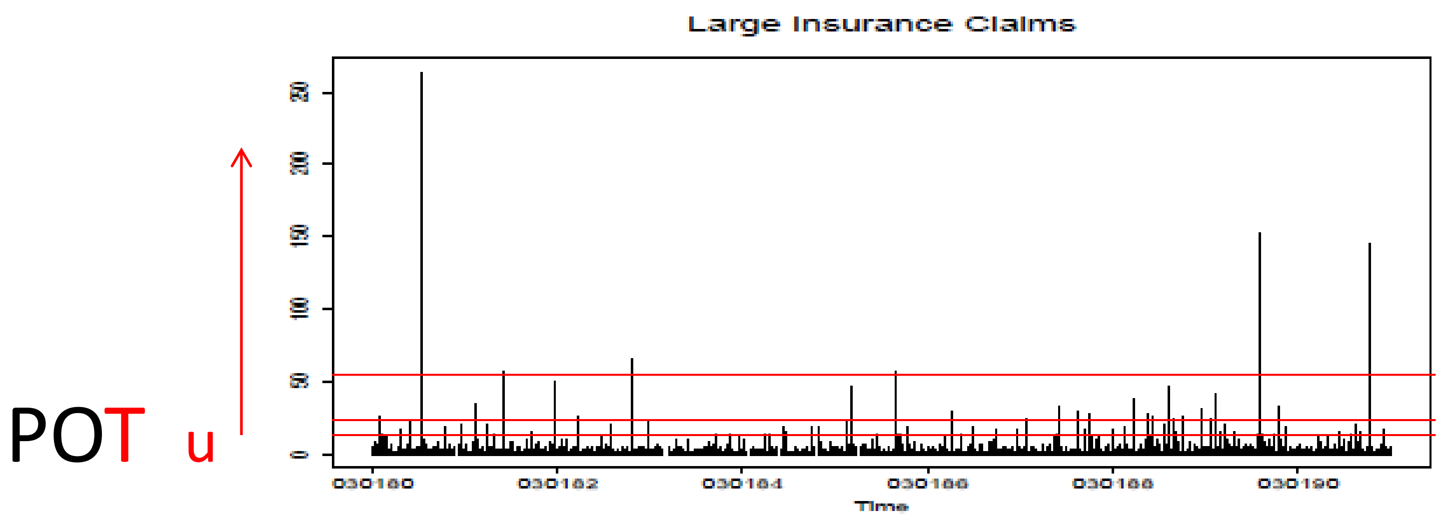
For details and applications, see





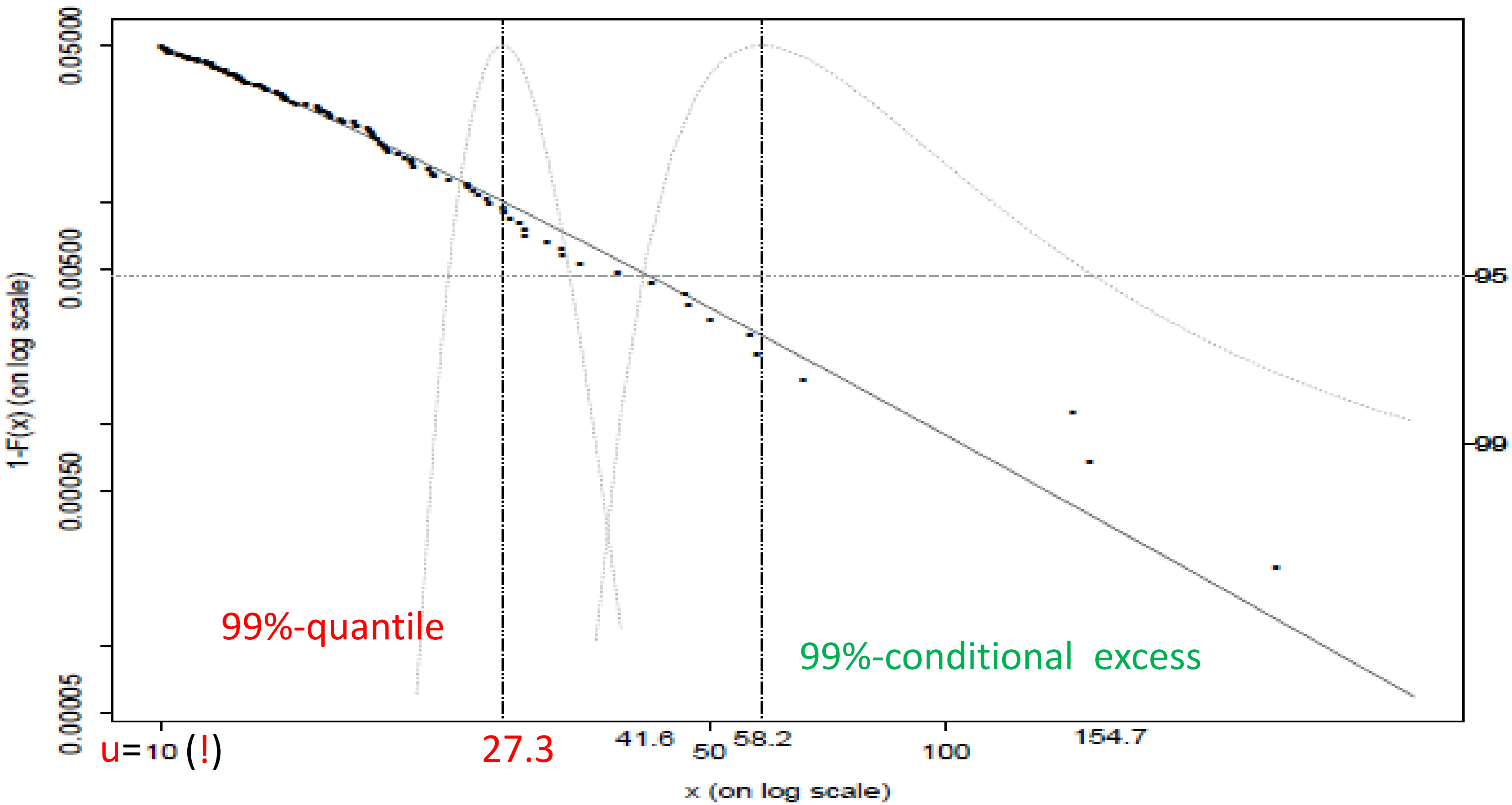
Danish Fire Loss Example

The Danish data consist of 2167 losses exceeding one million Danish Krone from the years 1980 to 1990. The loss figure is a total loss for the event concerned and includes damage to buildings, damage to contents of buildings as well as loss of profits. The data have been adjusted for inflation to reflect 1985 values.



99%-quantile with 95% aCI (Profile Likelihood): 27.3 (23.3, 33.1)

99% Conditional Excess: E(X I X > 27.3) with aCI



When does $F \in \text{MDA}(H_\xi)$ hold?

1. Fréchet Case: $(\xi > 0)$

Gedenko (1943) showed that for $\xi > 0$,

Power law

Beware!

$$F \in \text{MDA}(H_\xi) \iff 1 - F(x) = x^{-1/\xi} L(x) \text{ for } x > 0,$$

for some function $L(x)$ which is *slowly varying at* ∞.

A positive function L on $(0, \infty)$ is slowly varying at ∞ if $\forall t > 0 \quad \lim_{x\to\infty} \frac{L(tx)}{L(x)} = 1$. Example: $L(x) = \ln(x)$.

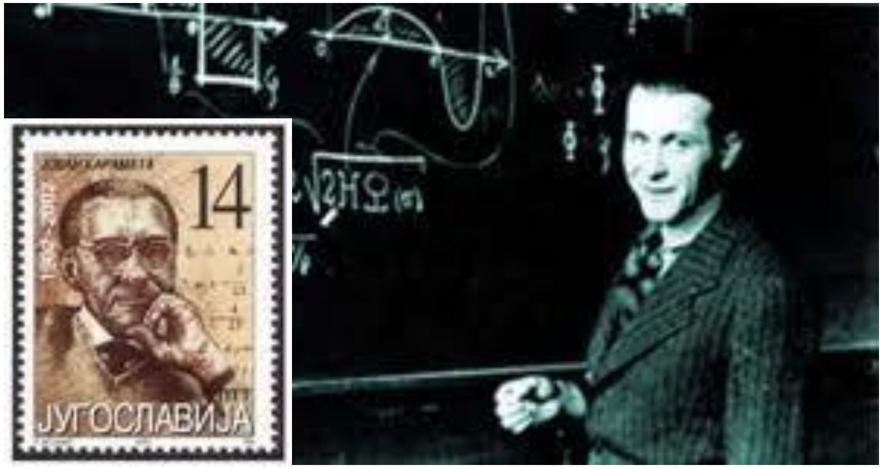
Summary:

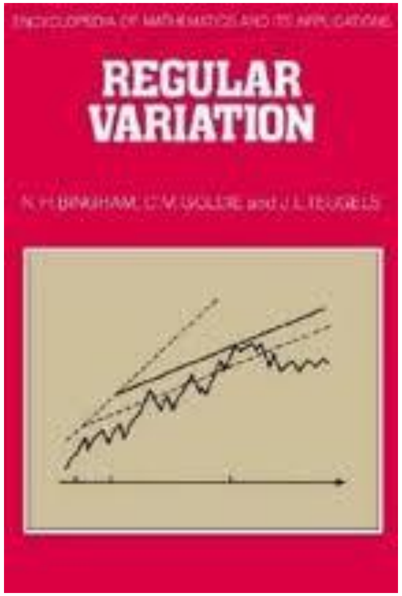
If the tail of the df F decays like a power function, then the distribution is in $\text{MDA}(H_\xi)$ for $\xi > 0$.

An interludium on Regular Variation, more in particular, on the Slowly Varying L in Gnedenko's Theorem:

One further name and a book:

Jovan
Karamata
(1902 -1967)





by N.H. Bingham, C.M. Goldie and J.L. Teugels (1987): contains ALL about L-functions!

The basic theorems on slow (SV) and regular variation (RV) (Karamata):

- The Uniform Convergence Theorem
- The Representation Theorem
- Karamata’s Theorem (Integrating SV and RV functions)
- Karamata’s Tauberian Theorem (The Laplace transfom of RV probability measures)
- The Monotone Density Theorem (Taking derivatives of RV functions) + other operations

A warning on EVT related rates of convergence:

$$\underbrace{\lim_{u \uparrow x_0} \sup_{x \in (0, x_0 - u)} \left| F_u(x) - G_{\xi, \beta(u)}(x) \right|}_{=: d(u)} = 0$$

Rate of convergence to the GPD for different distributions, as a function of the threshold u

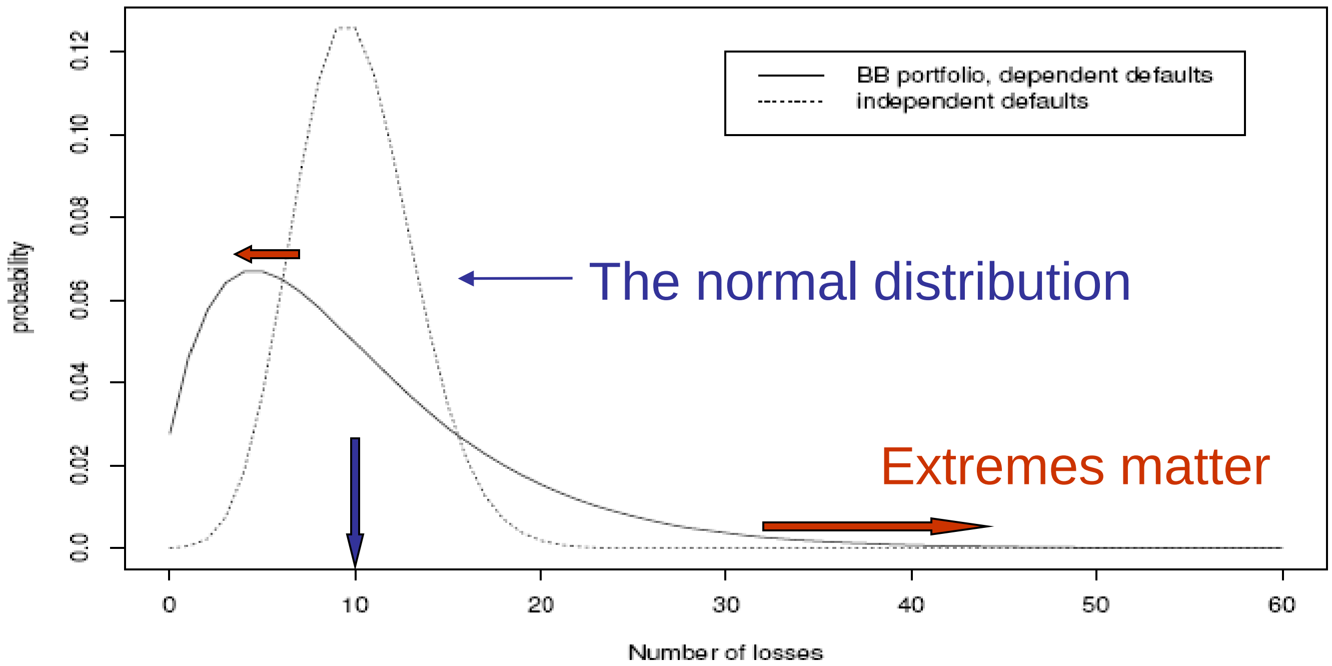
L matters!

Distribution	Parameters	$\overline{F}$	$d(u)$
Exponential(λ)	$\lambda > 0$	$e^{-\lambda x}$	0
Pareto(α)	$\alpha > 0$	$x^{-\alpha}$	0
Double exp. parent		e^{-e^x}	$O(e^{-u})$
Student t	$\nu > 0$	$\overline{t}_\nu(x)$	$O(\frac{1}{u^2})$
Normal(0, 1)		$\overline{\Phi}(x)$	$O(\frac{1}{u^2})$
Weibull(τ, c)	$\tau \in \mathbb{R}_+ \setminus \{1\}, c > 0$	$e^{-(cx)^\tau}$	$O(\frac{1}{u^\tau})$
Lognormal(μ, σ)	$\mu \in \mathbb{R}, \sigma > 0$	$\overline{\Phi}(\frac{\log x - \mu}{\sigma})$	$O(\frac{1}{\log u})$
Loggamma(γ, α)	$\alpha > 0, \gamma \neq 1$	$\overline{\Gamma}_{\alpha,\gamma}(x)$	$O(\frac{1}{\log u})$
g-and-h	$g, h > 0$	$\overline{\Phi}(k^{-1}(x))$	$O(\frac{1}{\sqrt{\log u}})$

A warning on micro-correlation and extreme-tail modelling:

micro-

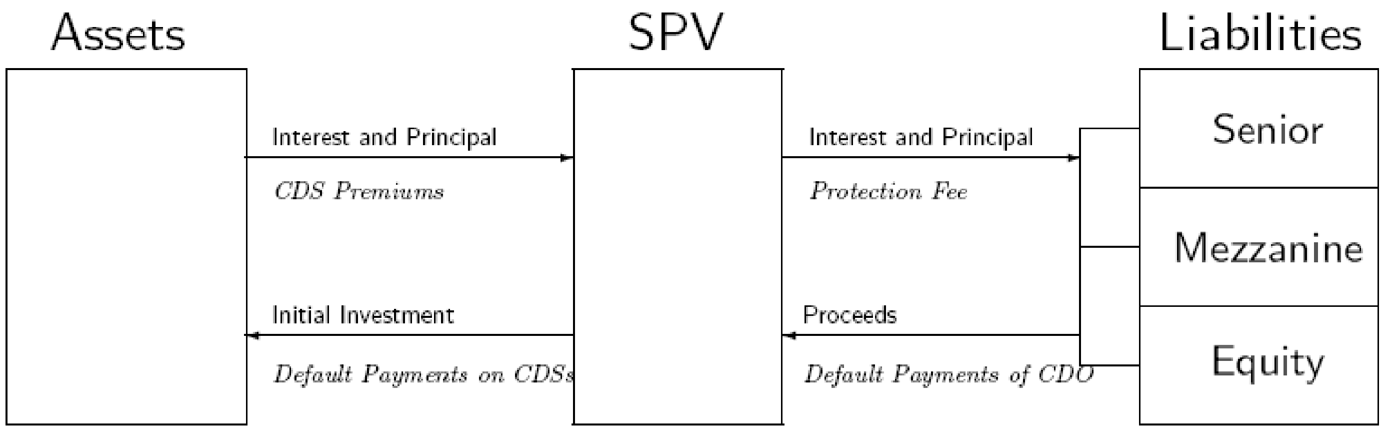
Impact of dependence on loss distribution



Distribution of number of defaults for homogeneous portfolio of 1000 BB loans with default probability $\approx 1\%$; Bernoulli mixture model with default correlation $\approx 0.22\%$ is compared with independent default model.

Correlation matters

CDOs - Basic Structure



Payments in a CDO structure; above arrow: asset-based structure; below arrow: *synthetic CDO*.

Distance!!!!

Quality????

The L'Aquila Earthquake

April 6, 2009

Question: Why this combination?

David Spiegelhalter

A final example:

31. Jan. 1953 – 1. Feb. 1953

- 1836 people killed
- 72000 people evacuated
- 49000 houses and farms floaded
- 201000 cattle drowned
- 500 km coastal defenses destroyed; more than 400 breaches of dykes
- 200000 ha land floaded

The Delta-Project

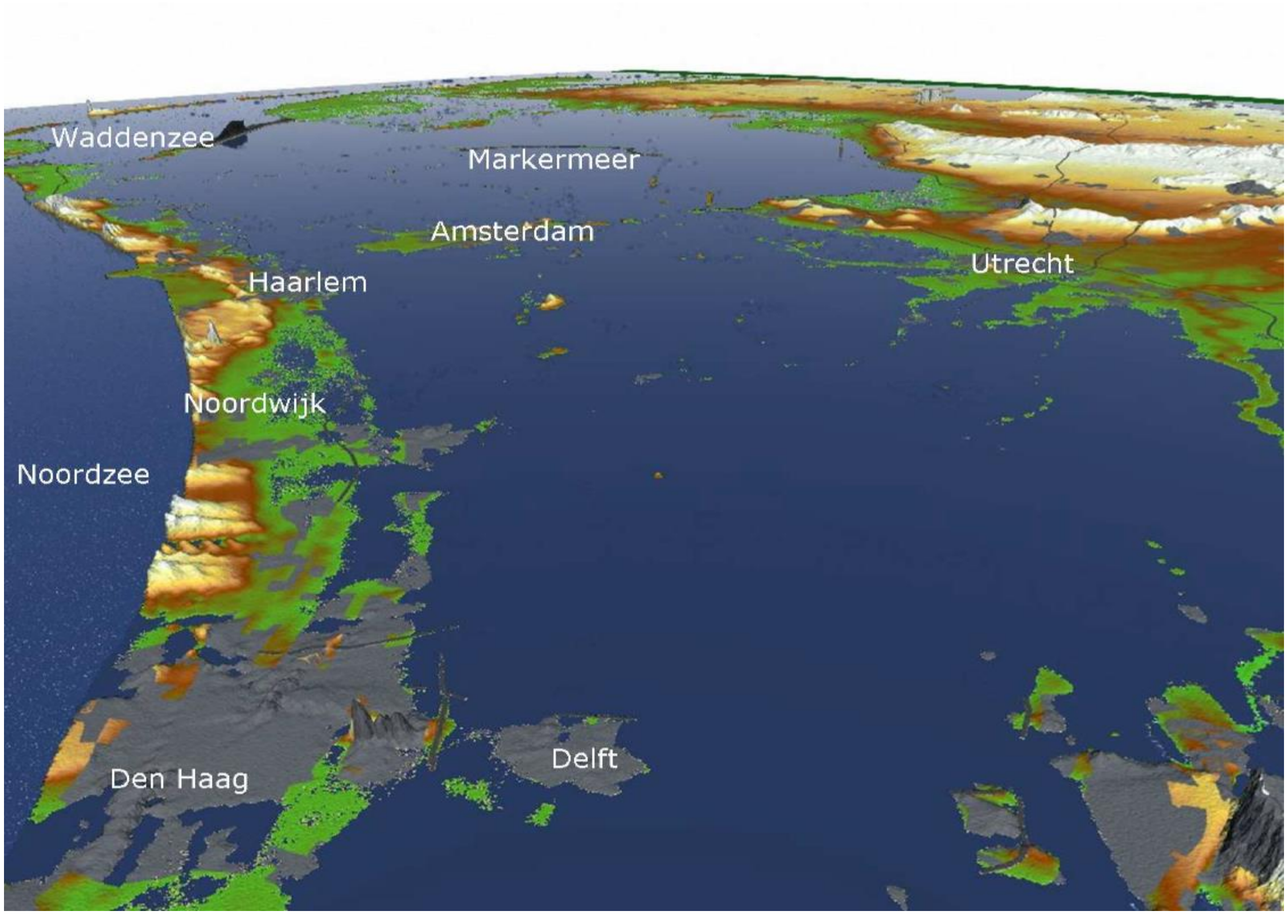
- Coastal floаd-protection
- Requested dyke height at l: $h_d(l)$
- Safety margin at l: **MYSS**(l) =

  Maximal **Y**early **S**ea **S**urge at l:
- Probability(**MYSS**(l) > $h_d(l)$)

  should be „small“, whereby „small“ is defined as (Risk):
  - 1 / 10 000 in the Randstad
  - 1 / 250 in the Deltaregion to the North
  - Similar requirements for rivers, but with 1/10 – 1/100
- For the Randstad (Amsterdam-Roterdam):

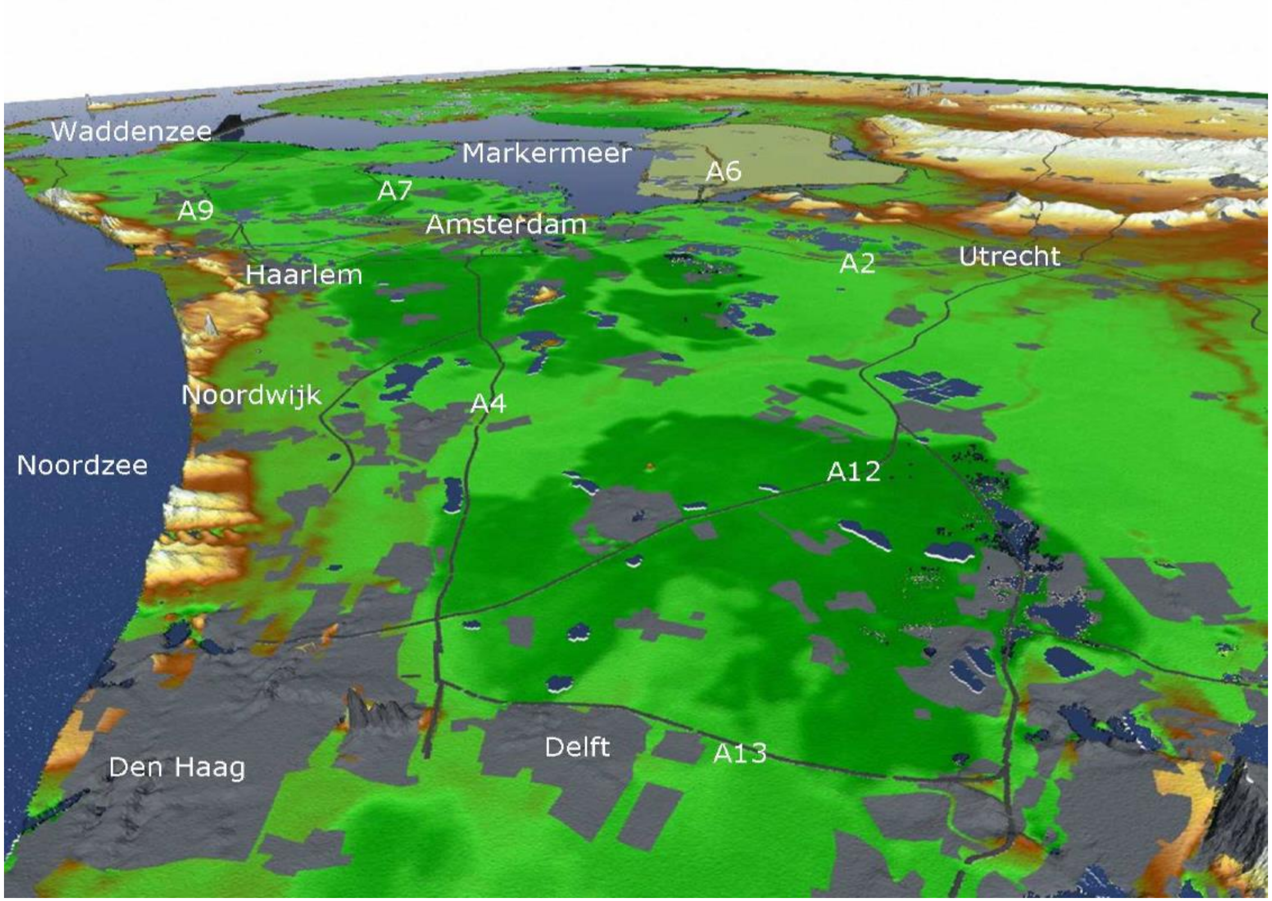
  Dyke height = Normal-level (= NAP) + 5.14 m

The Netherlands without dykes!



Henk van den Brink, KNMI, Fighting the arch-enemy with mathematics (de Haan) and climate models

The Netherlands with dykes:



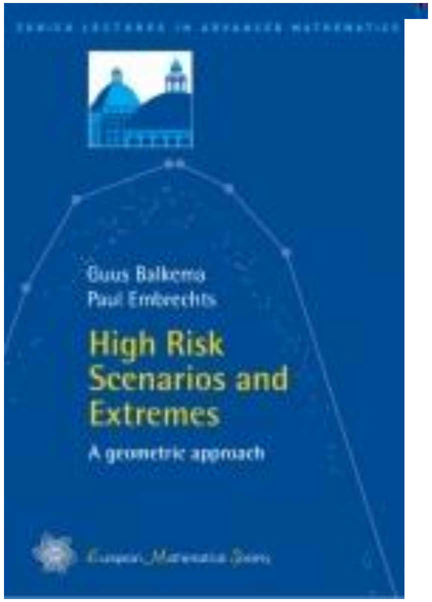
(Henk van den Brink, KNMI)

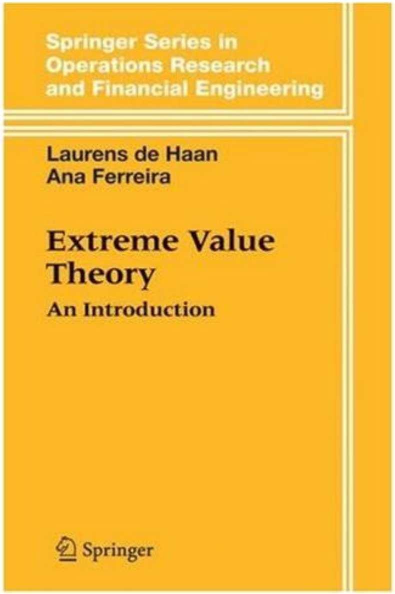
Remark: 1/10 000 → 1/100 000 (but also) → 1/200

Guus Balkema

PE

Laurens de Haan





Some issues:

- Extremes for discrete data: special theory
- No unique/canonical theory for multivariate extremes because of lack of standard ordering, hence theory becomes context dependent
- Interesting links with rare event simulation, large deviations and importance sampling
- High dimensionality, d > 3 or 4 (sic)
- Time dependence (processes), non-stationarity
- Extremal dependence (← financial crisis)
- And finally ... APPLICATIONS ... COMMUNICATION !!

Thank you!