An Academic Response to Basel 3.5

Risk Aggregation and Model Uncertainty

Paul Embrechts

RiskLab, Department of Mathematics, ETH Zurich

Senior SFI Professor

www.math.ethz.ch/~embrechts/

Joint work with A. Beleraj, G. Puccetti, L. Rüschendorf and R. Wang

Outline

1. Regulation
2. Basel 3.5 Question
3. VaR Aggregation
4. Model Uncertainty
5. Conclusion
6. References

Regulation

Four regulatory documents

R1: BCBS-Consultative Document, May 2012,
Fundamental review of the trading book (⇐ Basel 3.5)

R2: United States Senate, March 15, 2013,
JPMorgan Chase Whale trades: a case history of derivatives risks and abuses

R3: UK House of Lords/House of Commons, June 12, 2013,
Changing banking for good, Volumes I and II

R4: BCBS-Consultative Document, October 2013,
Fundamental review of the trading book: A revised market risk framework. (⇐ Basel 3.5)

- (In total, more than 1000 pages!)

Regulation

Some statements:

From R1: Page 20. *Choice of risk metric*:
"... However, a number of weaknesses have been identified with VaR, including its inability to capture "tail risk". The Committee therefore believes it is necessary to consider alternative risk metrics that may overcome these weaknesses."

From R2: Pages 13 and 172. *VaR models changes*:
"$7 billion, or more than 50% of the total $13 billion RWA reduction, could be achieved by modifying risk related models." "The change in VaR methodology effectively masked the significant changes in the portfolio."

Regulation

From R3: Volume II, page 119. *Output of a "stress test" excercise, from HBOS*:
"We actually got an external advisor [to assess how frequently a particular event might happen] and they came out with one in 100,000 years and we said "no", and I think we submitted one in 10,000 years. But that was a year and a half before it happened. It doesn't mean to say it was wrong: it was just unfortunate that the 10,000th year was so near."

Basel 3.5 Question

In this talk we focus on the following question raised by the Basel Committee:

From R1, Page 41, Question 8:

"What are the likely constraints with moving from VaR to ES, including any challenges in delivering robust backtesting, and how might these be best overcome?"

- A challenge for financial mathematicians and financial statisticians!

From R4, Page 3:

"the Committee has its intention to pursue two key confirmed reforms ... Move from Value-at-Risk (VaR) to Expected Shortfall (ES)."

Basel 3.5 Question

We focus on the mathematical and statistical aspects, avoiding discussion on practicalities and operational issues.

From R1, Page 3:

"The Committee recognises that moving to ES could entail certain operational challenges; nonetheless it believes that these are outweighed by the benefits of replacing VaR with a measure that better captures tail risk."

VaR and ES

Definition

$\mathrm{VaR}_\alpha(X)$, for $\alpha \in (0,1)$,

$$\mathrm{VaR}_\alpha(X) = F_X^{-1}(\alpha) = \inf\{x \in \mathbb{R} : F_X(x) \geq \alpha\}.$$

Definition

$\mathrm{ES}_\alpha(X)$, for $\alpha \in (0,1)$, if $\mathbb{E}[X] < \infty$,

$$\mathrm{ES}_\alpha(X) = \frac{1}{1-\alpha}\int_\alpha^1 \mathrm{VaR}_\delta(X)d\delta \underset{(F\text{ cont.})}{=} \mathbb{E}\left[X | X > \mathrm{VaR}_\alpha(X)\right].$$

VaR versus ES, extreme value theory

- For all $\alpha \in (0,1) \Rightarrow \text{ES}_\alpha(X) \geq \text{VaR}_\alpha(X)$.
- For light tailed distributions (such as $X \sim N(\mu, \sigma^2)$),

$$\lim_{\alpha \to 1} \frac{\text{ES}_\alpha(X)}{\text{VaR}_\alpha(X)} = 1.$$

- For heavy tailed distributions:
$P(X > x) = x^{-1/\xi} L(x),\ 0 < \xi < 1,\ L$ slowly varying,

$$\lim_{\alpha \to 1} \frac{\text{ES}_\alpha(X)}{\text{VaR}_\alpha(X)} = \frac{1}{1-\xi}.$$

VaR versus ES, 0.99 vs 0.975

From R4: Page 22, *Moving to expected shortfall*:
"... using an ES model, the Committee believes that moving to a confidence level of 97.5% (relative to the 99th percentile confidence level for the current VaR measure) is appropriate."

$\text{VaR}_{0.99}$ vs $\text{ES}_{0.975}$

- Example: $X \sim$ Normal(0,1).

$$\text{ES}_{0.975}(X) = 2.3378,$$

$$\text{VaR}_{0.99}(X) = 2.3263.$$

They are quite close for all normal models!

VaR versus ES, 0.99 vs 0.975

From EVT: approximately,

- for heavy-tailed risks, $\mathrm{ES}_{0.975}$ yields a more conservative value than $\mathrm{VaR}_{0.99}$;
- for light-tailed distributions, $\mathrm{ES}_{0.975}$ yields an equivalent regulation principle as $\mathrm{VaR}_{0.99}$;
- for risks that do not have a very heavy tail, it holds $\mathrm{ES}_{0.975}(X) \approx \mathrm{VaR}_{0.99}(X)$.

details

VaR Aggregation

Consider:

- One-period risk positions $X_1, \dots, X_d$ with known distribution functions (dfs) F_i, $i = 1, \dots, d$;
- Portfolio position $X_d^+ = X_1 + \cdots + X_d$;
- $\mathrm{VaR}_\alpha(X_i)$, $i = 1, \dots, d$, the marginal VaR's at the common confidence level $\alpha \in (0, 1)$.

Task:

$$\text{Calculate } \mathrm{VaR}_\alpha(X_d^+)$$

Problem:

- We need a *joint* model for the random vector $\mathbf{X} = (X_1, \dots, X_d)'$

VaR Aggregation

- **X** elliptical

$$\mathrm{VaR}_\alpha(X_d^+) \leq \sum_{1=1}^{d} \mathrm{VaR}_\alpha(X_i)$$

  Examples: multivariate Gaussian, multivariate Student t.

- **X** comonotone i.e. there exist increasing functions ψ_i, $i = 1, \ldots, d$ and a random variable Z so that

$$X_i = \psi_i(Z)$$

  then

$$\mathrm{VaR}_\alpha(X_d^+) = \sum_{i=1}^{d} \mathrm{VaR}_\alpha(X_i)$$

  i.e. VaR_α (like ES_α) is comonotone additive.

- Diversification benefit: one often uses

$$(1-\delta)\sum_{i=1}^{d} \mathrm{VaR}_\alpha(X_i),\ 0 < \delta < 1.$$

VaR Bounds

The Fréchet (unconstrained) problem

$$\underline{\mathrm{VaR}}_\alpha(X_d^+) = \inf_F \{\mathrm{VaR}_\alpha(X_1^F + \cdots + X_d^F) : X_i \overset{d}{\sim} F_i, i = 1, \ldots, d\}$$

$$\overline{\mathrm{VaR}}_\alpha(X_d^+) = \sup_F \{\mathrm{VaR}_\alpha(X_1^F + \cdots + X_d^F) : X_i \overset{d}{\sim} F_i, i = 1, \ldots, d\}$$

VaR Bounds

Equivalently, for $\mathcal{C}_d$ the space of all d-copulas

$$\underline{\mathrm{VaR}}_\alpha(X_d^+) = \inf_{C\in\mathcal{C}_d}\{\mathrm{VaR}_\alpha(X_1^C + \cdots + X_d^C) : X_i \overset{d}{\sim} F_i, \; i = 1,\ldots,d\}$$

$$\overline{\mathrm{VaR}}_\alpha(X_d^+) = \sup_{C\in\mathcal{C}_d}\{\mathrm{VaR}_\alpha(X_1^C + \cdots + X_d^C) : X_i \overset{d}{\sim} F_i, \; i = 1,\ldots,d\}$$

Recall from Sklar's Theorem: $F = C(F_1,\ldots,F_d)$.

VaR Bounds

$d = 2$

The sharp bounds $\overline{\text{VaR}}_\alpha(X_2^+)$ and $\underline{\text{VaR}}_\alpha(X_2^+)$ are known for *any* type of marginal distributions F_1, F_2. Analytic formulas are given in Makarov (1981) and Rüschendorf (1982).

▸ details

VaR Bounds

$d \geq 3$, Homogeneous case

- $\overline{\text{VaR}}_\alpha(X_d^+)$: A dual bound technique introduced in Embrechts and Puccetti (2006).
- Analytical results obtained for both $\overline{\text{VaR}}_\alpha(X_d^+)$ and $\underline{\text{VaR}}_\alpha(X_d^+)$ under a tail-monotone condition on F (mostly satisfied in practice) by Wang, Peng and Yang (2013), based on the concept of complete mixability.
- Sharpness of the dual bound of $\overline{\text{VaR}}_\alpha(X_d^+)$ under same conditions obtained by Puccetti and Rüschendorf (2013).

details

VaR Bounds

$d \geq 3$, Heterogeneous case

- Rearrangement Algorithm of Embrechts, Puccetti, Rüschendorf (2013) yields a powerful computational tool for the calculation of $\overline{\mathrm{VaR}}_\alpha(X_d^+)$ and $\underline{\mathrm{VaR}}_\alpha(X_d^+)$, and possibly $d \geq 1000$.
- Analytical approximation and connection with convex order are given by Bernard, Jiang and Wang (2014).

Dependence Uncertainty

Worst-dependence scenarios:

$$\overline{\mathrm{VaR}}_\alpha(X_d^+) = \sup_F \{\mathrm{VaR}_\alpha(X_1^F + \cdots + X_d^F) : X_i \overset{d}{\sim} F_i,\ 1 \leq i \leq d\}.$$

$$\overline{\mathrm{ES}}_\alpha(X_d^+) = \sup_F \{\mathrm{ES}_\alpha(X_1^F + \cdots + X_d^F) : X_i \overset{d}{\sim} F_i,\ 1 \leq i \leq d\}$$

$$= \sum_{i=1}^d \mathrm{ES}_\alpha(X_i).$$

Dependence Uncertainty

Two important measures

Measure 1 Superadditivity ratio

$$\overline{\triangle}_{\alpha,d}(X_d^+) = \frac{\overline{\mathrm{VaR}}_\alpha(X_d^+)}{\sum_{i=1}^d \mathrm{VaR}_\alpha(X_i)}.$$

Measure 2 Ratio between worst-ES and worst-VaR

$$\mathcal{B}_{\alpha,d}(X_d^+) = \frac{\overline{\mathrm{ES}}_\alpha(X_d^+)}{\overline{\mathrm{VaR}}_\alpha(X_d^+)} = \frac{\sum_{i=1}^d \mathrm{ES}_\alpha(X_i)}{\overline{\mathrm{VaR}}_\alpha(X_d^+)}.$$

Dependence Uncertainty

Superadditivity ratio: some examples

- Short tailed risks
  - LogNormal(2,1)-distributed risks $\Rightarrow \overline{\triangle}_{0.999,d}(X_d^+) \approx 1.4$.
  - Gamma(3,1)-distributed risks $\Rightarrow \overline{\triangle}_{0.999,d}(X_d^+) \approx 1.1$.
- Heavy tailed risks
  - Pareto(2)-distributed risks $\Rightarrow \overline{\triangle}_{0.999,d}(X_d^+) \approx 2$.

In QRM applications often Pareto(θ) with $\theta \in [0.5, 5]$:

- $[0.5, 1]$ catastrophe insurance,
- $[3, 5]$ market return data,
- $\theta \geq 0.5$ for operational risk.

VaR versus ES: Dependence Uncertainty

Asymptotic equivalence for large dimensions of the risk portfolio, under some general conditions:

$$\lim_{d\to\infty} \frac{\overline{\mathrm{ES}}_\alpha(X_d^+)}{\overline{\mathrm{VaR}}_\alpha(X_d^+)} = 1$$

▸ details

- In the case of F_i being identical:

$$\overline{\triangle}_{\alpha,d}(X_d^+) \approx \frac{\mathrm{ES}_\alpha(X_1)}{\mathrm{VaR}_\alpha(X_1)}.$$

Application: Operational Risk

Definition

Operational risk is the risk of losses resulting from inadequate or failed internal processes, people and systems, or external events.

Remark: This definition includes legal risk but excludes reputational and strategic risk.

Application: Operational Risk

The LDA Operational risk capital calculation under Basel II

The ingredients:

- Risk measure VaR_α
- Holding period: 1 year
- Confidence level: 99.9%, $\alpha = 0.999$
- The data 7×8 matrix; 8 Business lines, 7 Loss types
- Often: aggregate column-wise $\Rightarrow \mathrm{VaR}_\alpha^{(1)}, \ldots, \mathrm{VaR}_\alpha^{(8)}$

$$\text{Aggregate:} \quad \sum_{i=1}^{8} \mathrm{VaR}_\alpha^{(i)} = \mathrm{VaR}_\alpha^{+}.$$

Example: Pareto(2) risks

Sharp bounds on VaR and ES for the sum of d Pareto(2) distributed rvs for $\alpha = 0.999$; VaR_α^+ corresponds to the comonotonic case.

	$d = 8$	$d = 56$
$\underline{\mathrm{VaR}}_\alpha$	31	53
$\underline{\mathrm{ES}}_\alpha$	178	472
VaR_α^+	245	1715
$\overline{\mathrm{VaR}}_\alpha$	465	3454
$\overline{\mathrm{ES}}_\alpha$	498	3486
$\overline{\triangle}_\alpha(X_d^+)$	1.898	2.014
$\mathcal{B}_\alpha(X_d^+)$	1.071	1.009

An inhomogeneous Portfolio

Dependence-uncertainty spreads of VaR and ES for an inhomogeneous portfolio $X_d^+ = X_1 + \cdots + X_d$, where $X_i \sim \text{Pareto}(2 + 0.1i)$, $i = 1, \ldots, 5$; $X_i \sim \text{Exp}(i-5)$, $i = 6, \ldots, 10$; $X_i \sim \text{Log–Normal}(0, (0.1(i-10))^2)$, $i = 11, \ldots, 20$.

	$d=5$			$d=20$		
	best	worst	spread	best	worst	spread
$\text{ES}_{0.975}$	22.48	44.88	22.40	29.15	102.35	73.20
$\text{VaR}_{0.975}$	9.79	41.46	31.67	21.44	100.65	79.21
$\text{VaR}_{0.9875}$	12.06	56.21	44.16	22.12	126.63	104.51
$\text{VaR}_{0.99}$	12.96	62.01	49.05	22.29	136.30	114.01
$\frac{\overline{\text{ES}}_{0.975}}{\overline{\text{VaR}}_{0.975}}$		1.08			1.02	

Generally, $\text{VaR}_\alpha(X_d^+)$ has a larger DU-spread compared to $\text{ES}_\beta(X_d^+)$ for $\alpha \geq \beta$; see Embrechts, Wang and Wang (2014).

Backtesting

Recall from R1, Page 41, Question 8
"... *robust backtesting* ..."

Backtesting:

(i) estimate a risk measure from past observations;
(ii) test whether (i) is appropriate using future observations;
(iii) purpose: test and update risk measure forecasts.

Backtesting

Example - VaR backtesting:

(1) suppose the estimated/modeled VaR_α is V at $t = 0$;

(2) consider $A_t = \mathrm{I}_{\{X_t > V\}}$ based new iid observations X_t, $t > 0$;

(3) standard hypothesis testing methods for H_0: A_t are iid Bernoulli$(1 - \alpha)$ random variables.

For ES such simple and intuitive backtesting techniques do not exist!

Backtesting

Elicitability

- A new notion for comparing risk measure forecasts: elicitability; Gneiting (2011).
- Roughly speaking, a risk measure (statistical functional) $\rho : \mathcal{P} \to \mathbb{R}$ is elicitable if ρ is the unique solution to the following equation:

$$\rho(L) = \operatorname*{argmin}_{x \in \mathbb{R}} \mathbb{E}[s(x, L)],$$

where

- $s : \mathbb{R}^2 \to [0, \infty)$ is a strictly consistent scoring function;
- for example, the mean is elicitable with $s = (x - L)^2$.

Backtesting

Elicitability and backtesting

- suppose the estimated/modeled ρ is ρ_0 at $t = 0$;
- based on new iid observations X_t, $t > 0$, consider the statistics $s(\rho_0, X_t)$; for instance, test statistic can typically be chosen as $T_n(\rho_0) = \frac{1}{n}\sum_{t=1}^{n} s(\rho_0, X_t)$;
- $T_n(\rho_0)$: a statistic which indicates the goodness of forecasts.
- updating ρ: look at a minimizer for $T_n(\rho)$;
- the above procedure is model-independent.

Elicitable statistics are straightforward to backtest.

Backtesting

VaR vs ES: elicitability

Theorem (Gneiting (2011)).

Under general conditions,

- VaR is elicitable;
- ES is not elicitable.

Backtesting

Remarks:

- under specific EVT-based conditions, backtesting of ES is possible; see McNeil et al. (2005), p.163;
- the relevance of elicitability for risk management purposes is heavily contested:
  - Emmer, Kratz and Tasche (2014): alternative method for backtesting ES; favors ES.
  - Davis (2014): backtesting based on prequential principle; favors quantile-based statistics (VaR-type).

Robustness

Robustness - some quotes

A precise definition matters!

- Cont et al. (2010): "Our results illustrate in particular, that using recently proposed risk measures such as CVaR/Expected Shortfall leads to a less robust risk measurement procedure than Value-at-Risk."
- Kou et al. (2013): "Coherent risk measures are not robust", proposed Median Shortfall (VaR-like).
- Emmer et al. (2014): "The fact that VaR does not cover tail risks 'beyond' VaR is a more serious deficiency although ironically it makes VaR a risk measure that is more robust than the other risk measures we have considered."

Robustness

Example: different internal models

- Same data set, two different parametric models (e.g. normal vs student-t).
- Estimation of parameters, and compare the VaR and ES for two models.
- VaR is more robust in this setting, since it does not take the tail behavior into account (normal and student-t do not make a big difference).
- ES is less robust (heavy reliance on the model's tail behavior).
- Capital requirements: heavily depends on the internal models.

Robustness

Opposite opinions

- Cambou and Filipovic (2014): " ES is robust, and VaR is non-robust based on the notion of ϕ-divergence".
- Krätschmer et al. (2014): "We argue here that Hampel's classical notion of qualitative robustness is not suitable for risk measurement ..." (Introduce an index of qualitative robustness).
- BCBS (2013, R4): "This confidence level [97.5th ES] will provide a broadly similar level of risk capture as the existing 99th percentile VaR threshold, while providing a number of benefits, including generally more stable model output and often less sensitivity to extreme outlier observations."

Much more work is needed!

VaR versus ES: Summary

Value-at-Risk

1. Always exists
2. Only frequency
3. Non-coherent risk measure (non-subadditive)
   - Heavy tailed
   - Very skew
   - Special dependencies
4. Backtesting straightforward
5. Estimation (EVT)
6. Model uncertainty
7. Robust with respect to weak topology

Expected Shortfall

1. Needs first moment
2. Frequency and severity
3. Coherent risk measure (diversification benefit)
4. Backtesting an issue (non-elicitability)
5. Estimation (EVT)
6. Model uncertainty
7. Robust with respect to Wasserstein distance

Conclusion

C1 Q8 and Basel 3.5: a short question with many ramifications. No clear answer so far.

C2 On ES or VaR? ES! . . . however . . .

C3 Concerning MU and VaR bounds:

- Find sharp couplings
- Are they realistic in practice?
- Impose extra dependence assumptions
- Add statistical uncertainty

C4 Many more examples needed

C5 Expectiles as an alternative?

more discussions on risk measures

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THANK YOU!

VaR versus ES, 0.99 vs 0.975

- In general: for $\xi \in [0,1)$ ($\xi = 0$ indicates a light tail),

$$\frac{\mathrm{ES}_{0.975}(X)}{\mathrm{VaR}_{0.975}(X)} \approx \frac{1}{1-\xi},$$

and

$$\frac{\mathrm{VaR}_{0.99}(X)}{\mathrm{VaR}_{0.975}(X)} \approx 2.5^{\xi}.$$

Putting the above together,

$$\frac{\mathrm{VaR}_{0.99}(X)}{\mathrm{ES}_{0.975}(X)} \approx 2.5^{\xi}(1-\xi).$$

VaR versus ES, 0.99 vs 0.975

- $\xi \in [0, 1)$,

$$\frac{\mathrm{VaR}_{0.99}(X)}{\mathrm{ES}_{0.975}(X)} \approx 2.5^{\xi}(1-\xi) \leq e^{\xi}(1-\xi) \leq 1.$$

  Approximately, $\mathrm{ES}_{0.975}$ yields a more conservative regulation principle than $\mathrm{VaR}_{0.99}$.
- For a particular X, it is not always $\mathrm{ES}_{0.975}(X) \geq \mathrm{VaR}_{0.99}(X)$.

VaR versus ES, 0.99 vs 0.975

- Light-tailed distributions: as $\xi \to 0$,

$$\frac{\mathrm{VaR}_{0.99}(X)}{\mathrm{ES}_{0.975}(X)} \approx 2.5^{\xi}(1-\xi) \to 1.$$

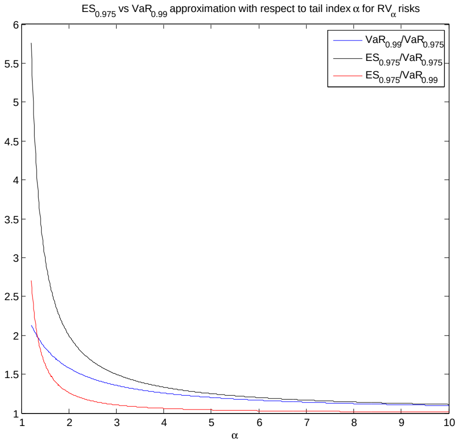
  For light-tailed distributions, $\mathrm{ES}_{0.975}$ yields an (approximately) equivalent regulation principle as $\mathrm{VaR}_{0.99}$.

- It seems that the value

$$c = 2.5 = (1-0.975)/(1-0.99)$$

  is chosen such that c is close to $e \approx 2.72$, so that the approximation $c^{\xi}(1-\xi) \approx 1$ holds most accurate for small ξ; note that $e^{-\xi} \approx 1-\xi$ for small ξ.

VaR versus ES, 0.99 vs 0.975 ($\alpha = 1/\xi$)



VaR Bounds

Makarov and Rüschendorf

For $d = 2$, sharp tail bound for any $s \in \mathbb{R}$ is:

$$\sup\{P(X_1 + X_2 \geq s) : X_i \sim F_i\} = \inf_{x \in \mathbb{R}} \{\overline{F}_1(x-) + \overline{F}_2(s - x)\},$$

where $\overline{F}_i(x) = 1 - F_i(x) = P(X_1 > x)$ and $\overline{F}_1(x-) = P(X_1 \geq x)$.

back

VaR Bounds

Sharp VaR bounds (Wang, Peng and Yang (2013))

Suppose that the density function of F is decreasing on $[b,\infty)$ for some $b \in \mathbb{R}$. Then, for $\alpha \in [F(b),1)$, and $X \overset{d}{\sim} F$,

$$\overline{\mathrm{VaR}}_\alpha(X_d^+) = d\mathbb{E}[X|X \in [F^{-1}(\alpha+(d-1)c_{d,\alpha}), F^{-1}(1-c_{d,\alpha})]],$$

$c_{d,\alpha}$ is the smallest number in $[0, \frac{1}{d}(1-\alpha)]$ s.t.

$$\int_{a+(d-1)c}^{1-c} F^{-1}(t)dt \geq \frac{1-\alpha-dc}{d}(F^{-1}(\alpha+(d-1)c)+F^{-1}(1-c)).$$

Red part clearly has an ES-type form ($c_{d,\alpha}=0$ leads to ES).

VaR Bounds

Sharp VaR bounds II

Suppose that the density function of F is decreasing on its support. Then for $\alpha \in (0,1)$ and $X \overset{d}{\sim} F$,

$$\underline{\mathrm{VaR}}_\alpha(X_d^+) = \max\{(d-1)F^{-1}(\alpha) + F^{-1}(0), d\mathbb{E}[X|X \le F^{-1}(\alpha)]\}.$$

Red part has a Left-Tail-ES-type form.

Complete Mixability

Definition (Complete mixability, Wang and Wang (2011))

A distribution function F on $\mathbb{R}$ is called d-completely mixable (d-CM) if there exist d random variables $X_1, \ldots, X_d$ identically distributed as F such that

$$P(X_1 + \cdots + X_d = dk) = 1, \tag{1}$$

for some $k \in \mathbb{R}$.

- Some examples of CM distributions: Normal, Student t, Cauchy, Uniform, Binomial.
- Most relevant result: F has a monotone density on a finite interval with a mean condition (depends on d) is d-CM.
  - Examples: truncated Pareto, Gamma, Log-normal.

back

Asymptotic Equivalence

Theorem (Embrechts, Wang and Wang (2014))

Suppose the continuous distributions F_i, $i \in \mathbb{N}$ satisfy that for $X_i \sim F_i$ and some $\alpha \in (0,1)$,

(i) $\mathbb{E}[|X_i - \mathbb{E}[X_i]|^k]$ is uniformly bounded for some $k > 1$;

(ii) $\liminf_{d\to\infty} \frac{1}{d}\sum_{i=1}^{d} \mathrm{ES}_\alpha(X_i) > 0$.

Then

$$\lim_{d\to\infty} \frac{\overline{\mathrm{ES}}_\alpha(X_d^+)}{\overline{\mathrm{VaR}}_\alpha(X_d^+)} = 1.$$

back

The Holy Triangle of Risk Measures

There are many desired properties of a good risk measure.
Some properties are without debate:

- cash-invariance: $\rho(X+c)=\rho(X)+c$, $c\in\mathbb{R}$;
- monotonicity: $\rho(X)\le\rho(Y)$ if $X\le Y$;
- identity: $\rho(1)=1$;
- law-invariance: $\rho(X)=\rho(Y)$ if $X=_d Y$.

(A standard risk measure; those properties are not restrictive)

The Holy Triangle of Risk Measures

In my opinion, in addition to being standard, the three key elements of being a good risk measure are

(C) Coherence (subadditivity): $\rho(X+Y) \leq \rho(X) + \rho(Y)$. [diversification benefit/capturing the tail/convex optimization/capital allocation]

(A) Comonotone additivity: $\rho(X+Y) = \rho(X) + \rho(Y)$ if X and Y are comonotone. [economical interpretation/distortion representation/non-diversification identity]

(E) Elicitability [robust estimation/backtesting straightforward].

The War of the Two Kingdoms

- Some financial mathematicians
  - appreciate coherence (subadditivity);
  - favor ES in general.
- Some financial statisticians
  - appreciate backtesting and statistical advantages;
  - favor VaR in general.

A natural question is to find a standard risk measure which is both coherent (subadditive) and elicitable.

Expectiles

Expectiles

- For $0 < \tau < 1$ and $X \in L^2$,

$$e_\tau(X) = \underset{x \in \mathbb{R}}{\operatorname{argmin}}\ \mathbb{E}[\tau \max(X - x, 0)^2 + (1 - \tau)\max(x - X, 0)^2].$$

- $e_\tau(X)$ is the unique solution x of the equation for $X \in L^1$:

$$\tau \mathbb{E}[(X - x)^+] = (1 - \tau)\mathbb{E}[(x - X)^+].$$

- $e_{1/2}(X) = \mathbb{E}[X]$.

Expectiles

The risk measure e_τ has the following properties:

1. homogeneous and standard,
2. subadditive for $1/2 \leq \tau < 1$, superadditive for $0 < \tau \leq 1/2$,
3. elicitable,
4. coherent for $1/2 \leq \tau < 1$,
5. not comonotone additive in general.

Bellini et al. (2014), Ziegel (2014), Delbaen (2014).

The War of the Three Kingdoms

In summary:

- VaR has (A) and (E): often criticized for not being subadditive: diversification/aggregation problems and inability to capture the tail!
- ES has (C) and (A): criticized for estimation, backtesting and robustness problems!
- Expectile has (C) and (E): criticized for lack of economical meaning, distributional computation and over-diversification benefits!

The War of the Three Kingdoms

The following holds (Bellini and Bignozzi (2014), Ziegel (2014)):

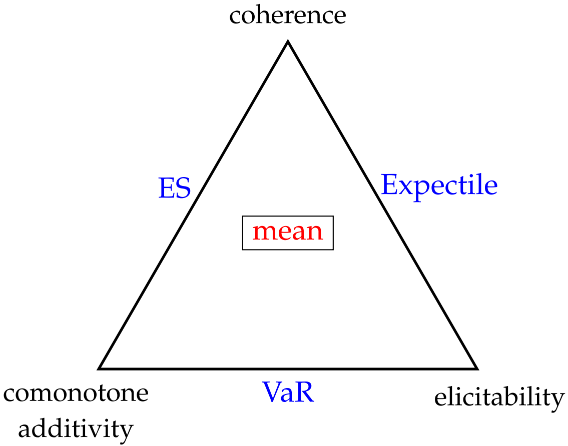
- if ρ is coherent, and elicitable with a convex scoring function, then ρ is an expectile;
- any spectral risk measure (coherent and comonotone additive) must not be elicitable, expect for the mean.

In summary:

The only standard risk measure that has (C), (A) and (E) is the mean, which is not a tail risk measure, and does not have a risk loading.

- Remark: the very old-school risk measure/pricing principle $\rho(X) = (1+\theta)\mathbb{E}[X]$, $\theta > 0$ has (C-subadditivity), (A) and (E), although it is not standard.

The Holy Triangle of Risk Measures



Extreme-aggregation Measure

- For any risk measure ρ, denote its worst-case value under dependence uncertainty as $\overline{\rho}(X_d^+)$.
- For $X \sim F$, let

$$\Gamma_\rho(X) = \limsup_{d\to\infty} \frac{1}{d}\overline{\rho}(X_d^+),$$

where $X_d^+ = X_1 + \cdots + X_d$ and $X_i \sim F$, $i = 1, \ldots, d$.

- Γ_ρ is called an extreme-scenario measure induced by ρ.
- Γ_ρ represents the limiting worst-case value of ρ for a homogeneous portfolio.
- Special case: $\Gamma_{\mathrm{VaR}_\alpha} = \mathrm{ES}_\alpha$.

Extreme-aggregation Measure

Theorem (Wang, Bignozzi and Tsanakas (2014))

For commonly used classes of risk measures ρ, Γ_ρ is a coherent risk measure. Moreover, it is

(a) *the smallest subadditive risk measure that dominates ρ;*

(b) *a spectral risk measure if ρ is a distortion risk measure;*

(c) *an expectile if ρ is a shortfall risk measure;*

(d) *the mean if ρ is a superadditive distortion risk measure.*

Extreme-aggregation Measure

When a non-coherent risk measure is used for a portfolio, its extreme behavior under dependence uncertainty leads to coherence.

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