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The Rearrangement Algorithm

a new tool for computing bounds on risk measures

joint work with
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1. QRM framework

2. The Rearrangement Algorithm

3. Applications

1. QRM framework

QRM framework under Basel 2,3,Solvency 2,...

$$L_1 \sim F_1, \quad L_2 \sim F_2, \quad \dots, \quad L_d \sim F_d$$

one period risks with statistically estimated marginals.

$L^+ = L_1 + \cdots + L_d$ total loss exposure

$\rho(L_1 + \cdots + L_d)$ amount of capital to be reserved

If a dependence model is not specified there exist infinitely many values for the risk measure which are consistent with the choice of the marginals

$\underline{\rho}$ $\overline{\rho}$

$$\underline{\rho} = \inf\{\, \rho(L_1 + \cdots + L_d) : L_j \sim F_j, 1 \le j \le d\}$$

$$\overline{\rho} = \sup\{\, \rho(L_1 + \cdots + L_d) : L_j \sim F_j, 1 \le j \le d\}$$

Risk measures: definition

Value-at-Risk (VaR)

- $\mathrm{VaR}_\alpha(L^+) = F_{L^+}^{-1}(\alpha) = \inf\{x \in \mathbb{R} : F_{L^+}(x) \geq \alpha\}, \ \alpha \in (0,1)$

  i.e. $P(L^+ > \mathrm{VaR}_\alpha(L^+)) \leq 1 - \alpha$

Expected Shortfall (ES)

- $\mathrm{ES}_\alpha(L^+) = \dfrac{1}{1-\alpha} \displaystyle\int_\alpha^1 \mathrm{VaR}_q(L^+)\, dq, \ \alpha \in (0,1)$

  i.e. $\mathrm{ES}_\alpha(L^+) = E[L^+ | L^+ > \mathrm{VaR}_\alpha(L^+)]$ if L^+ is continuous

QRM framework

$$L_1 \sim F_1, \quad L_2 \sim F_2, \quad \ldots, \quad L_d \sim F_d$$

one period risks with statistically estimated marginals.

model uncertainty for VaR

$\underline{\mathrm{VaR}}_\alpha(L^+)$ $\overline{\mathrm{VaR}}_\alpha(L^+)$

model uncertainty for ES

$\underline{\mathrm{ES}}_\alpha(L^+)$ $\overline{\mathrm{ES}}_\alpha(L^+)$

Known bounds

- Subadditivity of ES implies that

$$\overline{\mathrm{ES}}_\alpha(L^+) = \sum_{j=1}^{d} \mathrm{ES}_\alpha(L_j) = \sum_{j=1}^{d} \frac{1}{1-\alpha} \int_\alpha^1 \mathrm{VaR}_q(L_j)\, dq$$

- $\underline{\mathrm{ES}}_\alpha(L^+), \underline{\mathrm{VaR}}_\alpha(L^+), \overline{\mathrm{VaR}}_\alpha(L^+)$

are known only in the case $d = 2$ and for $d \geq 3$ under special assumptions, e.g. for identically distributed risks having monotone densities; see Puccetti (2013) and Puccetti, G. and L. Rüschendorf (2013).

For general inhomogenous marginals, there does not exist an analytical tool to compute them.

2. The Rearrangement Algorithm

The Rearrangement Algorithm

Question: Given the matrix ***X***, rearrange the second column to obtain rowwise sums with minimal variance

$$\boldsymbol{X} = \begin{bmatrix} 1 & 1 \\ 2 & 4 \\ 3 & 3 \\ 4 & 2 \\ 5 & 5 \end{bmatrix} \quad \begin{matrix} \mathbf{2} \\ \mathbf{6} \\ \mathbf{6} \\ \mathbf{6} \\ \mathbf{10} \end{matrix} \qquad \boldsymbol{Y} = \begin{bmatrix} 1 & 5 \\ 2 & 4 \\ 3 & 3 \\ 4 & 2 \\ 5 & 1 \end{bmatrix} \quad \begin{matrix} \mathbf{6} \\ \mathbf{6} \\ \mathbf{6} \\ \mathbf{6} \\ \mathbf{6} \end{matrix}$$

Solution: Arrange the second column oppositely to the first

The Rearrangement Algorithm

Question (more difficult): Given the matrix ***X***, rearrange the entries within each column to obtain rowwise sums with minimal variance

$$X = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & 1 \\ 3 & 3 & 4 \\ 4 & 2 & 3 \\ 5 & 5 & 5 \end{bmatrix} \quad \begin{matrix} \mathbf{4} \\ \mathbf{7} \\ \mathbf{10} \\ \mathbf{9} \\ \mathbf{15} \end{matrix} \qquad \longrightarrow \qquad \begin{bmatrix} 5 & 1 & 2 \\ 3 & 4 & 1 \\ 2 & 3 & 4 \\ 4 & 2 & 3 \\ 1 & 5 & 5 \end{bmatrix} \quad \begin{matrix} \mathbf{3\ 8} \\ \mathbf{5\ 8} \\ \mathbf{7\ 9} \\ \mathbf{5\ 9} \\ \mathbf{10\ 11} \end{matrix}$$

Strategy: rearrange the entries of column *j* oppositely to the sum of the other columns. Then iterate for all *j*.

$$\begin{bmatrix} 5 & 1 & 2 \\ 3 & 4 & 1 \\ 2 & 3 & 4 \\ 4 & 2 & 3 \\ 1 & 5 & 5 \end{bmatrix} \quad \begin{matrix} \mathbf{8} \\ \mathbf{8} \\ \mathbf{9} \\ \mathbf{9} \\ \mathbf{11} \end{matrix}$$

$$\begin{bmatrix} 5 & 1 & 2 \\ 3 & 5 & 1 \\ 2 & 3 & 4 \\ 4 & 2 & 3 \\ 1 & 4 & 5 \end{bmatrix} \quad \begin{matrix} \mathbf{8} \\ \mathbf{9} \\ \mathbf{9} \\ \mathbf{9} \\ \mathbf{10} \end{matrix}$$

$$\boldsymbol{Y} = \begin{bmatrix} 5 & 1 & 2 \\ 3 & 5 & 1 \\ 2 & 3 & 4 \\ 4 & 2 & 3 \\ 1 & 4 & 5 \end{bmatrix} \quad \begin{matrix} \mathbf{8} \\ \mathbf{9} \\ \mathbf{9} \\ \mathbf{9} \\ \mathbf{10} \end{matrix}$$

Rearrangement Algorithm:

Rearrange the entries in the columns of ***X*** until you find an *ordered* matrix ***Y***, i.e. a matrix in which

each column is oppositely ordered to the sum of the others.

$$
\boldsymbol{X} = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & 1 \\ 3 & 3 & 4 \\ 4 & 2 & 3 \\ 5 & 5 & 5 \end{bmatrix} \quad \begin{matrix} \mathbf{4} \\ \mathbf{7} \\ \mathbf{10} \\ \mathbf{9} \\ \mathbf{15} \end{matrix} \qquad \boldsymbol{Y} = \begin{bmatrix} 5 & 1 & 2 \\ 3 & 5 & 1 \\ 2 & 3 & 4 \\ 4 & 2 & 3 \\ 1 & 4 & 5 \end{bmatrix} \quad \begin{matrix} \mathbf{8} \\ \mathbf{9} \\ \mathbf{9} \\ \mathbf{9} \\ \mathbf{10} \end{matrix}
$$

Let ***+(X)*** and ***+(Y)*** be the vectors having as components the componentwise sum of each row of ***X*** and, respectively, ***Y***.

$$
\boldsymbol{+(X)} = \begin{bmatrix} \mathbf{4} \\ \mathbf{7} \\ \mathbf{10} \\ \mathbf{9} \\ \mathbf{15} \end{bmatrix} \quad \geq_{cx} \quad \boldsymbol{+(Y)} = \begin{bmatrix} \mathbf{8} \\ \mathbf{9} \\ \mathbf{9} \\ \mathbf{9} \\ \mathbf{10} \end{bmatrix}
$$

The **convex order** $Y \leq_{cx} X$ is defined as

$$Y \leq_{cx} X \quad \text{iff} \quad \mathbb{E}[f(Y)] \leq \mathbb{E}[f(X)]$$

for all convex functions f such that the expectations exist.

$$Y \leq_{cx} X$$

implies

$$\mathbb{E}(Y) = \mathbb{E}(X) \;\; \text{and} \;\; \mathrm{var}(Y) \leq \mathrm{var}(X)$$

and is equivalent to

$$\mathrm{ES}_\alpha(Y) \leq \mathrm{ES}_\alpha(X), \;\; \alpha \in (0, 1)$$

Associate to a $(N \times d)$ matrix $\boldsymbol{X}$ the N-discrete d-variate distribution giving probability mass $1/N$ to each one of its N row vectors.

$$\boldsymbol{X} = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & 1 \\ 3 & 3 & 4 \\ 4 & 2 & 3 \\ 5 & 5 & 5 \end{bmatrix} \begin{matrix} \mathbf{4} \\ \mathbf{7} \\ \mathbf{10} \\ \mathbf{9} \\ \mathbf{15} \end{matrix} \qquad \boldsymbol{Y} = \begin{bmatrix} 5 & 1 & 2 \\ 3 & 5 & 1 \\ 2 & 3 & 4 \\ 4 & 2 & 3 \\ 1 & 4 & 5 \end{bmatrix} \begin{matrix} \mathbf{8} \\ \mathbf{9} \\ \mathbf{9} \\ \mathbf{9} \\ \mathbf{10} \end{matrix}$$

Theorem (see Puccetti, 2013)

Let $\boldsymbol{Y}$ be the matrix obtained by applying the RA to $\boldsymbol{X}$. Then, the distribution associated to $\boldsymbol{Y}$ has the same univariate marginals of the distribution associated to $\boldsymbol{X}$.

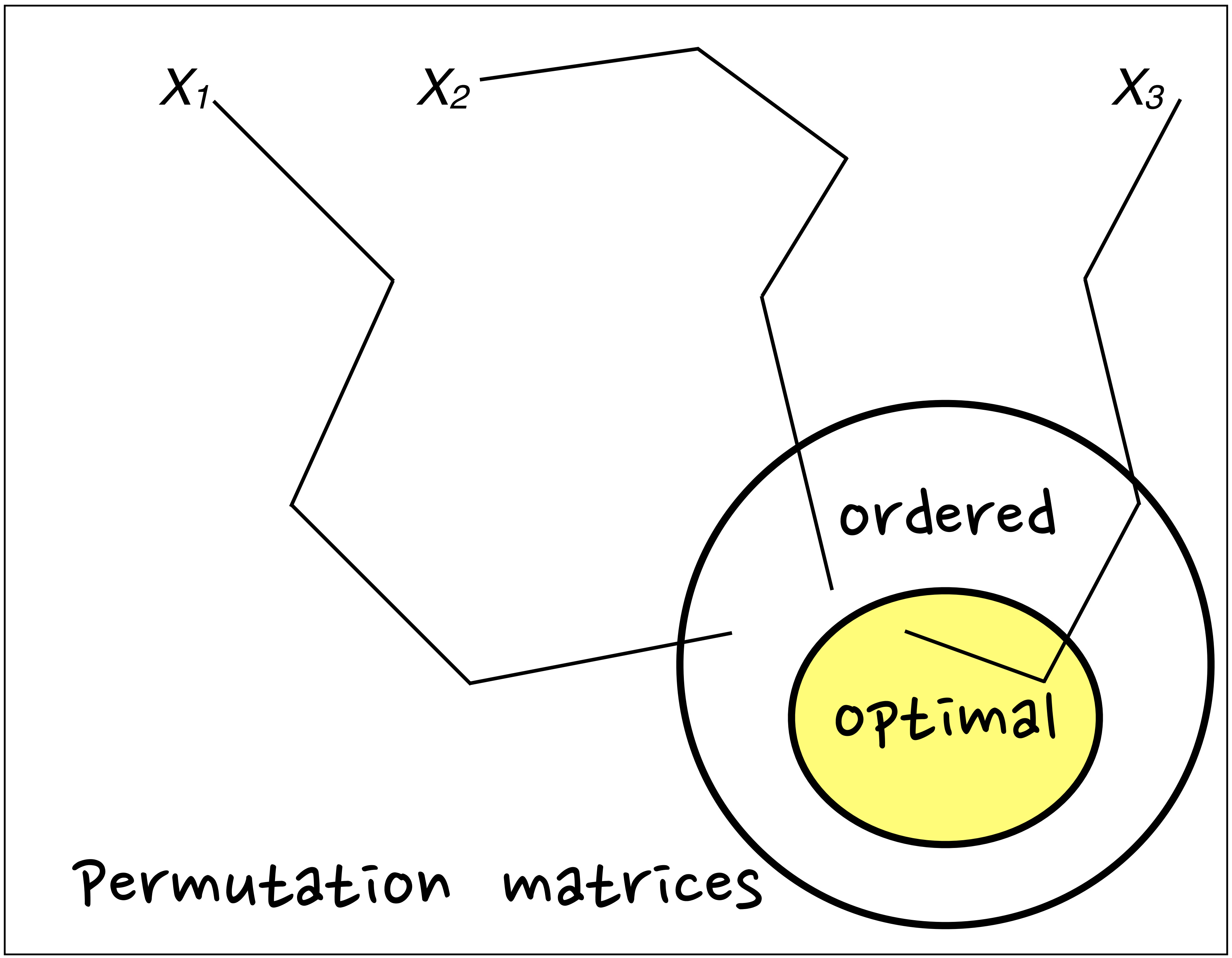
Moreover, if $(X_1, \ldots, X_d) \sim \boldsymbol{X}$ and $(Y_1, \ldots, Y_d) \sim \boldsymbol{Y}$, then

$$Y_1 + \cdots + Y_d \leq_{cx} X_1 + \cdots + X_d \,,$$

and $\mathrm{ES}_\alpha(Y_1 + \cdots + Y_d) \leq \mathrm{ES}_\alpha(X_1 + \cdots + X_d), \; \alpha \in (0, 1)$

The RA finds a finite sequence of matrices with a decreasing expected shortfall for the the sum of the components of the random vectors having the associated distributions.

It may fail in general to minimize ES



ordered

5	1	2	8
3	5	1	9
2	3	4	9
4	2	3	9
1	4	5	10

optimal

5	3	1	9
3	4	2	9
1	5	3	9
4	1	4	9
2	2	5	9

3. Applications

General distributions

Pareto (4)

0.00000000
0.01709526
0.03509834
0.05409255
0.07417231
0.09544512
0.11803399
0.14208048
0.16774842
0.19522861
0.22474487
0.25656172
0.29099445
0.32842233
0.36930639
0.41421356
0.46385011
0.51910905
0.58113883
0.65144565
0.73205081
0.82574186
0.93649167
1.07019668
1.23606798
1.44948974
1.73861279
2.16227766
2.87298335
4.47722558
Inf

1) Approximate the support of each marginal F_j from above and below:

$$\underline{F}_j \geq F_j \geq \overline{F}_j$$

and create two matrices ***X*** and ***Y*** with *N* rows and *d* columns.

2) Iteratively rearrange the column of each matrix and find two matrices ***X**** and ***Y**** with each column oppositely ordered to the sum of the other columns.

3) If $(X_1, \ldots, X_d) \sim \boldsymbol{X}^*$ and $(Y_1, \ldots, Y_d) \sim \boldsymbol{Y}^*$, then

$$\mathrm{ES}_\alpha(X_1 + \cdots + X_d) \simeq \underline{\mathrm{ES}}_\alpha(L^+) \leq \mathrm{ES}_\alpha(Y_1 + \cdots + Y_d)$$

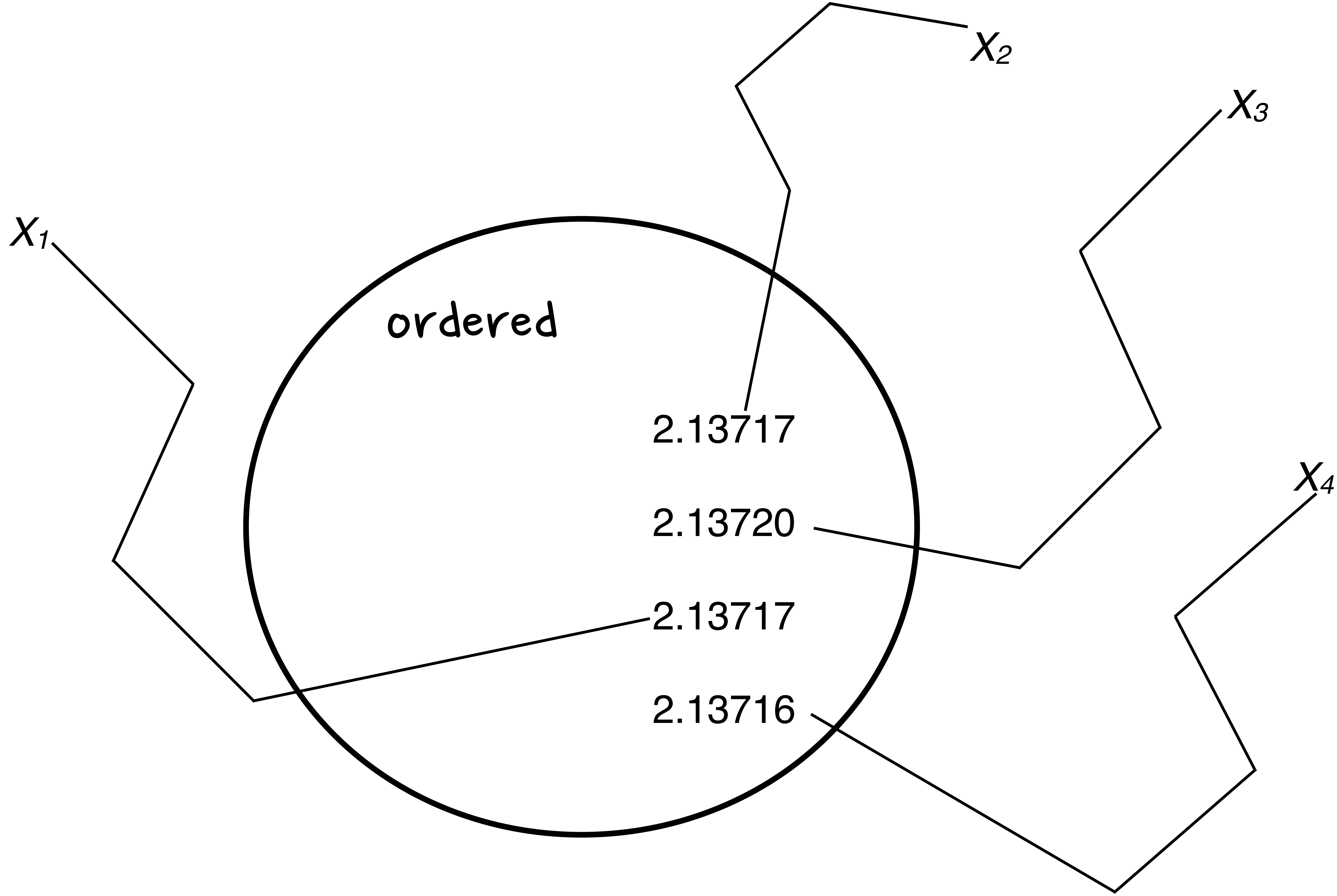
4) Run the algorithm with *N* large enough.

Pareto(4) marginals and $\alpha = 0.90$

	1	2	3	SUM
1	0.00000000	0.00000000	0.00000000	0.0000000
2	0.00851141	0.00851141	0.00851141	0.0255342
3	0.01739783	0.01739783	0.01739783	0.0521935
4	0.02669010	0.02669010	0.02669010	0.0800703
5	0.03642284	0.03642284	0.03642284	0.1092685
6	0.04663514	0.04663514	0.04663514	0.1399054
7	0.05737126	0.05737126	0.05737126	0.1721138
8	0.06868166	0.06868166	0.06868166	0.2060450
9	0.08062409	0.08062409	0.08062409	0.2418723
10	0.09326511	0.09326511	0.09326511	0.2797953
11	0.10668192	0.10668192	0.10668192	0.3200458
12	0.12096464	0.12096464	0.12096464	0.3628939
13	0.13621937	0.13621937	0.13621937	0.4086581
14	0.15257205	0.15257205	0.15257205	0.4577161
15	0.17017366	0.17017366	0.17017366	0.5105210
16	0.18920712	0.18920712	0.18920712	0.5676213
17	0.20989674	0.20989674	0.20989674	0.6296902
18	0.23252142	0.23252142	0.23252142	0.6975643
19	0.25743343	0.25743343	0.25743343	0.7723003
20	0.28508585	0.28508585	0.28508585	0.8552576
21	0.31607401	0.31607401	0.31607401	0.9482220
22	0.35120015	0.35120015	0.35120015	1.0536005
23	0.39157884	0.39157884	0.39157884	1.1747365
24	0.43881781	0.43881781	0.43881781	1.3164534
25	0.49534878	0.49534878	0.49534878	1.4860463
26	0.56508458	0.56508458	0.56508458	1.6952537
27	0.65487546	0.65487546	0.65487546	1.9646264
28	0.77827941	0.77827941	0.77827941	2.3348382
29	0.96798967	0.96798967	0.96798967	2.9039690
30	1.34034732	1.34034732	1.34034732	4.0210420
SUM	8.59595166	8.59595166	8.59595166	NA

ORDERED MATRIX

	1	2	3	SUM
1	0.18920712	0.20989674	0.35120015	0.750304
2	0.17017366	0.39157884	0.18920712	0.750960
3	0.08062409	0.23252142	0.43881781	0.751963
4	0.20989674	0.15257205	0.39157884	0.754048
5	0.43881781	0.10668192	0.20989674	0.755396
6	0.09326511	0.17017366	0.49534878	0.758788
7	0.56508458	0.09326511	0.10668192	0.765032
8	0.10668192	0.56508458	0.09326511	0.765032
9	0.12096464	0.08062409	0.56508458	0.766673
10	0.31607401	0.13621937	0.31607401	0.768367
11	0.49534878	0.12096464	0.15257205	0.768885
12	0.39157884	0.25743343	0.12096464	0.769977
13	0.28508585	0.31607401	0.17017366	0.771334
14	0.13621937	0.35120015	0.28508585	0.772505
15	0.35120015	0.18920712	0.23252142	0.772929
16	0.23252142	0.28508585	0.25743343	0.775041
17	0.25743343	0.43881781	0.08062409	0.776875
18	0.05737126	0.06868166	0.65487546	0.780928
19	0.65487546	0.05737126	0.06868166	0.780928
20	0.06868166	0.65487546	0.05737126	0.780928
21	0.15257205	0.49534878	0.13621937	0.784140
22	0.03642284	0.04663514	0.77827941	0.861337
23	0.04663514	0.77827941	0.03642284	0.861337
24	0.77827941	0.03642284	0.04663514	0.861337
25	0.01739783	0.96798967	0.02669010	1.012078
26	0.96798967	0.02669010	0.01739783	1.012078
27	0.02669010	0.01739783	0.96798967	1.012078
28	1.34034732	0.00851141	0.00000000	1.348859
29	0.00851141	0.00000000	1.34034732	1.348859
30	0.00000000	1.34034732	0.00851141	1.348859
SUM	8.59595166	8.59595166	8.59595166	NA



With $N = 2 \times 10^6$, we obtain the first three decimals of $\underline{\mathrm{ES}}_{0.9}(L^+) = 2.1377$ in 2 mins.

$\mathrm{VaR}_{0.99}(L_1)$

Pareto (2)

	1	2	3	Σ
1	9.00000	9.00000	9.00000	27.0000
2	9.17095	9.17095	9.17095	27.5129
3	9.35098	9.35098	9.35098	28.0530
4	9.54093	9.54093	9.54093	28.6228
5	9.74172	9.74172	9.74172	29.2252
6	9.95445	9.95445	9.95445	29.8634
7	10.18034	10.18034	10.18034	30.5410
8	10.42080	10.42080	10.42080	31.2624
9	10.67748	10.67748	10.67748	32.0325
10	10.95229	10.95229	10.95229	32.8569
11	11.24745	11.24745	11.24745	33.7423
12	11.56562	11.56562	11.56562	34.6969
13	11.90994	11.90994	11.90994	35.7298
14	12.28422	12.28422	12.28422	36.8527
15	12.69306	12.69306	12.69306	38.0792
16	13.14214	13.14214	13.14214	39.4264
17	13.63850	13.63850	13.63850	40.9155
18	14.19109	14.19109	14.19109	42.5733
19	14.81139	14.81139	14.81139	44.4342
20	15.51446	15.51446	15.51446	46.5434
21	16.32051	16.32051	16.32051	48.9615
22	17.25742	17.25742	17.25742	51.7723
23	18.36492	18.36492	18.36492	55.0948
24	19.70197	19.70197	19.70197	59.1059
25	21.36068	21.36068	21.36068	64.0820
26	23.49490	23.49490	23.49490	70.4847
27	26.38613	26.38613	26.38613	79.1584
28	30.62278	30.62278	30.62278	91.8683
29	37.72983	37.72983	37.72983	113.1895
30	53.77226	53.77226	53.77226	161.3168
Σ	494.99920	494.99920	494.99920	NA

$\mathrm{VaR}_1(L_1)$

Fix $\alpha \in (0,1)$ and assume that each $F_j^{-1}|[\alpha, 1]$ takes only N values all having the same probability $(1-\alpha)/N$.

$$P\left(\sum_{j=1}^{d} L_j \geq \min(\mathrm{rowSums}(X))\right) \geq 1-\alpha$$

$$\mathrm{VaR}_\alpha(L_1 + \cdots + L_d) \geq \min(\mathrm{rowSums}(X))$$

$$\overline{\mathrm{VaR}}_\alpha(L^+) = \max_{\tilde{X} \in \mathcal{P}(X)} \min(\mathrm{rowSums}(\tilde{X}))$$

(proof in Puccetti and Rüschendorf (2012b))

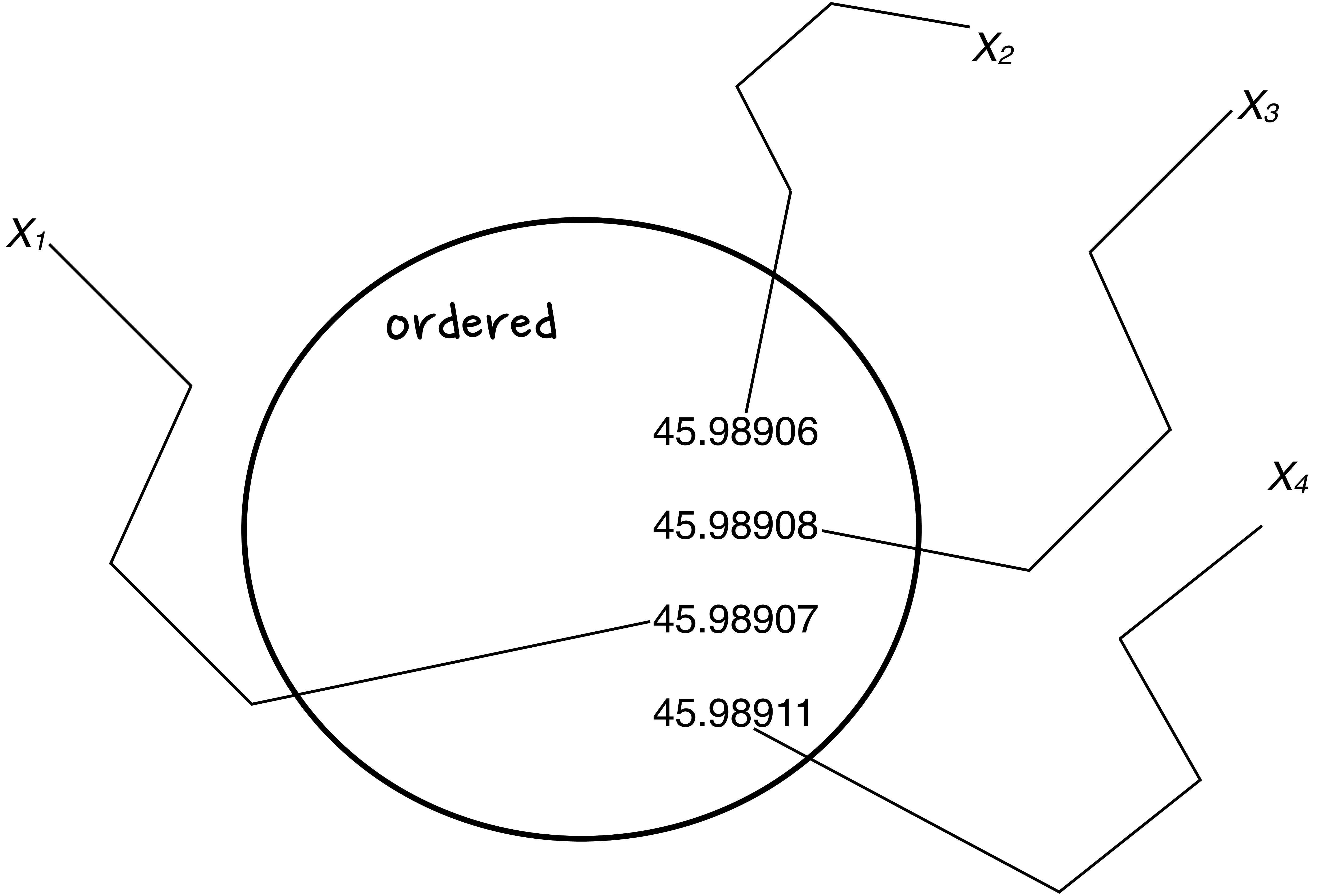
Analogous procedure for $\underline{\mathrm{VaR}}_\alpha(L^+)$

Pareto(2) marginals and $\alpha = 0.99$

	1	2	3	Σ
1	9.00000	9.00000	9.00000	27.0000
2	9.17095	9.17095	9.17095	27.5129
3	9.35098	9.35098	9.35098	28.0530
4	9.54093	9.54093	9.54093	28.6228
5	9.74172	9.74172	9.74172	29.2252
6	9.95445	9.95445	9.95445	29.8634
7	10.18034	10.18034	10.18034	30.5410
8	10.42080	10.42080	10.42080	31.2624
9	10.67748	10.67748	10.67748	32.0325
10	10.95229	10.95229	10.95229	32.8569
11	11.24745	11.24745	11.24745	33.7423
12	11.56562	11.56562	11.56562	34.6969
13	11.90994	11.90994	11.90994	35.7298
14	12.28422	12.28422	12.28422	36.8527
15	12.69306	12.69306	12.69306	38.0792
16	13.14214	13.14214	13.14214	39.4264
17	13.63850	13.63850	13.63850	40.9155
18	14.19109	14.19109	14.19109	42.5733
19	14.81139	14.81139	14.81139	44.4342
20	15.51446	15.51446	15.51446	46.5434
21	16.32051	16.32051	16.32051	48.9615
22	17.25742	17.25742	17.25742	51.7723
23	18.36492	18.36492	18.36492	55.0948
24	19.70197	19.70197	19.70197	59.1059
25	21.36068	21.36068	21.36068	64.0820
26	23.49490	23.49490	23.49490	70.4847
27	26.38613	26.38613	26.38613	79.1584
28	30.62278	30.62278	30.62278	91.8683
29	37.72983	37.72983	37.72983	113.1895
30	53.77226	53.77226	53.77226	161.3168
Σ	494.99920	494.99920	494.99920	NA

ORDERED MATRIX

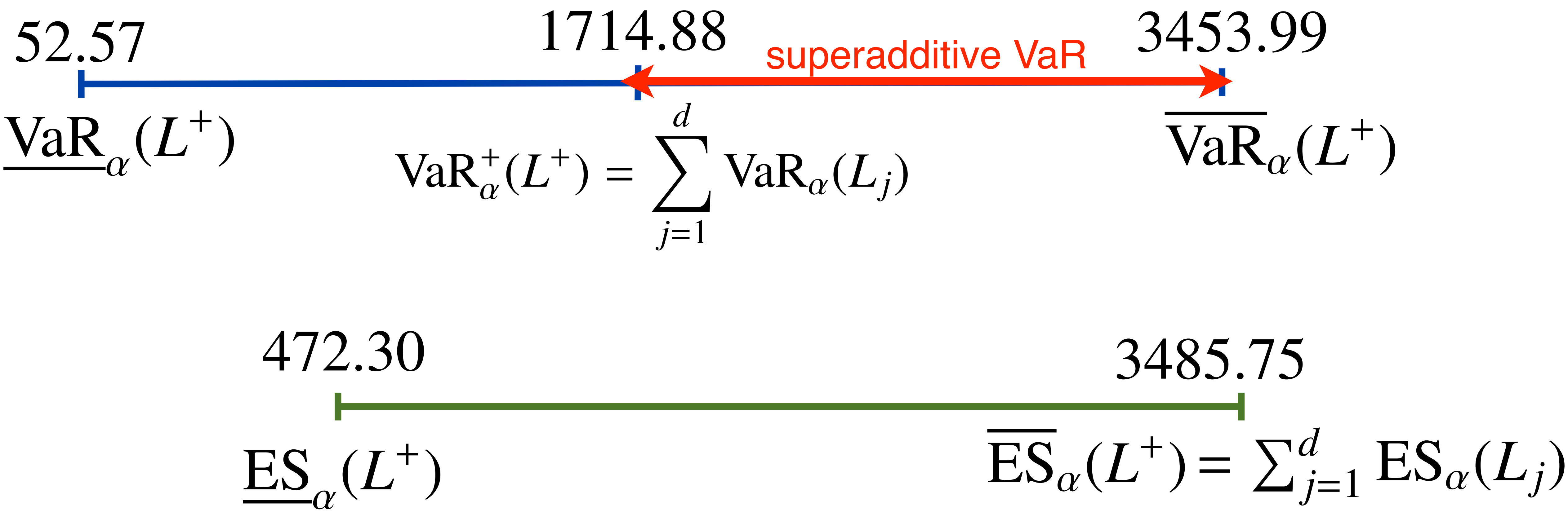
	1	2	3	
[1,]	16.32051	14.19109	13.63850	44.1501
[2,]	13.14214	17.25742	14.19109	44.5906
[3,]	12.28422	12.69306	19.70197	44.6793
[4,]	18.36492	13.63850	12.69306	44.6965
[5,]	11.56562	18.36492	14.81139	44.7419
[6,]	19.70197	13.14214	11.90994	44.7540
[7,]	12.69306	14.81139	17.25742	44.7619
[8,]	17.25742	11.24745	16.32051	44.8254
[9,]	11.90994	11.56562	21.36068	44.8362
[10,]	11.24745	21.36068	12.28422	44.8924
[11,]	21.36068	12.28422	11.24745	44.8924
[12,]	13.63850	19.70197	11.56562	44.9061
[13,]	15.51446	16.32051	13.14214	44.9771
[14,]	14.81139	11.90994	18.36492	45.0862
[15,]	10.95229	10.67748	23.49490	45.1247
[16,]	10.67748	23.49490	10.95229	45.1247
[17,]	23.49490	10.95229	10.67748	45.1247
[18,]	14.19109	15.51446	15.51446	45.2200
[19,]	26.38613	10.42080	10.18034	46.9873
[20,]	10.42080	10.18034	26.38613	46.9873
[21,]	10.18034	26.38613	10.42080	46.9873
[22,]	30.62278	9.74172	9.95445	50.3190
[23,]	9.95445	30.62278	9.74172	50.3190
[24,]	9.74172	9.95445	30.62278	50.3190
[25,]	9.54093	37.72983	9.35098	56.6217
[26,]	37.72983	9.35098	9.54093	56.6217
[27,]	9.35098	9.54093	37.72983	56.6217
[28,]	9.00000	9.17095	53.77226	71.9432
[29,]	9.17095	53.77226	9.00000	71.9432
[30,]	53.77226	9.00000	9.17095	71.9432
[31,]	494.99920	494.99920	494.99920	NA



With *N=10^5*, we obtain the first three decimals of $\overline{\mathrm{VaR}}_{0.99}(L^+) = 45.9898$ in 0.2 sec.

Model uncertainty for id risks

d=56, $L_i \sim$ Pareto(2), $\alpha = 99.9\%$



Several inhomogeneous examples are given in Embrechts, P., Puccetti, G. and L. Rüschendorf (2013).

For a risk vector $(L_1, \ldots, L_d)$, we define the *superadditivity ratio*

$$\delta_\alpha(d) = \frac{\overline{\mathrm{VaR}}_\alpha(L^+)}{\mathrm{VaR}_\alpha^+(L^+)}$$

Assume that the random variables L_j are positive, identically distributed like F, an unbounded continuous distribution having an ultimately decreasing density and **finite mean**. Then

$$\lim_{d\to\infty} \delta_\alpha(d) = \frac{\mathrm{ES}_\alpha(L_1)}{\mathrm{VaR}_\alpha(L_1)}$$

α	$\theta = 1.1$	$\theta = 1.5$	$\theta = 2$	$\theta = 3$	$\theta = 4$
0.99	11.154337	3.097350	2.111111	1.637303	1.487492
0.995	11.081599	3.060242	2.076091	1.603135	1.454080
0.999	11.018773	3.020202	2.032655	1.555556	1.405266

Table 3: Values for the constant d_α for Pareto(θ) distributions.

For a risk vector $(L_1, \ldots, L_d)$, we define the *superadditivity ratio*

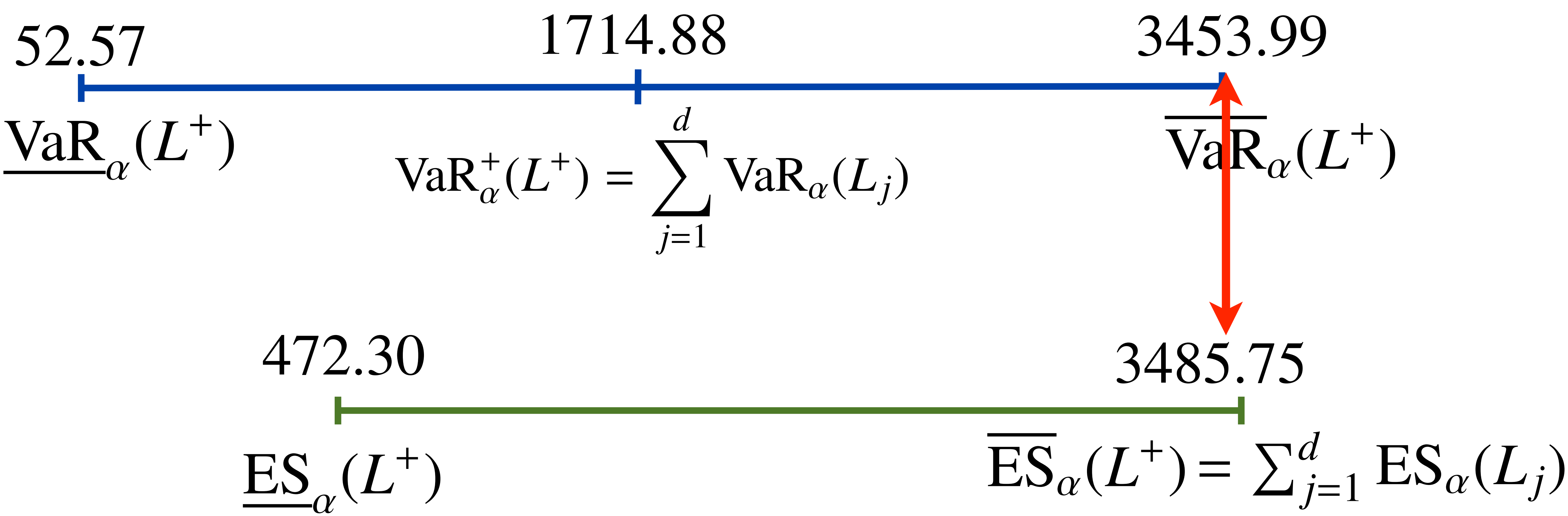
$$\delta_\alpha(d) = \frac{\overline{\mathrm{VaR}}_\alpha(L^+)}{\mathrm{VaR}_\alpha^+(L^+)}$$

Assume that the random variables L_j are positive, identically distributed like F, an unbounded continuous distribution having an ultimately decreasing density and **infinite mean**. Then

$$\lim_{d \to \infty} \delta_\alpha(d) = \infty$$

Model uncertainty for id risks

d=56, $L_i \sim$ Pareto(2), $\alpha = 99.9\%$



Several inhomogeneous examples are given in Embrechts, P., Puccetti, G. and L. Rüschendorf (2013).

For any portfolio $(L_1, \ldots, L_d)$, of course we have that

$$\frac{\overline{\mathrm{ES}}_\alpha(L_1 + \cdots + L_d)}{\overline{\mathrm{VaR}}_\alpha(L_1 + \cdots + L_d)} \geq 1.$$

Assume that the random variables L_j are positive, identically distributed like F, an unbounded continuous distribution having an ultimately decreasing density and **finite mean**. Then

Theorem (see Puccetti and Rüschendorf, 2013pp)

$$\lim_{d\to\infty} \frac{\overline{\mathrm{ES}}_\alpha(L_1 + \cdots + L_d)}{\overline{\mathrm{VaR}}_\alpha(L_1 + \cdots + L_d)} = 1.$$

Conjecture: the same result holds also for non id rvs

Application to inhomogeneous data

- marginal losses are distributed like a Generalized Pareto Distribution (GPD), that is

$$F_i(x) = 1 - \left(1 + \xi_i \frac{x}{\beta_i}\right)^{-1/\xi_i}, \ x \geq 0, \ 1 \leq i \leq d.$$

- Moscadelli (2004) contains an analysis of the Basel II data on Operational Risk coming out of the second Quantitative Impact Study (QIS)

Business line	i	ξ_i	β_i
Corporate Finance	1	1.19	774
Trading & Sales	2	1.17	254
Retail Banking	3	1.01	233
Commercial Banking	4	1.39	412
Payment & Settlement	5	1.23	107
Agency Services	6	1.22	243
Asset Management	7	0.85	314
Retail Brokerage	8	0.98	124

Model uncertainty for non id risks

α	$\underline{\mathrm{VaR}}_\alpha(L^+)$	$\mathrm{VaR}^+_\alpha(L^+)$	$\mathrm{VaR}_\alpha(L^{\Pi,+})$	$\overline{\mathrm{VaR}}_\alpha(L^+)$
0.99	1.78×10^5	5.14×10^5	7.08×10^5	2.56×10^6
0.995	4.68×10^5	1.22×10^6	1.68×10^6	5.96×10^6
0.999	4.38×10^6	9.33×10^6	1.28×10^7	4.34×10^7

Table 3: Estimates for $\mathrm{VaR}_\alpha(L^+)$ for a random vector of $d = 8$ GPD-distributed risks having the parameters in Table 2 and different dependence assumptions, i.e. (from left to right) best-case dependence, comonotonicity, independence, worst-case dependence. Each estimate for $\underline{\mathrm{VaR}}_\alpha(L^+)$ and $\overline{\mathrm{VaR}}_\alpha(L^+)$ has been obtained via the RA in about 9 mins using $N = 2 \times 10^6$ and $\epsilon = 0.1$.

Adding additional information

- we show that additional positive dependence information added on top of the marginal distributions does not improve the VaR bounds substantially;

- we show that additional information on higher dimensional sub-vectors of marginals leads to possibly much narrower VaR bounds;

.

α	$\underline{\mathrm{VaR}}_\alpha(L^+)$	$\underline{\mathrm{VaR}}^r_\alpha(L^+)$	$\mathrm{VaR}^+_\alpha(L^+)$	$\mathrm{VaR}_\alpha(L^{\Pi,+})$	$\overline{\mathrm{VaR}}^r_\alpha(L^+)$	$\overline{\mathrm{VaR}}_\alpha(L^+)$
0.99	1.78×10^5	2.26×10^5	5.14×10^5	7.08×10^5	2.06×10^6	2.56×10^6
0.995	4.68×10^5	5.36×10^5	1.22×10^6	1.68×10^6	4.82×10^6	5.96×10^6
0.999	4.38×10^6	4.72×10^6	9.33×10^6	1.28×10^7	3.56×10^7	4.34×10^7

Table 7: Estimates for $\mathrm{VaR}\alpha(L^+)$ for the Moscadelli example under different dependence assumptions, i.e. (from left to right) best-case dependence, best-case under additional information, comonotonicity, independence, worst-case under additional information (risks are two-by-two independent), worst-case dependence.

Summary

The rearrangement algorithm computes numerically sharp bounds on the ES/VaR of a sum of dependent random variables.

- it can be used with *any* set of *inhomogeneous* marginals, with dimensions *d* up in the several hundreds and for any quantile level α

- using the connection to convex order, it can be used also to compute moment bounds on supermodular functions (-> asset pricing)

- accuracy/speed can be increased by introducing a randomized starting condition and a termination condition based on the required accuracy.

The main message coming from our papers is that currently a whole toolkit of analytical and numerical techniques is available to better understand the aggregation and diversification properties of non-coherent risk measures such as Value-at-Risk.

4. Further mathematical links

WORST ES SCENARIO

	1	2	3	Σ
1	9.00000	9.00000	9.00000	27.0000
2	9.17095	9.17095	9.17095	27.5129
3	9.35098	9.35098	9.35098	28.0530
4	9.54093	9.54093	9.54093	28.6228
5	9.74172	9.74172	9.74172	29.2252
6	9.95445	9.95445	9.95445	29.8634
7	10.18034	10.18034	10.18034	30.5410
8	10.42080	10.42080	10.42080	31.2624
9	10.67748	10.67748	10.67748	32.0325
10	10.95229	10.95229	10.95229	32.8569
11	11.24745	11.24745	11.24745	33.7423
12	11.56562	11.56562	11.56562	34.6969
13	11.90994	11.90994	11.90994	35.7298
14	12.28422	12.28422	12.28422	36.8527
15	12.69306	12.69306	12.69306	38.0792
16	13.14214	13.14214	13.14214	39.4264
17	13.63850	13.63850	13.63850	40.9155
18	14.19109	14.19109	14.19109	42.5733
19	14.81139	14.81139	14.81139	44.4342
20	15.51446	15.51446	15.51446	46.5434
21	16.32051	16.32051	16.32051	48.9615
22	17.25742	17.25742	17.25742	51.7723
23	18.36492	18.36492	18.36492	55.0948
24	19.70197	19.70197	19.70197	59.1059
25	21.36068	21.36068	21.36068	64.0820
26	23.49490	23.49490	23.49490	70.4847
27	26.38613	26.38613	26.38613	79.1584
28	30.62278	30.62278	30.62278	91.8683
29	37.72983	37.72983	37.72983	113.1895
30	53.77226	53.77226	53.77226	161.3168
Σ	494.99920	494.99920	494.99920	NA

WORST VAR SCENARIO

	1	2	3	
[1,]	16.32051	14.19109	13.63850	44.1501
[2,]	13.14214	17.25742	14.19109	44.5906
[3,]	12.28422	12.69306	19.70197	44.6793
[4,]	18.36492	13.63850	12.69306	44.6965
[5,]	11.56562	18.36492	14.81139	44.7419
[6,]	19.70197	13.14214	11.90994	44.7540
[7,]	12.69306	14.81139	17.25742	44.7619
[8,]	17.25742	11.24745	16.32051	44.8254
[9,]	11.90994	11.56562	21.36068	44.8362
[10,]	11.24745	21.36068	12.28422	44.8924
[11,]	21.36068	12.28422	11.24745	44.8924
[12,]	13.63850	19.70197	11.56562	44.9061
[13,]	15.51446	16.32051	13.14214	44.9771
[14,]	14.81139	11.90994	18.36492	45.0862
[15,]	10.95229	10.67748	23.49490	45.1247
[16,]	10.67748	23.49490	10.95229	45.1247
[17,]	23.49490	10.95229	10.67748	45.1247
[18,]	14.19109	15.51446	15.51446	45.2200
[19,]	26.38613	10.42080	10.18034	46.9873
[20,]	10.42080	10.18034	26.38613	46.9873
[21,]	10.18034	26.38613	10.42080	46.9873
[22,]	30.62278	9.74172	9.95445	50.3190
[23,]	9.95445	30.62278	9.74172	50.3190
[24,]	9.74172	9.95445	30.62278	50.3190
[25,]	9.54093	37.72983	9.35098	56.6217
[26,]	37.72983	9.35098	9.54093	56.6217
[27,]	9.35098	9.54093	37.72983	56.6217
[28,]	9.00000	9.17095	53.77226	71.9432
[29,]	9.17095	53.77226	9.00000	71.9432
[30,]	53.77226	9.00000	9.17095	71.9432
[31,]	494.99920	494.99920	494.99920	NA

The worst-VaR scenario (and the best-ES scenario) yields a dependence in which:

- either the rvs are very close to each other and sum up to something very close to the worst-VaR estimate (*complete mixability*)

- or one of the components is large and the others are small (*mutual exclusivity*)

These scenarios exhibit

negative dependence!

Complete mixability

Definition

A distribution *F* is called *d*-completely mixable if there exist *d* random variables $X_1, \ldots, X_d$ identically distributed as *F* such that

$$P(X_1 + \cdots + X_d = \text{constant}) = 1$$

Examples

- *F* is continuous with a *monotone* density on a bounded support and satisfies a moderate mean condition; see Wang and Wang (2011).

- *F* is continuous with a *concave* density on a bounded support; see Puccetti, Wang and Wang (2012).

Applications

Plays the role of the lower Frèchet bound in multidimensional optimization problems comonotonicity | complete mixability

$$\boldsymbol{X} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \\ 4 & 4 & 4 \\ 5 & 5 & 5 \end{bmatrix} \quad \begin{matrix} \mathbf{3} \\ \mathbf{6} \\ \mathbf{9} \\ \mathbf{12} \\ \mathbf{15} \end{matrix} \qquad \boldsymbol{X}' = \begin{bmatrix} 5 & 3 & 1 \\ 3 & 4 & 2 \\ 1 & 5 & 3 \\ 4 & 1 & 4 \\ 2 & 2 & 5 \end{bmatrix} \quad \begin{matrix} \mathbf{9} \\ \mathbf{9} \\ \mathbf{9} \\ \mathbf{9} \\ \mathbf{9} \end{matrix}$$

rearrangement = dependence

For N large enough, it is possible to approximate any dependence between N-discrete marginals by a proper rearrangement of the columns of $\boldsymbol{X}$; see Rüschendorf (1983) and Durante, F. and J.F. Sánchez (2012)

Overall conclusions

- We are able to compute reliable bounds on the VaR/ES of a sum.

- Rearrangements provide an effective way to handle dependence (alternative/complementary to copulas).

- The concept of complete mixability enters many important optimization problems as an extension of the lower Fréchet bound in dimensions $d \geq 3$.

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