Personal Finance and Life Insurance under Separation of Risk Aversion and Elasticity of Substitution

Ninna Reitzel Jensen* & Mogens Steffensen†

Abstract

In a classical Black-Scholes market, we establish a connection between two seemingly different approaches to continuous-time utility optimization. We study the optimal consumption, investment, and life insurance decision of an investor with power utility and an uncertain lifetime. To separate risk aversion from elasticity of inter-temporal substitution, we introduce certainty equivalents. We propose a time-inconsistent global optimization problem, and we present a verification theorem for an equilibrium control.

In the special case without mortality risk, we discover that our optimization approach is equivalent to recursive utility optimization with Epstein-Zin preferences. We find this interesting since our optimization problem has a more natural interpretation than the recursive utility optimization problem and since recursive utility, in contrast to our approach, gives rise to severe differentiability problems. Also, our optimization approach can there be seen as a generalization of recursive utility optimization with Epstein-Zin preferences to include mortality risk and life insurance.

Keywords: Recursive utility, lifetime uncertainty, stochastic control, generalized Hamilton-Jacobi-Bellman equation.

*PhD student, Department of Mathematical Sciences, University of Copenhagen, Universitetsparken 5, DK- 2100 København Ø, Denmark, e-mail: ninna@math.ku.dk.

†Professor, Department of Mathematical Sciences, University of Copenhagen, Universitetsparken 5, DK- 2100 København Ø, Denmark.

1 Introduction

In a classical Black-Scholes market, we establish a connection between two seemingly different approaches to continuous-time utility optimization for a certain-lived investor. One approach is recursive utility optimization with Epstein-Zin preferences, studied in Duffie and Epstein (1992) and Kraft and Seifried (2010) for general preferences. The other approach is non-linear expected power utility optimization with dynamic updating, studied in this paper for an uncertain-lived investor. This approach is apt for a set-up with mortality risk and utility from inheritance, and because of the established connection for a certain-lived investor, we propose to regard our approach as a generalization of the recursive utility approach to a set-up with mortality risk and life insurance.

Over time, the optimal consumption and investment decisions of a certain-lived investor has been treated in various papers. An important, early example is Merton (1971) who considers time-additive utility optimization in continuous time. Using dynamic programming techniques, the value function of the time-additive optimization problem can be characterized by a partial differential equation. The equation is called a Hamilton-Jacobi-Bellman equation, and it includes a term $u(c)$ where u is the investor's utility function for consumption and c is the consumption rate.

Richard (1975) generalized the work by Merton (1971) to include mortality risk and life insurance. The value function, V, of the generalized optimization problem is characterized by a partial differential equation similar to the original Hamilton-Jacobi-Bellman equation. The main alteration consists in addition of the term

$$\mu(t)\,\tilde{u}(b+x) - \mu(t)\,V(t,x) \ , \tag{1}$$

where μ is the investor's mortality intensity, $\tilde{u}$ is the investor's utility function for inheritance, b is a term insurance sum paid out upon death, and x is wealth. Also, there is an effect on the wealth dynamics due to financing of the term insurance. We note that $\mu(t)\,\tilde{u}(b+x)$ can be interpreted as the investor's probability weighted utility gain associated with death. Similarly, $\mu(t)\,V(t,x)$ can be interpreted as the investor's probability weighted utility loss associated with death. The term in (1) is therefore the investor's probability weighted net-gain associated with death.

Unfortunately, time-additive utility has the disadvantage that it mixes preferences for risk and preferences for inter-temporal substitution. The recursive utility approach and our approach both deal with this problem, in two seemingly different ways.

Recursive utility is founded in discrete time, and it allows for separation of preferences for risk and inter-temporal substitution through a recursive definition, a (utility) certainty equivalent and a time-aggregator. In Duffie and Epstein (1992), recursive utility is extended to continuous time where it is called stochastic differential utility. The link to discrete-time recursive utility is vague though, and in Kraft and Seifried (2010), the extension is refined and called continuous-time recursive utility. In both papers, the optimal consumption and investment decision of a certain-lived investor is studied. The value function, V, of the recursive optimization problem is characterized by a Hamilton-Jacobi-Bellman equation (in the following 'pseudo-Bellman equation') where the term $u(c,t)$ is replaced by a term $f(c,V(t,x))$. Here, f is the normalized aggregator representing the investor's preferences. In particular, Epstein-Zin preferences are represented by the aggregator

$$f(c,V)=\theta\delta V\left(\left(\frac{c}{((1-\gamma)\,V)^{\frac{1}{1-\gamma}}}\right)^{\frac{1-\gamma}{\theta}}-1\right).$$

The recursive optimization problem is less intuitive than the time-additive optimization problem, and to our knowledge, the literature contains no attempt to extend the recursive utility problem to a set-up with mortality risk and life insurance. However, inspired by the mortality extension in Richard (1975), it is natural to suggest a pseudo-Bellman equation where we combine $f(c, V)$ defined above with the additional term $\mu(t)\,\tilde{u}(b+x) - \mu(t)\,V(t,x)$.

For Epstein-Zin preferences, we present another suggestion—namely an alteration of the normalized aggregator (and no additional term). The altered aggregator arises from the following optimization approach: We consider an uncertain-lived investor with power utility. To separate preferences for risk and preferences for inter-temporal substitution, we introduce consumption certainty equivalents, and we propose a time-global optimization problem that is about maximizing an infinite sum of infinitesimally small certainty equivalents for future consumption and inheritance. The problem is non-linear in expectation, and it is therefore time-inconsistent (see e.g. Björk et al. (2012) for a description of time-inconsistency). To deal with the time-inconsistency, we search for an equilibrium control instead of a classical optimal control, and we present a verification theorem for a particular equilibrium control. The corresponding value function is characterized by a pseudo-Bellman equation where the term $f(c, V(t,x))$ is replaced by the term $\tilde{f}(t, c, x+b, V(t,x))$. Here, the altered aggregator $\tilde{f}$ is given by

$$\tilde{f}(t,c,y,V) = \theta\delta V\left(\left(\frac{c^{1-\gamma}}{V(1-\gamma)}\right)^{\frac{1}{\kappa}} + \left(\frac{\varepsilon(t)\,\mu(t)\,y^{1-\gamma}}{V(1-\gamma)}\right)^{\frac{1}{\kappa}}\right)^{\frac{\kappa}{\theta}} - (\mu(t)+\theta\delta)\,V\,.$$

For a certain-lived investor (i.e. $\mu = 0$), the two aggregators f and $\tilde{f}$ coincide. We say that our approach is equivalent to recursive utility optimization with Epstein-Zin preferences, for a certain-lived investor. Because of the equivalence, we propose the aggregator $\tilde{f}$ as a mortality extension of the normalized Epstein-Zin aggregator—that is, we propose to see our approach as a generalization of the recursive utility approach with Epstein-Zin preferences to a set-up with mortality risk and life insurance. This proposal is supported by the fact that our optimization problem has a natural interpretation, both with and without mortality risk. Furthermore, our approach is a generalization of the time-additive utility optimization in Richard (1975) to time-non-additive power utility.

Structure of the paper

In Section 2, we propose an optimization problem and define the concept of equilibrium controls. We present a verification theorem for a particular equilibrium control, and we derive closed-form expressions for the control and the corresponding value function. Finally, we compare our results to Richard (1975).

In Section 3, we give a short introduction to recursive utility, and we demonstrate the identity of our pseudo-Bellman equation and the pseudo-Bellman equation in Duffie and Epstein (1992). Also, we outline perspectives of the established equivalence.

In Section 4, we derive a stochastic differential equation for the optimal consumption rate from Section 2, and we construct numerical examples to illustrate how it differs from the optimal consumption rate from time-additive utility. The numerical examples all arise from the special case without market risk.

2 Optimization problem

2.1 Set-up

We consider an investor making decisions concerning consumption, investment, and life insurance in continuous time. We adopt the classical survival model and model the death of the investor by a mortality intensity μ. By N and $I = 1 - N$, we indicate whether the investor is dead or alive at a given point in time (e.g. $N(t) = 1$ if the investor is dead at time t). We treat N and I as stochastic processes on an abstract probability space $(\Omega, \mathcal{F}, P)$.

The investor has access to a classical Black-Scholes market consisting of a bank account, B, with risk free short rate r, and a stock, S, with excess return λ and volatility σ. The asset prices are described by the stochastic differential equations (SDEs)

$$\begin{aligned}
\mathrm{d}B(t) &= B(t)\, r \mathrm{d}t\ , \quad t \geq 0\ , \quad B(0) = 1\ , \\
\mathrm{d}S(t) &= S(t)\left[(r+\lambda)\,\mathrm{d}t + \sigma \mathrm{d}W(t)\right]\ , \quad t \geq 0\ , \quad S(0) = s_0\ ,
\end{aligned}$$

where $r, \lambda, \sigma > 0$ are constants, and W is a standard Brownian motion on the probability space $(\Omega, \mathcal{F}, P)$.

Also, the investor can trade term insurance contracts with a life insurance company. Note that there is no loss of generality in only considering term insurance, since all available life insurance products are linear combinations of term insurance contracts and a savings plan. A death sum b triggers premiums payments at rate $b\hat{\mu}$. Here, $\hat{\mu}$ is the mortality intensity used by the insurance company for pricing, and it may or may not be equal to μ. For simplicity, we assume that the insurance company does not build up reserves or pay out bonus. The term insurance completes the market.

We fix a time-horizon n that we think of as the investor's maximum remaining lifetime. The investor has wealth X and invests a proportion

π of X in the stock and a proportion $(1-\pi)$ of X in the bank account. As long as the investor is alive, she consumes at rate c, earns money at rate w (deterministic), and buys life insurance at premium rate $b\hat{\mu}$. When the investor dies, her inheritors receive the death sum b and the remaining wealth. While the investor is alive, her wealth is formalized by the SDE

$$\begin{aligned} \mathrm{d}X(t) &= X(t)\left[(r+\pi(t)\lambda)\,\mathrm{d}t + \pi(t)\,\sigma \mathrm{d}W(t)\right] \\ &\qquad - \left(c(t) + b(t)\,\hat{\mu}(t) - w(t)\right)\mathrm{d}t\ , \qquad t \in [0,n]\ , \\ X(0) &= x_0\ , \end{aligned} \tag{2}$$

where x_0 is the initial wealth of the investor, w is a continuous, deterministic function, and c, π, b are stochastic processes, i.e.

$$c, \pi, b : [0,n] \times \Omega \to \mathbb{R}\ . \tag{3}$$

In addition to the investor's monetary wealth, we also formalize the investor's human wealth which we denote by L. We do this here because the quantity arises in the solution to problems similar to ours. The investor's human wealth is the financial value of her future labour income, and it is given by

$$L(t) = \int_t^n w(s)\, e^{-\int_t^s (r+\hat{\mu}(v))\,\mathrm{d}v}\,\mathrm{d}s\ , \quad t \in [0,n]\ . \tag{4}$$

We note that $\hat{\mu}$ (and not μ) appears in (4) because $\hat{\mu}$ is the intensity used for pricing the term insurance, and this asset completes the market.

Since the investor cannot look into the future, it is natural to require that the set of control processes (c, π, b) is adapted to the wealth process X. However, for computational convenience, we go one step further and require that (c, π, b) is of feedback form, i.e.

$$\left(c(t), \pi(t), b(t)\right) = \left(\tilde{c}(t, X(t)), \tilde{\pi}(t, X(t)), \tilde{b}(t, X(t))\right)\ , \quad t \in [0,n]\ ,$$

for deterministic, measurable functions

$$\tilde{c}, \tilde{\pi}, \tilde{b} : [0,n] \times \mathbb{R} \to \mathbb{R}\ . \tag{5}$$

For simplicity, we redefine $(c, \pi, b) \equiv \left(\tilde{c}, \tilde{\pi}, \tilde{b}\right)$ and speak of the function (c, π, b) as a control. We thereby leave out the tildes in (5) and overtype the processes in (3). Now the SDE in (2) reads

$$\begin{aligned} \mathrm{d}X(t) &= X(t)\left[(r + \pi(t, X(t))\lambda)\,\mathrm{d}t + \pi(t, X(t))\,\sigma\mathrm{d}W(t)\right] \\ &\quad - \left(c(t, X(t)) + b(t, X(t))\,\hat{\mu}(t) - w(t)\right)\mathrm{d}t\ , \qquad t \in [0, n]\ , \\ X(0) &= x_0\ , \end{aligned} \tag{6}$$

where c, π, b are deterministic, measurable functions of time and wealth.

Remark 2.1. *To ensure that* (6) *makes sense, we only consider controls* (c, π, b) *for which the SDE in* (6) *has a unique solution. Also, we require that the investor's total wealth* $X + L$*, consumption rate* c*, and inheritance* $X + b$ *never fall below 0. To ensure this, we only consider controls* (c, π, b) *for which* $(c(t, x), \pi(t, x), b(t, x))$ *belongs to the set*

$$\Gamma(t, x) \equiv \begin{cases} [0, \infty) \times \mathbb{R} \times [-x, \infty) & \text{if } x + L(t) > 0\ , \\ \{0\} \times \{0\} \times \{-x\} & \text{if } x + L(t) = 0, L(t) > 0\ , \\ \{0\} \times \mathbb{R} \times \{0\} & \text{if } x = L(t) = 0\ . \end{cases}$$

It is easy to verify that this constraint ensures the required non-negativity. We say that a control (c, π, b) *is* admissible *if it meets the requirements above, and by* $\mathcal{U}$ *we denote the set of admissible controls. In Subsection 2.3, we impose some additional constraints on the admissible controls.*

2.2 Formulation

For a moment, we think of the investor as certain-lived, i.e. we let $\mu = \hat{\mu} = 0$ in the set-up from the previous subsection. Then a classical optimization problem on behalf of the investor is that of maximizing expected time-additive power utility of consumption, i.e.

$$\sup_{c, \pi} E\left[\int_0^n e^{-\delta t} \frac{1}{1-\gamma} c^{1-\gamma}(t, X(t))\ \mathrm{d}t\right]\ , \tag{7}$$

where $\delta \geq 0$ is a subjective utility discount rate, $\gamma > 0, \gamma \neq 1$, is thought of as risk aversion, and (c, π) is chosen among a suitable set of admissible controls. This problem can be dealt with by considering the value function

$$W(t,x) = \sup_{c,\pi} E_{t,x}\left[\int_t^n e^{-\delta s} \frac{1}{1-\gamma} c^{1-\gamma}(s, X(s))\, \mathrm{d}s\right] ,$$

where $E_{t,x}$ denotes conditional expectation given $X(t) = x$. By application of dynamic programming techniques, the value function can be characterized by the Hamilton-Jacobi-Bellman equation, i.e. a partial differential equation containing a local optimization problem at each point (t, x). Using the linearity of the expectation operator and the law of iterated expectation, it can be proven that the solution (c, π) to the continuum of local optimization problems is also a solution to the global optimization problem (see e.g. Chapter 19 in Björk (2009)). In the following, the linearity (in expectation) of the optimization problem is spoiled, and then there is no longer coincidence between local and global optimization.

We mentioned that γ is thought of as risk aversion, but γ also plays a role in the time-additivity of (7). The parameter γ does not only represent aversion towards risk, it is also related to the Elasticity of Inter-temporal Substitution (EIS). Whereas risk aversion expresses the investor's willingness to gamble, EIS expresses the investor's willingness to substitute consumption over time. To illustrate this, we take away the investor's option to invest in the stock. We are then faced with the deterministic optimization problem

$$\sup_c \int_0^n e^{-\delta t} \frac{1}{1-\gamma} c^{1-\gamma}(t, X(t))\, \mathrm{d}t \ , \tag{8}$$

where X is now a deterministic process. Since there is no risk left in the set-up, the solution to (8) should not be related to the investor's aversion towards risk, but the solution does depend on γ. Hence, we have found a way to formalize EIS in the case of no risk, and this motivates our formalization of EIS below, in the presence of risk.

In this paper, we separate risk aversion from EIS by forming certainty equivalents

$$u^{-1}\left(E\left[u\left(c\left(t, X\left(t\right)\right)\right)\right]\right) \ , \tag{9}$$

where u is a utility function representing the investor's preferences for risk. We then add certainty equivalents (while taking EIS into account) instead of adding utility. The entity in (9) is deterministic and expresses which certain time-t consumption rate the investor requires at time 0 in order to give up the uncertain time-t consumption rate $c\left(t, X\left(t\right)\right)$. In the case of power utility, i.e. $u\left(c\right) = \frac{1}{1-\gamma}c^{1-\gamma}$, the certainty equivalent in (9) equals

$$\left(E\left[c^{1-\gamma}\left(t, X\left(t\right)\right)\right]\right)^{\frac{1}{1-\gamma}} \ .$$

For the addition of certainty equivalents, we introduce an EIS-parameter $\phi > 0, \phi \neq 1$, and formalize EIS as in (8). This gives us the problem

$$\sup_{c,\pi} \int_0^n e^{-\delta t} \frac{1}{1-\phi} \left(E\left[c^{1-\gamma}\left(t, X\left(t\right)\right)\right]\right)^{\frac{1}{\theta}} \mathrm{d}t \tag{10}$$

with $\theta = \frac{1-\gamma}{1-\phi}$. The special case $\gamma = \phi$ corresponds to the problem in (7). Given basic knowledge of dynamic programming, it is clear that the problem in (10) cannot be dealt with using classical dynamic programming techniques. This is due to the power $\frac{1}{\theta}$. While we are at spoiling linearity, we make yet another transformation and face the problem

$$\sup_{c,\pi} \frac{1}{1-\gamma} \left(\int_0^n \delta e^{-\delta t} \left(E\left[c^{1-\gamma}\left(t, X\left(t\right)\right)\right]\right)^{\frac{1}{\theta}} \mathrm{d}t\right)^{\theta} \ . \tag{11}$$

This problem is equivalent to the problem in (10)—that is, if $\delta > 0$ and $\left(1-\phi\right)\left(1-\gamma\right) > 0$. By 'equivalent' we mean that the control $\left(c, \pi\right)$ realizing the supremum in (10) is identical to the control $\left(c, \pi\right)$ realizing the supremum in (11). From now on, we assume that $\delta > 0$ and $\left(1-\phi\right)\left(1-\gamma\right) > 0$, and it turns out that the problem in (11) is more convenient to work with than the problem in (10). The constants δ and $\frac{1}{1-\gamma}$ match the powers $-\delta$ and $1-\gamma$ (which is convenient for differentiation), and in some ways, the power θ offsets the complications from the power $\frac{1}{\theta}$. We note that the factor $\frac{1}{1-\gamma}$ is placed outside the integral (and the parentheses) because the factor can be negative and should therefore not be taken to the power θ or $\frac{1}{\theta}$.

Finally, we go back to the original set-up with mortality risk. We assume that the processes N and I are independent of the process W, and we propose to consider the generalized optimization problem

$$\sup_{(c,\pi,b)\in\mathcal{U}} \frac{1}{1-\gamma}\left(\int_0^n \delta e^{-\delta t}\left(\begin{array}{c}\left(E\left[c^{1-\gamma}\left(t,X^{c,\pi,b}(t)\right)\frac{I(t)\,\mathrm{d}t}{\mathrm{d}t}\right]\right)^{\frac{1}{\kappa}}+\\ \left(E\left[\begin{array}{c}\varepsilon(t)\frac{\mathrm{d}N(t)}{\mathrm{d}t}\times\\ \left(\begin{array}{c}X^{c,\pi,b}(t)+\\ b\left(t,X^{c,\pi,b}(t)\right)\end{array}\right)^{1-\gamma}\end{array}\right]\right)^{\frac{1}{\kappa}}\end{array}\right)^{\frac{\kappa}{\theta}}\mathrm{d}t\right)^{\theta}, \tag{12}$$

where the expectation operates on all underlying random variables (i.e. W, N, and I), $\mathcal{U}$ is the set of admissible controls defined in Remark 2.1, and ε is a non-negative, continuous, deterministic weight function. Up to a scaling, the first mean value is the expected utility from consumption, and the second mean value is the expected utility from inheritance. We have included the function ε to allow for a different weight on inheritance than on consumption and to allow for a changing weight on inheritance throughout life. We have introduced the additional parameter $\kappa>0$ in order to separate risk aversion from elasticity of substitution between consumption and inheritance, and we have decorated X with superscript c,π,b to emphasize that it is the wealth process stemming from the control (c,π,b). *Altogether, the generalized problem in* (12) *is a question of maximizing an infinite sum of infinitesimal certainty equivalents for future consumption and inheritance.* The problem is complicated, and inspired by dynamic programming, one could try to look at the value function

$$W\left(t,x\right)=\sup_{\left(c,\pi,b\right)\in\mathcal{U}}Z^{c,\pi,b}\left(t,x\right) \tag{13}$$

where the objective function $Z^{c,\pi,b}:[0,n]\times\mathbb{R}\to\mathbb{R}$ is given by

$$\begin{aligned}
&Z^{c,\pi,b}\left(t,x\right)= \qquad (14)\\
&\frac{1}{1-\gamma}\left(\int_0^n\delta e^{-\delta(s-t)}\left(\begin{array}{c}\left(E^0_{t,x}\left[c^{1-\gamma}\left(t,X^{c,\pi,b}\left(t\right)\right)\frac{I(t)\,\mathrm{d}t}{\mathrm{d}t}\right]\right)^{\frac{1}{\kappa}}+\\ \left(E^0_{t,x}\left[\begin{array}{c}\varepsilon\left(t\right)\frac{\mathrm{d}N(t)}{\mathrm{d}t}\times\\ \left(\begin{array}{c}X^{c,\pi,b}\left(t\right)+\\ b\left(t,X^{c,\pi,b}\left(t\right)\right)\end{array}\right)^{1-\gamma}\end{array}\right]\right)^{\frac{1}{\kappa}}\end{array}\right)^{\frac{\kappa}{\theta}}\mathrm{d}t\right)^{\theta}=\\
&\frac{1}{1-\gamma}\left(\int_t^n\delta e^{-\delta(s-t)}e^{-\frac{1}{\theta}\int_t^s\mu(v)\,\mathrm{d}v}\left(\begin{array}{c}\left(m^{c,\pi,b}\left(t,s,x\right)\right)^{\frac{1}{\kappa}}+\\ \left(n^{c,\pi,b}\left(t,s,x\right)\right)^{\frac{1}{\kappa}}\end{array}\right)^{\frac{\kappa}{\theta}}\mathrm{d}s\right)^{\theta}
\end{aligned}$$

with

$$\begin{aligned}
m^{c,\pi,b}\left(t,s,x\right)&=E_{t,x}\left[c^{1-\gamma}\left(s,X^{c,\pi,b}\left(s\right)\right)\right]\ ,\\
n^{c,\pi,b}\left(t,s,x\right)&=E_{t,x}\left[\varepsilon\left(s\right)\mu\left(s\right)\left(X^{c,\pi,b}\left(s\right)+b\left(s,X^{c,\pi,b}\left(s\right)\right)\right)^{1-\gamma}\right]\ .
\end{aligned}$$

The operators $E_{t,x}$ and $E^0_{t,x}$ denote conditional expectation given $X^{c,\pi,b}\left(t\right)=x$ and $\left(X^{c,\pi,b}\left(t\right),N\left(t\right)\right)=\left(x,0\right)$, respectively. The second equality in (14) follows from independence between $\left(N,I\right)$ and W. By construction, $\left(1-\gamma\right)Z^{c,\pi,b}$ is non-negative, and in general, we assume that $Z^{c,\pi,b}\left(t,x\right)$ is non-zero for $x+L\left(t\right)>0$ and $t<n$.

Given the non-linearity (in conditional expectation) of $Z^{c,\pi,b}$, the solution to (12) is likely to be inconsistent with the solution to (13) for $t>0$. By 'inconsistent' we mean that the decision we make at time t based on (13) is not the same as the decision we plan to make at time t based on (12), for the same realization of the wealth process. More formally, if we denote the two solutions by (c^0, π^0, b^0) and (c^t, π^t, b^t), it might be that

$$\begin{aligned}&\left(c^0\left(t, X^{c^0,\pi^0,b^0}(t)\right), \pi^0\left(t, X^{c^0,\pi^0,b^0}(t)\right), c^0\left(t, X^{c^0,\pi^0,b^0}(t)\right)\right)\\ &\quad\neq \left(c^t\left(t, X^{c^0,\pi^0,b^0}(t)\right), \pi^t\left(t, X^{c^0,\pi^0,b^0}(t)\right), c^t\left(t, X^{c^0,\pi^0,b^0}(t)\right)\right) .\end{aligned}$$

We dislike this time-inconsistency, and we do not wish to introduce pre-commitment. Instead, we take inspiration from Björk et al. (2012), discard the optimization problem in (12)–(13), and search for an equilibrium control for the objective function $Z^{c,\pi,b}$, $(c,\pi,b) \in \mathcal{U}$. The equilibrium formulation arises from a game theoretic approach to stochastic control problems, and rewriting Definition 2.1 in Björk et al. (2012) in the language of this paper, we get the following definition:

Definition 2.1 (Equilibrium)**.** Consider a set of admissible controls $\bar{\mathcal{U}}$ and a control (c^*, π^*, b^*) in $\bar{\mathcal{U}}$ (informally viewed as a candidate equilibrium control). Choose a fixed control $\left(\bar{c}, \bar{\pi}, \bar{b}\right) \in \bar{\mathcal{U}}$, a real number $h > 0$, and a initial point $(u, y) \in [0, n] \times \mathbb{R}$. Define the control $\left(c^h, \pi^h, b^h\right)$ by

$$\left(c^h, \pi^h, b^h\right)(t,x) = \begin{cases} (\bar{c}, \bar{\pi}, \bar{b})(t,x) , & u \le t < u+h, x \in \mathbb{R} , \\ (c^*, \pi^*, b^*)(t,x) , & u+h \le t \le n, x \in \mathbb{R} . \end{cases}$$

If for all controls $\left(\bar{c}, \bar{\pi}, \bar{b}\right) \in \bar{\mathcal{U}}$ and all points $(u, y) \in [0, n] \times \mathbb{R}$

$$\liminf_{h\to 0} \frac{Z^{c^*,\pi^*,b^*}(u,y) - Z^{c^h,\pi^h,b^h}(u,y)}{h} \ge 0 , \tag{15}$$

we say that (c^*, π^*, b^*) is an *equilibrium control* for the function $Z^{c,\pi,b}$, $(c, \pi, b) \in \bar{\mathcal{U}}$. The corresponding *equilibrium value function* V is given by

$$V(t,x) = Z^{c^*,\pi^*,b^*}(t,x) .$$

Remark 2.2. *We stress that an equilibrium control is not optimal in the sense that it realizes the supremum in* (12) *(or* (13) *for that matter). However, the control is optimal in the 'intuitive' sense that it maximizes the* *investor's total utility given that the investor continues to use the control. Therefore, we use the terms equilibrium control and* optimal control *interchangeably. With this convention, there might be several or even no optimal controls because Björk et al. (2012) prove neither existence nor uniqueness of the equilibrium control.*

In the next subsection, we present a verification theorem for a particular optimal control and the corresponding equilibrium value function. Furthermore, we present closed form expressions for the control and the corresponding value function. To get the proof running, we need to introduce of set of non-standard assumptions, see page 40. These assumption serve to prove that the equilibrium condition in (15) is satisfied. Also, we need to impose some additional constraints on the set of admissible controls and on the candidate equilibrium control, but these are all are standard regularity conditions, see the theorem below.

2.3 Solution

Theorem 2.1 (Verification theorem)**.** *Define the set of admissible controls, $\mathcal{U}^e$, as those controls (c,π,b) in $\mathcal{U}$ (see Remark 2.1) for which the partial differential equations (PDEs) in* (29) *have solutions in $\mathcal{C}^{1,0,2}$ and the stochastic integrals in* (30)–(31) *are martingales. Also, define the function $f:[0,n]\times(0,\infty)^2\times\left(1_{\{\gamma<1\}}(0,\infty)\cup 1_{\{\gamma>1\}}(-\infty,0)\right)\to\mathbb{R}$ by*

$$\begin{aligned} f(t,c,y,z) = {} & \theta\delta z\left(\left(\frac{c^{1-\gamma}}{z(1-\gamma)}\right)^{\frac{1}{\kappa}}+\left(\frac{\varepsilon(t)\,\mu(t)\,y^{1-\gamma}}{z(1-\gamma)}\right)^{\frac{1}{\kappa}}\right)^{\frac{\kappa}{\theta}} \\ & -(\mu(t)+\theta\delta)\,z\ . \end{aligned} \tag{16}$$

Assume that there exist functions (U,l_1,l_2) in

$$\mathcal{C}^{1,2}([0,n]\times\mathbb{R})\times\mathcal{C}^{1,0,2}\left([0,n]^2\times\mathbb{R}\right)\times\mathcal{C}^{1,0,2}\left([0,n]^2\times\mathbb{R}\right)$$

such that the function U solves the pseudo-Bellman equation

$$U_t(t,x) = \inf_{(c,\pi,b)\in\Gamma(x,t)} \left[\begin{array}{c} -f(t,c,x+b,U(t,x)) \\ -((r+\pi\lambda)x - c - \hat{\mu}(t)b + w(t))U_x \\ -\frac{1}{2}\sigma^2\pi^2x^2U_{xx}(t,x) \end{array} \right], \quad (17)$$

$$U(n,x) = 0 \ ,$$

and such that the functions l_1 and l_2, for each fixed s, solve the PDEs

$$(l_i)_t(t,s,x) = -\left[\begin{array}{c} x(r+\pi^*(t,x)\lambda) - c^*(t,x) \\ -\hat{\mu}(t,)b^*(t,x) + w(t) \end{array} \right] \times (l_i)_x(t,s,x)$$
$$- \frac{1}{2}(\pi^*(t,x))^2\sigma^2x^2(l_i)_{xx}(t,s,x), \ i = 1,2 \ , \quad (18)$$
$$l_1(s,s,x) = (c^*)^{1-\gamma}(s,x) \ ,$$
$$l_2(s,s,x) = \varepsilon(s)\mu(s)(x+b^*(s,x))^{1-\gamma} \ ,$$

where (c^*,π^*,b^*) is the function of (t,x) that realizes the infimum in (17).

Also, assume that the stochastic integrals in (39)–(40) are martingales, that the SDE in (6) has a unique solution for (c^*,π^*,b^*), and that the stochastic integrals in (30)–(31) are martingales for (c^*,π^*,b^*). Finally, assume that the assumptions on page 40 are satisfied.

Then (c^*,π^*,b^*) is a control in $\mathcal{U}^e$, and it is an optimal control for the function $Z^{c,\pi,b}$, $(c,\pi,b)\in\mathcal{U}^e$, defined in (14). The corresponding equilibrium value function V is given by

$$V(t,x) = U(t,x) \ ,$$

and it holds that

$$m^{c^*,\pi^*,b^*}(t,s,x) = l_1(t,s,x) \ ,$$
$$n^{c^*,\pi^*,b^*}(t,s,x) = l_2(t,s,x) \ .$$

Proof. The proof is presented in Appendix A on page 35. To get all the way from a more general pseudo-Bellman equation in the proof in (37) to the pseudo-Bellman equation in (17), we assume, at some point, that the solution is separable in wealth. However, we have built Theorem 2.1 around the version in (17) in order to visualize the connection to recursive utility. The solution we present in Theorem 2.2 below is separable and makes the assumption appear innocent. □

We note that we have replaced the global optimization problem in (13) with the continuum of local optimization problems in (17). Also, we recognize f as a generalization of the normalized continuous-time Epstein-Zin aggregator. We comment more on this in Section 3. We call the PDE in (17) a pseudo-Bellman equation because it bears resemblance to—but is different from—the Hamilton-Jacobi-Bellman equation known from dynamic programming.

Theorem 2.2 (Optimal control)**.** *Define the function $g : [0, n] \to \mathbb{R}$ by*

$$g(t) = \delta \left(\int_t^n \tilde{\mu}(s) \, e^{-\int_t^s \tilde{r}(v) \, \mathrm{d}v} \, \mathrm{d}s \right)^{\phi} \ , \quad t \leq n \ ,$$

where

$$\tilde{r}(v) = -\frac{1}{\phi} \left[(1-\phi) \left(r + \frac{1}{2} \frac{1}{\gamma} \frac{\lambda^2}{\sigma^2} + \hat{\mu}(v) - \frac{\mu(v)}{1-\gamma} \right) - \delta \right] \ ,$$

$$\tilde{\mu}(s) = \left(1 + \left(\frac{\varepsilon(s) \, \mu(s)}{\hat{\mu}^{1-\gamma}(s)} \right)^{\frac{1}{\gamma+\kappa-1}} \right)^{\frac{(\kappa-1+\gamma)(1-\phi)}{(1-\gamma)\phi}} \ .$$

Moreover, define the functions $h_1, h_2 : [0, n]^2 \to \mathbb{R}$ by

$$h_i(t, s) = b_i(s) \, e^{-\int_t^s a(v) \, \mathrm{d}v} \ , \quad i = 1, 2 \ , \quad t \leq s \ ,$$

where

$$
\begin{aligned}
a(v) = &-(1-\gamma)\left(r+\hat{\mu}(v)+\frac{1}{2}\frac{\lambda^2}{\gamma\sigma^2}\right)\\
&-(1-\gamma)\left(-\delta^{\frac{1}{\phi}}g^{-\frac{1}{\phi}}(v)\left(1+\left(\frac{\varepsilon(v)\,\mu(v)}{\hat{\mu}^{1-\gamma}(v)}\right)^{\frac{1}{\gamma+\kappa-1}}\right)^{\frac{(\kappa-1+\gamma)(1-\phi)}{(1-\gamma)\phi}}\right) ,\\
b_1(s) = &\left(\delta^{\frac{1}{\phi}}g^{-\frac{1}{\phi}}(s)\right)^{1-\gamma}\left(1+\left(\frac{\varepsilon(s)\,\mu(s)}{\hat{\mu}^{1-\gamma}(s)}\right)^{\frac{1}{\gamma+\kappa-1}}\right)^{\frac{\kappa-\phi\kappa-1+\gamma}{\phi}} ,\\
b_2(s) = &\, b_1(s)\left(\frac{\varepsilon(s)\,\mu(s)}{\hat{\mu}^{1-\gamma}(s)}\right)^{\frac{\kappa}{\gamma+\kappa-1}} .
\end{aligned}
$$

The optimal control from Theorem 2.1 is given by

$$
\begin{aligned}
c^*(t,x) &= \delta^{\frac{1}{\phi}}g^{-\frac{1}{\phi}}(t)\left(1+\left(\frac{\varepsilon(t)\,\mu(t)}{\hat{\mu}^{1-\gamma}(t)}\right)^{\frac{1}{\gamma+\kappa-1}}\right)^{\frac{\kappa-\phi\kappa-1+\gamma}{(1-\gamma)\phi}}(x+L(t)) ,\\
\pi^*(t,x)\,x &= \frac{\lambda}{\gamma\sigma^2}(x+L(t)) , \qquad (19)\\
b^*(t,x)+x &= c^*(t,x)\left(\frac{\varepsilon(t)\,\mu(t)}{\hat{\mu}^{\kappa}(t)}\right)^{\frac{1}{\gamma+\kappa-1}} ,
\end{aligned}
$$

and it holds that

$$
\begin{aligned}
V(t,x) &= \frac{1}{1-\gamma}(x+L(t))^{1-\gamma}g^{\theta}(t) ,\\
m^{c^*,\pi^*,b^*}(t,s,x) &= (x+L(t))^{1-\gamma}h_1(t,s) ,\\
n^{c^*,\pi^*,b^*}(t,s,x) &= (x+L(t))^{1-\gamma}h_2(t,s) .
\end{aligned}
$$

Proof. The proof is presented in Appendix A on page 35. □

We note that c^*, π^*, and b^*+x are all directly proportional to the investor's total wealth $x+L$. The optimal proportion π^* of wealth to invest in the stock is independent of the elasticity parameters κ and ϕ, and it is the same as in the well-known case of time-additive utility. The expressions for the optimal consumption rate and the optimal inheritance are more complicated, but the optimal consumption is directly proportional to the optimal inheritance, and the optimal consumption rate can be written as

$$c^*(t,x) = \frac{x + \int_t^n w(s)\, e^{-\int_t^s (r+\hat{\mu}(v))\,\mathrm{d}v}\,\mathrm{d}s}{\int_t^n \tilde{\mu}(s)\, e^{-\int_t^s \tilde{r}(v)\,\mathrm{d}v}\,\mathrm{d}s}\,\tilde{\mu}^{\frac{\kappa-\phi\kappa-1+\gamma}{(\kappa-1+\gamma)(1-\phi)}}(t) \ .$$

Also, we note that in the case $\varepsilon = 0$ (i.e. the investor does not care about inheritance, for example because she does not have dependants), it holds that $b^*(\cdot,x) = -x$. This means that the investor continuously *sells* term insurance with a death sum equal to her wealth. Thereby, she jeopardizes her wealth in the case of death, in return for a higher consumption rate while alive. This is the design of a life annuity, and it is a reasonable life insurance decision for an investor without dependants.

2.4 Comparison to Richard (1975)

In this subsection, we consider the special case of time-additive utility, i.e. the case $\phi = \gamma$ and $\kappa = 1$. Letting $\phi = \gamma = K$ and (innocently) dividing by $e^{\delta t}$, the global optimization problem in (13) reduces to

$$\sup_{(c,\pi,b)\in\mathcal{U}} E_{t,x}\left[\int_t^n \delta e^{-\delta s} e^{-\int_t^s \mu(v)\,\mathrm{d}v} \left(\begin{array}{c} \frac{c^{1-K}\left(s,X^{c,\pi,b}(s)\right)}{1-K} + \\ \varepsilon(s)\,\mu(s) \times \\ \frac{\left(X^{c,\pi,b}(s)+b\left(s,X^{c,\pi,b}(s)\right)\right)^{1-K}}{1-K} \end{array}\right) \mathrm{d}s\right]. \quad (20)$$

This simpler problem of maximizing expected time-additive utility for an uncertain-lived investor is treated in Richard (1975) (without an explicit state dependent constraint on the controls in $\mathcal{U}$). Richard (1975) allows for a much broader variety of utility functions than power utility functions, but in Section 4, focus is limited to (weighted) power utility. If we, in

Section 4 of Richard (1975), let the constant relative risk aversion be given by $\gamma = 1 - K$, and if we let the weights be given by

$$h(t) = \delta e^{-\delta t}\ , \qquad m(t) = \varepsilon(t)\,\delta e^{-\delta t}\ ,$$

then the optimization problem in Richard (1975) coincides with the optimization problem in (20). Due to the time-additivity of the simplified problem, time-inconsistency is no longer an issue, and we wonder how our 'equilibrium' optimal control relates to the 'classical' optimal control in Richard (1975). With $\phi = \gamma = K$ and $\kappa = 1$, our optimal control (c^*, π^*, b^*) is given by

$$\begin{aligned}
\frac{c^*(t,x)}{x + L(t)} &= \delta^{\frac{1}{K}} g^{-\frac{1}{K}}(t)\ ,\\
\frac{\pi^*(t,x)\,x}{x + L(t)} &= \frac{\lambda}{K\sigma^2}\ ,\\
\frac{b^*(t,x) + x}{x + L(t)} &= \left(\frac{\varepsilon(t)\,\mu(t)}{\hat{\mu}(t)}\right)^{\frac{1}{K}} \delta^{\frac{1}{K}} g^{-\frac{1}{K}}(t)\ ,
\end{aligned}$$

where

$$L(t) = \int_t^n w(s)\, e^{-\int_t^s (r + \hat{\mu}(v))\,\mathrm{d}v}\,\mathrm{d}s\ ,$$

and

$$g(t) = e^{\delta t}\left(\int_t^n \begin{pmatrix} \left(1 + \varepsilon^{\frac{1}{K}}(s)\,\mu(s)\left(\frac{\mu(s)}{\hat{\mu}(s)}\right)^{\frac{1-K}{K}}\right)\times \\ \left(\delta e^{-\delta s}\right)^{\frac{1}{K}} e^{-\int_t^s \mu(v)\,\mathrm{d}v}\times \\ e^{\frac{1-K}{K}\left(r + \frac{1}{2}\frac{1}{K}\frac{\lambda^2}{\sigma^2}\right)(s-t) + \frac{1-K}{K}\int_t^s (\hat{\mu}(v) - \mu(v))\,\mathrm{d}v} \end{pmatrix} \mathrm{d}s\right)^K\ .$$

When writing down expressions for the optimal control in Richard (1975), we make use of the following correspondence between our notation and Richard's notation:

Us	λ	b	μ	$\hat{\mu}$	X	π	L	$\hat{\mu} - \mu$	$e^{-\int_0^t \mu(s)\,\mathrm{d}s}$
Richard	$\alpha - r$	$P\mu^{-1}$	λ	μ	W	w	b	η	$G(t)$

With $h(t) = \delta e^{-\delta t}$, $m(t) = \varepsilon(t)\,\delta e^{-\delta t}$, and $\gamma = 1 - K$ in Section 4 of Richard (1975), the 'classical' optimal control $(c^{**}, \pi^{**}, b^{**})$ is given by

$$\begin{aligned}
\frac{c^{**}(t,x)}{x+L(t)} &= \left(\delta e^{-\delta t}\right)^{\frac{1}{K}} a^{-\frac{1}{K}}(t) \ , \\
\frac{\pi^{**}(t,x)\,x}{x+L(t)} &= \frac{\lambda}{K\sigma^2} \ , \\
\frac{b^{**}(t,x)+x}{x+L(t)} &= \left(\frac{\mu(t)}{\hat{\mu}(t)}\right)^{\frac{1}{K}} \left(\varepsilon(t)\,\delta e^{-\delta t}\right)^{\frac{1}{K}} a^{-\frac{1}{K}}(t) \ ,
\end{aligned}$$

where

$$a(t) = \left(\int_t^n \left(\begin{array}{c} \left(\left(\frac{\mu(s)}{\hat{\mu}(s)}\right)^{\frac{1-K}{K}} \mu(s) \left(\varepsilon(s)\,\delta e^{-\delta s}\right)^{\frac{1}{K}} + \left(\delta e^{-\delta s}\right)^{\frac{1}{K}}\right) \times \\ e^{-\int_t^s \mu(v)\,\mathrm{d}v} e^{\frac{1-K}{K}\left(r+\frac{1}{2}\frac{1}{K}\frac{\lambda^2}{\sigma^2}\right)(s-t)+\frac{1-K}{K}\int_t^s(\hat{\mu}(v)-\mu(v))\,\mathrm{d}v} \end{array}\right) \mathrm{d}s\right)^K .$$

Actually, Richard (1975) writes down

$$a(t) = \left(\int_t^n \left(\begin{array}{c} \left(\left(\frac{\mu(s)}{\hat{\mu}(s)}\right)^{\frac{1-K}{K}} \mu(s)^{\boldsymbol{\frac{1}{K}}} \left(\varepsilon(s)\,\delta e^{-\delta s}\right)^{\frac{1}{K}} + \left(\delta e^{-\delta s}\right)^{\frac{1}{K}}\right) \times \\ e^{-\int_t^s \mu(v)\,\mathrm{d}v} e^{\frac{1-K}{K}\left(r+\frac{1}{2}\frac{1}{K}\frac{\lambda^2}{\sigma^2}\right)(s-t)+\frac{1-K}{K}\int_t^s(\hat{\mu}(v)-\mu(v))\,\mathrm{d}v} \end{array}\right) \mathrm{d}s\right)^K$$

—but from his derivation, it appears that the bold power $\boldsymbol{\frac{1}{K}}$ must be an error. This is supported by formula (1a) in Kraft and Steffensen (2008).

We see that $g(t) = e^{\delta t} a(t)$. Plugging this into e.g. our optimal control, we discover that the two optimal controls match perfectly. We consider this to be an interesting discovery since we have not proven our optimal control to be optimal in the usual sense. Also, our work can be seen as an extension of the utility optimization in Richard (1975) to time-*non*-additive utility, and this is one of our most important insights since the literature, to our knowledge, contains no other attempts in that direction. However, the extension is only for power utility.

3 Link to recursive utility

3.1 Motivation

In the previous section, we introduced certainty equivalents in order to separate risk aversion from elasticity of inter-temporal substitution. This draws our attention in the direction of recursive utility studied in e.g. Duffie and Epstein (1992) and Kraft and Seifried (2010). In advance, we have no reason to believe that our optimization approach is equivalent to continuous-time recursive utility optimization, but in the special case of no mortality risk, it turns out that the pseudo-Bellman equation characterizing our equilibrium value function coincides with the pseudo-Bellman equation characterizing the value function of the recursive utility optimization problem in Duffie and Epstein (1992) for Epstein-Zin preferences. In the following subsections, we give an introduction to recursive utility, demonstrate the identity of pseudo-Bellman equations, and outline the perspectives of our findings.

3.2 A short introduction to recursive utility

Let $(\Omega, \mathcal{F}, P)$ be a probability space endowed with a filtration $\{\mathcal{F}_t\}_{t\in[0,n]}$ satisfying the usual conditions. Fix a set $\mathcal{C} \subset \mathbb{R}^k$ of consumption rates and denote by C a class of predictable $\mathcal{C}$-valued processes with time-horizon $[0, n]$. The backbone of recursive utility is the construction of a mapping $\mathfrak{u} : \mathsf{C} \to \mathbb{R}$ that ranks consumption streams in such a way that $\mathfrak{u}(c) \geq \mathfrak{u}(c')$ if and only if the consumption stream c is weakly preferred to the consumption stream c'. This is done by means of a *utility process* V^c associated to c by setting

$$\mathfrak{u}(c) = V^c(0) \ , \quad c \in \mathsf{C} \ .$$

The utility process is assumed to take values in a subinterval $\mathcal{V} \subset \mathbb{R}$ of the real line, and $\mathfrak{u}$ is referred to as a *recursive utility function.*

3.2.1 Discrete-time recursive utility

Recursive utility is first defined discrete time, and in Section 3 of Kraft and Seifried (2010), we find a brief review of discrete-time recursive utility. Let $\{t_0, t_1, \ldots, t_m\}$ be a partition of $[0, n]$, and let $c = \{c(t_k)\}_{k=1,\ldots,m}$ be a discrete-time consumption stream in C. Then the utility process V^c is defined through the backward recursion

$$\begin{aligned} V^c(t_m) &= 0 \ , \\ V^c(t_k) &= W\left(t_{k+1} - t_k, c(t_k), \mathfrak{m}\left(\mathcal{L}\left(V^c(t_{k+1}) | \mathcal{F}_{t_k}\right)\right)\right) \ , \\ & \qquad\qquad\qquad k = 0, \ldots, m-1 \ . \end{aligned} \tag{21}$$

Here, $W : [0, \infty) \times \mathcal{C} \times \mathcal{V} \to \mathcal{V}$ is a continuous function with $W(0, c, v) = v$ for $c \in \mathcal{C}, v \in \mathcal{V}$, $\mathcal{L}(V^c(t_{k+1}) | \mathcal{F}_{t_k})$ is the conditional distribution of $V^c(t_{k+1})$ given the information $\mathcal{F}_{t_k}$, and $\mathfrak{m}$ is a certainty equivalent on $\mathcal{V}$. Letting $\mathcal{M}_1(\mathcal{V})$ denote the set of probability measures on $\mathcal{B}(\mathcal{V})$ with moments of all orders, a functional $\mathfrak{m} : \mathcal{M}_1(\mathcal{V}) \to \mathbb{R}$ is called a certainty equivalent on $\mathcal{V}$ if $\mathfrak{m}(\delta_v) = v$ for all $v \in \mathcal{V}$ where δ_v is the Dirac measure at v.

W is often referred to as the time-aggregator because in a set-up without risk (implying $\mathfrak{m}(\mathcal{L}(V^c(t_{k+1}) | \mathcal{F}_{t_k})) = V^c(t_{k+1})$), it describes the intertemporal aggregation of present consumption c_{t_k} and the value of future consumption $V^c(t_{k+1})$. Similarly, $\mathfrak{m}$ is referred to as the risk-aggregator since it describes the risk weighted aggregation of possible future values of $V^c(t_{k+1})$. The pair $(W, \mathfrak{m})$ completely describes an investor's preferences for discrete-time stochastic consumption streams, and we call $(W, \mathfrak{m})$ a *discrete-time aggregator.*

A special class of certainty equivalents are those given by

$$\mathfrak{m}(\mu) = h^{-1}\left(\int_{\mathcal{V}} h \, \mathrm{d}\mu\right) \ , \quad \mu \in \mathcal{M}_1(\mathcal{V}) \ ,$$

for a strictly increasing, polynomially bounded $\mathcal{C}^2$-function $h : \mathcal{V} \to \mathbb{R}$. Here, $\mathfrak{m}$ is called an expected utility (EU) certainty equivalent. If h is the identity, then $\mathfrak{m}$ is called risk-neutral.

3.2.2 Continuous-time recursive utility

Duffie and Epstein (1992) denote their approach to recursive utility in continuous time by *stochastic differential utility* (SDU). They start from the discrete-time formulation in (21) and use a heuristic limiting argument to motivate their formulation of SDU, but SDU is defined in continuous time and does not rely on the heuristic derivation.

Kraft and Seifried (2010) set the heuristic limiting argument from Duffie and Epstein (1992) on a rigorous basis and denote their approach to recursive utility in continuous time by *continuous-time recursive utility* (CRU). Thereby, CRU is directly related to discrete-time recursive utility, and CRU is defined in a broader set-up than SDU.

We choose not to write down exactly how SDU and CRU are defined since the general definitions are complicated and since we gain sufficient insight from Lemma 3.2 below. In both SDU and CRU, the utility process V^c is generated by a *continuous-time aggregator* $(f, \mathfrak{m})$ on $\mathcal{V}$, where $f : \mathcal{C} \times \mathcal{V} \to \mathbb{R}$ is a Borel-measurable function, and $\mathfrak{m}$ is a certainty equivalent on $\mathcal{V}$. Also, both approaches have the disadvantage that they rely on the almost sure differentiability of the function $s \mapsto \mathfrak{m}\left(\mathcal{L}\left(V^c_{t+s} \middle| \mathcal{F}_t\right)\right)$ in $s = 0$.

We end this introduction with two lemmas. The first lemma describes the relation between discrete-time recursive utility and CRU. The second lemma shows that SDU and CRU are equivalent when the certainty equivalent is particularly simple. The lemmas follow from Corollary 6.3 and formula (7), (19), and (21) in Kraft and Seifried (2010):

Lemma 3.1. *Let* $(W, \mathfrak{m})$ *be a discrete-time aggregator on* $\mathcal{V}$*, assume that* W *is a* $\mathcal{C}^{1,0,1}$*-function, and define* $f : \mathcal{C} \times \mathcal{V} \to \mathbb{R}$ *by*

$$f(c, v) = \frac{\frac{\partial W}{\partial \Delta}(0, c, v)}{\frac{\partial W}{\partial v}(0, c, v)} . \tag{22}$$

Then $(f, \mathfrak{m})$ *is the CRU continuous-time aggregator corresponding to* $(W, \mathfrak{m})$*.*

Note that we cannot be sure that the aggregator $(f, \mathfrak{m})$ *actually generates a utility function, but if it does, then the discrete-time utility function and the continuous-time utility function represent the same preferences.*

Lemma 3.2. *Let* $(f, \mathfrak{m})$ *be a continuous-time aggregator on* $\mathcal{V} = \mathbb{R}$ *and assume that* $\{\mathcal{F}_t\}_{t \in [0,n]}$ *is generated by a standard Brownian motion, a Poisson random measure and the null sets,* $\mathfrak{m}$ *is the risk-neutral certainty equivalent, and* f *satisfies the Lipschitz and linear growth conditions*

$$\begin{aligned} |f(c, v) - f(c, w)| &\leq \alpha |v - w| & \forall c \in \mathcal{C}, v, w \in \mathbb{R} , \\ |f(c, 0)| &\leq \beta_1 + \beta_2 |c| & \forall c \in \mathcal{C} , \end{aligned}$$

for some $\alpha, \beta_0, \beta_1 > 0$*. Then SDU and CRU generate the same utility function* $\mathfrak{u} : \mathsf{C} \to \mathbb{R}$*, and it is given by* $\mathfrak{u}(c) = V^c(0)$ *where*

$$V^c(t) = E\left[\int_t^n f(c(s), V^c(s))\, \mathrm{d}s \middle| \mathcal{F}_t\right] \quad a.s .$$

We note that a continuous-time aggregator $(f, \mathfrak{m})$ is called normalized if $\mathfrak{m}$ is the risk-neutral certainty equivalent.

3.2.3 Example: Epstein-Zin preferences

An important class of recursive preferences are the Epstein-Zin preferences. In discrete time, these can be represented by a discrete-time aggregator

$(W, \mathfrak{m})$ on $\mathcal{V} = (0, \infty)$, where $\mathfrak{m}$ is the risk-neutral certainty equivalent, and W is given by

$$W(\Delta, c, v) = \frac{1}{1-\gamma}\left(\delta\Delta c^{1-\phi} + e^{-\delta\Delta}\left((1-\gamma)v\right)^{\frac{1-\phi}{1-\gamma}}\right)^{\frac{1-\gamma}{1-\phi}}$$

with $\gamma, \phi > 0$, $\gamma, \phi \neq 1$. Here, γ is the relative risk aversion, δ is the rate of time preference, and $\frac{1}{\phi}$ is the constant elasticity of inter-temporal substitution. Using formula (22), we find that the normalized continuous-time Epstein-Zin aggregator is given by $(f, \mathfrak{m})$, where

$$f(c, v) = \frac{\frac{\partial W}{\partial \Delta}(0, c, v)}{\frac{\partial W}{\partial v}(0, c, v)} = \frac{1-\gamma}{1-\phi}\delta v\left(\left(\frac{c}{\left((1-\gamma)v\right)^{\frac{1}{1-\gamma}}}\right)^{1-\phi} - 1\right).$$

It is easy to verify that f does not satisfy the Lipschitz and growth conditions of Lemma 3.2 for general ϕ and γ, so a priori we do not know if $(f, \mathfrak{m})$ generates a utility function. However, Duffie and Epstein (1992) mention in Example 3 that existence and uniqueness can be shown, and Kraft and Seifried (2010) make a similar comment in Remark 6.4.

3.3 Identity of pseudo-Bellman equations

For a while, we think of the investor from Section 2 as certain-lived, i.e. we fix $\mu = \hat{\mu} = 0$ in the set-up from Section 2. The investor's wealth process is now formalized by the SDE

$$\begin{aligned} \mathrm{d}X^{c,\pi}(t) &= X^{c,\pi}(t)\left[\left(r + \pi\left(t, X^{c,\pi}(t)\right)\lambda\right)\mathrm{d}t + \pi\left(t, X^{c,\pi}(t)\right)\sigma\mathrm{d}W(t)\right] \\ &\qquad - \left(c\left(t, X^{c,\pi}(t)\right) - w(t)\right)\mathrm{d}t, \\ X^{c,\pi}(0) &= x_0, \end{aligned}$$

where x_0 is the investor's initial wealth, w is a continuous, deterministic function of time, $r, \sigma, \lambda > 0$ are constants, and c, π are deterministic, measurable functions of time and wealth. The objective functions reads

$$Z^{c,\pi}(t,x) = \frac{1}{1-\gamma}\left(\int_t^n \delta e^{-\delta(s-t)} \left(E_{t,x}\left[c^{1-\gamma}(s, X^{c,\pi}(s))\right]\right)^{\frac{1}{\theta}} \mathrm{d}s\right)^{\theta} \ ,$$

where the parameters n, δ, γ, and θ are as in Section 2. We note that the death sum b has disappeared from both the wealth dynamics and the objective function. This is natural since the term insurance costs nothing (due to $\hat{\mu} = 0$) and pays out nothing (due to $\mu = 0$).

The problem of maximizing $Z^{c,\pi}$ is still time-inconsistent, so again we search for an equilibrium control for the function $Z^{c,\pi}$, $(c, \pi) \in \mathcal{U}_0^e$. Here, subscript 0 indicates that we have plugged in $\mu = \hat{\mu} = 0$ and left out b in the constraints defining $\mathcal{U}^e$. The same applies for $\mathcal{U}_0$ and Γ_0 below. We continue to use the terms optimal control and equilibrium control interchangeably. Plugging $\mu = \hat{\mu} = 0$ into Theorem 2.1 and leaving out all regularity assumptions, we get the following:

Theorem 3.1 (Certain-lived investor)**.** *Define the function* $f : (0,\infty) \times (1_{\{\gamma\in(0,1)\}}(0,\infty) \cup 1_{\{\gamma\in(1,\infty)\}}(-\infty,0)) \to \mathbb{R}$ *by*

$$f(c,Z) = \theta\delta Z\left(\left(\frac{c}{((1-\gamma)Z)^{\frac{1}{1-\gamma}}}\right)^{\frac{1-\gamma}{\theta}} - 1\right) \ . \tag{23}$$

Assume there exists a function U *in* $\mathcal{C}^{1,2}([0,n]\times\mathbb{R})$ *that solves the pseudo-Bellman equation*

$$U_t(t,x) = \inf_{(c,\pi,)\in\Gamma_0(t,x)} \begin{bmatrix} -f(c, U(t,x)) \\ -((r+\pi\lambda)x - c + w(t))U_x(t,x) \\ -\frac{1}{2}\sigma^2\pi^2x^2U_{xx}(t,x) \end{bmatrix} \ , \tag{24}$$

$$U(n,x) = 0 \ ,$$

and let (c^*, π^*) *be the function of* (t,x) *that realizes the infimum in* (24)*. Then* (c^*, π^*) *is an optimal control for the function* $Z^{c,\pi}$*,* $(c,\pi)\in\mathcal{U}_0^e$*, and* *the corresponding equilibrium value function V is given by*

$$V(t,x) = U(t,x) \ .$$

We recognize equation (24) from Proposition 9 in Duffie and Epstein (1992) as the pseudo-Bellman equation characterizing the value function of the continuous-time recursive utility optimization problem

$$\sup_{(c,\pi)\in\mathcal{D}} \mathfrak{u}\left(c^{c,\pi}\right) \ , \tag{25}$$

where $c^{c,\pi} = \{c(t, X^{c,\pi}(t))\}_{t\in[0,n]}$, $\mathcal{D}$ is the set of square-integrable, optional controls in $\mathcal{U}_0$, and $\mathfrak{u}$ is the utility function from Lemma 3.2 generated by the aggregator $(f, \mathfrak{m})$, where f is defined in (23), and $\mathfrak{m}$ is the risk-neutral certainty equivalent. In other words, $\mathfrak{u}(c^{c,\pi}) = V^{c,\pi}(0)$, where $V^{c,\pi}$ is defined via the backward equation

$$V^{c,\pi}(t) = E\left[\int_t^n f\left(c\left(s, X^{c,\pi}(s)\right), V^{c,\pi}(s)\right) \mathrm{d}s \middle| \mathcal{F}_t\right] \ .$$

Here, $\mathcal{F}_t$ denotes the augmentation of the σ-algebra generated by the sets $\{W(s) : 0 \leq s \leq t\}$.

We find the identity of pseudo-Bellman equations interesting since our optimization problem has a more natural interpretation than the recursive utility optimization problem in (25) and since our approach does not give rise to the differentiability problems mentioned in the previous subsection. Moreover, we recognize the aggregator $(f, \mathfrak{m})$ as the normalized continuous-time Epstein-Zin aggregator. This is again interesting since Epstein-Zin preferences are widely used in the literature.

When applying Proposition 9 in Duffie and Epstein (1992), we stumble on the fact that f does not satisfy certain Lipschitz and growth conditions, but Kraft et al. (2013) show that the proposition remains valid for e.g. $\phi \leq \gamma < 1$ and $\phi \geq \gamma > 1$, and in any case, the identity of pseudo-Bellman equations is noteworthy.

3.4 Perspectives

We have demonstrated that—in the special case without mortality risk—the pseudo-Bellman equation characterizing our equilibrium value function coincides with the pseudo-Bellman equation characterizing the value function of the recursive utility optimization problem in Duffie and Epstein (1992) for Epstein-Zin preferences. We formulate this by saying that our optimization approach (for a certain-lived investor) is equivalent to recursive utility optimization with Epstein-Zin preferences.

The equivalence between our optimization approach and recursive utility optimization (that is a well-established approach in diffusive markets) supports the use of our approach, also in cases that are not covered by recursive utility optimization. By 'not covered' we mean that neither SDU optimization nor CRU optimization is apt for an extended set-up with mortality risk and utility from inheritance since neither SDU nor CRU allows for utility from a lump sum at a random point in time. With our approach, we can provide such an extension for Epstein-Zin preferences. That is, our work can be seen as a generalization of recursive utility optimization with Epstein-Zin preferences to include mortality risk and life insurance. To our knowledge, the literature contains no other attempts in that direction.

4 The optimal consumption rate

4.1 Motivation

With the separation of preferences for risk and inter-temporal substitution, our utility optimization approach gives rise to a broader variety of optimal consumption curves than time-additive power utility optimization. To illustrate this, we derive an SDE for the optimal consumption rate from

Section 2, consider the special case of no market risk, and go through some numerical examples.

4.2 SDE

We assume that $\hat{\mu} = \alpha\mu$ for some constant $\alpha > 0$, μ is differentiable, and ε is constant. The optimal consumption rate is characterized by the SDE

$$\begin{aligned} \frac{\mathrm{d}c^*(t, X^*(t))}{c^*(t, X^*(t))} &= \frac{1}{\phi}\left(r - \delta + \left(\alpha - \frac{1}{\theta}\right)\mu(t) + (1+\phi)\frac{1}{2}\frac{\lambda^2}{\gamma\sigma^2} + \beta(t)\right)\mathrm{d}t \\ &\quad + \frac{\lambda}{\gamma\sigma}\mathrm{d}W(t)\ , \qquad (26) \\ c^*(0, X^*(0)) &= c^*(0, x_0)\ , \end{aligned}$$

where

$$\beta(t) = \frac{\kappa - \phi\kappa - 1 + \gamma}{1-\gamma}\,\frac{\gamma}{\gamma+\kappa-1}\,\frac{\mu_t(t)\,\varepsilon^{\frac{1}{\gamma+\kappa-1}}\alpha^{\frac{\gamma-1}{\gamma+\kappa-1}}\mu(t)^{\frac{\gamma}{\gamma+\kappa-1}-1}}{1+\varepsilon^{\frac{1}{\gamma+\kappa-1}}\alpha^{\frac{\gamma-1}{\gamma+\kappa-1}}\mu(t)^{\frac{\gamma}{\gamma+\kappa-1}}}\ .$$

The derivation is presented in Appendix B on page 44.

4.3 The special case without market risk

With $\lambda = 0$, there is no investment in the stock, and consequently, there is no market risk. The SDE in (26) reduces to the differential equation

$$\frac{\mathrm{d}c^*(t, X^*(t))}{c^*(t, X^*(t))} = \frac{1}{\phi}\left(r - \delta + \left(\alpha - \frac{1}{\theta}\right)\mu(t) + \beta(t)\right)\mathrm{d}t\ . \qquad (27)$$

The future optimal consumption rate is deterministic, and the initial value $c^*(0, x_0)$ is given by

$$\begin{aligned} c^*(0, x_0) &= \frac{x_0 + \int_0^n w(s)\, e^{-\int_0^s (r+\alpha\mu(v))\,\mathrm{d}v}\,\mathrm{d}s}{\int_0^n \tilde{\mu}(s)\, e^{-\int_0^s \tilde{r}(v)\,\mathrm{d}v}\,\mathrm{d}s} \\ &\quad \times \left(1 + \varepsilon^{\frac{1}{\gamma+\kappa-1}}\alpha^{\frac{\gamma-1}{\gamma+\kappa-1}}\mu(0)^{\frac{\gamma}{\gamma+\kappa-1}}\right)^{\frac{\kappa-\phi\kappa-1+\gamma}{(1-\gamma)\phi}}\ , \end{aligned} \qquad (28)$$

where

$$\tilde{r}(v) = -\frac{1}{\phi}\left[(1-\phi)\left(r+\left(\alpha-\frac{1}{1-\gamma}\right)\mu(v)\right)-\delta\right] ,$$
$$\tilde{\mu}(s) = \left(1+\varepsilon^{\frac{1}{\gamma+\kappa-1}}\alpha^{\frac{\gamma-1}{\gamma+\kappa-1}}\mu(s)^{\frac{\gamma}{\gamma+\kappa-1}}\right)^{\frac{(\kappa-1+\gamma)(1-\phi)}{(1-\gamma)\phi}} .$$

Since all market risk is eliminated, one might be surprised to see that the risk aversion parameter γ is still present, but this is due to mortality risk.

4.4 Numerics

4.4.1 Set-up

We consider an investor with the following characteristics:

- The investor is $t_0 = 25$ years old at time 0 and has an initial wealth of $x_0 = 10,000$ USD.
- Her death is governed by the mortality intensity[1]

$$\mu(t) = 5\cdot 10^{-4} + 5.3456\cdot 10^{-5}\cdot e^{0.087498(t_0+t)} .$$

- She starts off with a yearly labour income at rate 20,000 USD (we do not take taxes into account), and her labour income grows with the risk free short rate until the age of 65 where she retires, i.e.

$$w(t) = 20,000\cdot e^{rt}\cdot 1_{\{t_0+t\le 65\}} .$$

We only wish to focus on separation of risk aversion and EIS, so we fix $\alpha = \varepsilon = \kappa = 1$. Also, following (Kraft et al., 2013), we fix the risk free

[1] For the last three decades, this has served as a standard mortality intensity for adult women in Denmark.

short rate at $r = 0.05$ and the risk aversion at $\gamma = 2$. Finally, we fix the time-horizon $n = 85$ since there is very little probability that the investor survives the age of 110 with the G82 mortality. The fixed parameter values are summarized in the following table.

Parameter	α	ε	κ	r	γ	n
Fixed value	1	1	1	0.05	2	85

For a given choice of parameters, we first calculate the initial optimal consumption rate $c^* (0, x_0)$ by approximating the integrals in (28) with sums. We then calculate the future optimal consumption rates by approximating (27) with a difference equation.

4.4.2 Graphs

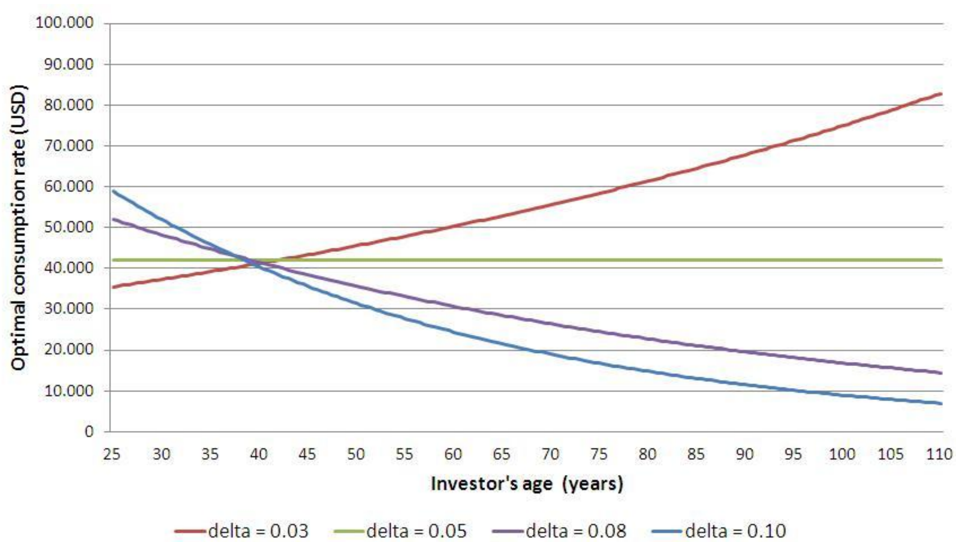

Figure 1: The optimal consumption rate as function of δ for fixed $\phi = 2$ $(= \gamma)$.

Fixing $\phi = 2$, we are in the time-additive case from (Richard, 1975), and letting δ vary, we get Figure 1. The investor's optimal yearly consumption rate is constant over time when δ is equal to r, and the rate is increasing (decreasing) when δ is smaller (larger) than r. This fits well with the intuition that δ is the investor's utility discount factor: if the investor discounts future consumption with a short rate that is larger than the risk free short rate, then she assigns a higher value to one unit of consumption 'now' than to one unit plus investment returns 'later'.

We notice that all the optimal consumption rates seem rather high compared to the investor's initial labour income and wealth. This is because the investor's labour income grows with the risk free short rate, and the plotted optimal consumption curves are expressed in nominal terms.

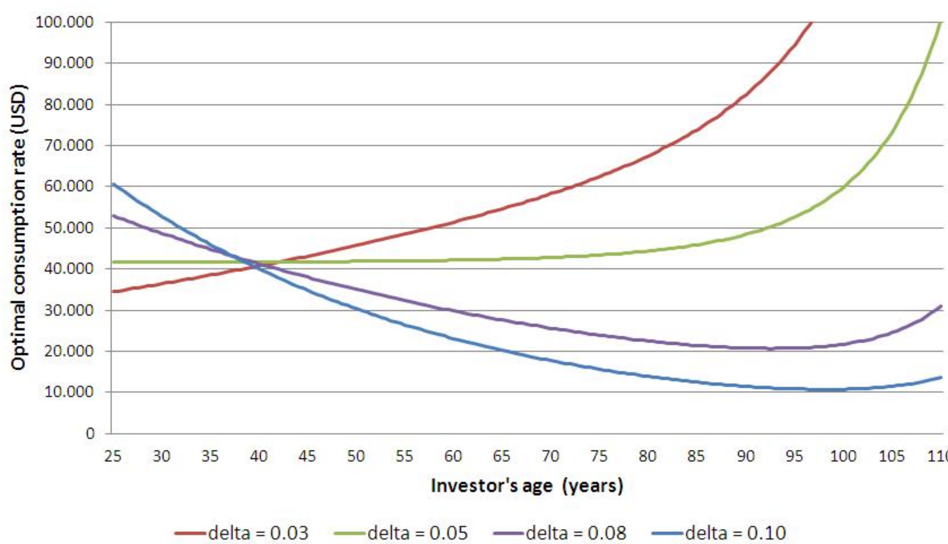

Figure 2: The optimal consumption rate as function of δ for fixed $\phi = 1.8$ $(< \gamma)$.

Fixing $\phi = 1.8$, we enable the separation of risk aversion and EIS that is special for this paper. Letting δ vary, we get Figure 2. The investor's optimal yearly consumption rate is increasing for δ smaller than $r = 0.05$ and non-monotone for δ larger than r. The non-monotone optimal consumption curves are first decreasing and then increasing. This seems like an odd phenomenon since we would expect the investor to consume less as she grows old. However, due to inflation, it might be reasonable with an increasing optimal consumption rate for high ages.

Fixing $\phi = 3$, we again enable separation of risk aversion and EIS. Letting δ vary, we get Figure 3. The investor's optimal yearly consumption rate is decreasing for δ larger than $r = 0.05$ and non-monotone for δ smaller than r. The non-monotone optimal consumption curves are first increasing and then decreasing. In the literature, this phenomenon is known as hump-shaped consumption.

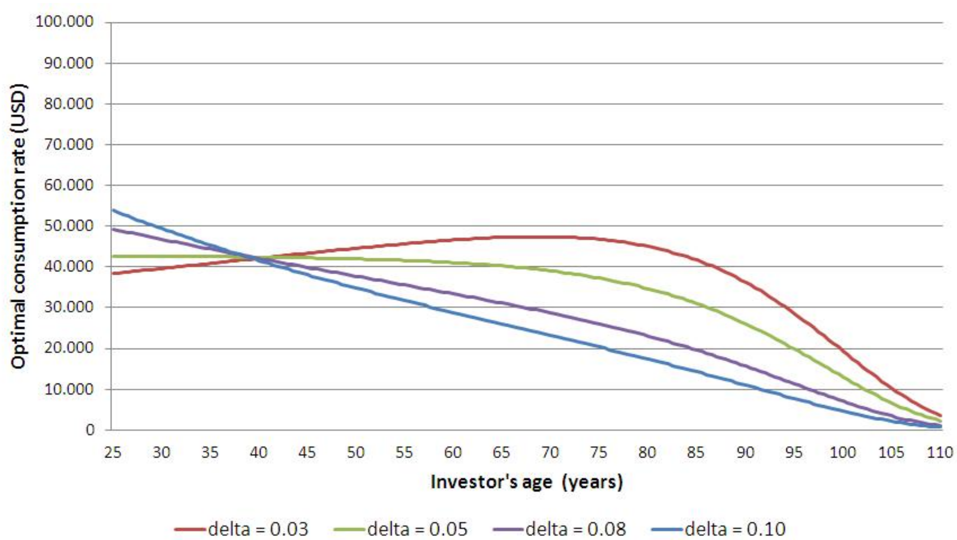

Figure 3: The optimal consumption rate as function of δ for fixed $\phi = 3$ ($> \gamma$).

Hump-shaped consumption is observed in realized consumption, and different articles contain different explanations for this. See e.g. Gourinchas and Parker (2002) who obtain the feature by income uncertainty. They fit to data a hump around age 50. Our hump is not fitted to any data, but the hump around 70 for $\delta = 0.03$ is not necessarily in conflict with their quantities since we illustrate consumption in nominal terms whereas they convert to 1987 dollars. We note that such hump-shaped consumption patterns cannot be obtained by standard recursive utility or time-additive utility under lifetime uncertainty. We do not claim to having found the most important source of hump-shapes, and we do not pursue this particular feature of our approach more for now. Yet, we find it interesting enough to stress that it is the very combination of separation of risk aversion and elasticity of substitution with an uncertain lifetime that takes us to this intriguing feature of realized consumption.

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A Proof of Theorem 2.1–2.2

Prerequisites

Fix a control $(c, \pi, b) \in \mathcal{U}^e$. First, we take a look at $m^{c,\pi,b}$ and $n^{c,\pi,b}$. We assume there exist functions $\Lambda_1^{c,\pi,b}$ and $\Lambda_2^{c,\pi,b}$ in $\mathcal{C}^{1,0,2}\left([0,n]^2 \times \mathbb{R}\right)$ such that

$$\begin{aligned}
\Lambda_{i,t}^{c,\pi,b}(t,s,x) = &- \left[x\left(r + \pi(t,x)\lambda\right) - c(t,x)\right]\Lambda_{i,x}^{c,\pi,b}(t,s,x) \\
&- \left[-\hat{\mu}(t)\, b(t,x) + w(t)\right]\Lambda_{i,x}^{c,\pi,b}(t,s,x) \\
&- \frac{1}{2}\pi^2(t,x)\,\sigma^2 x^2 \Lambda_{i,xx}^{c,\pi,b}(t,s,x) \ , \quad i = 1,2 \ , \\
\Lambda_1^{c,\pi,b}(s,s,x) = &\; c^{1-\gamma}(s,x) \ , \\
\Lambda_2^{c,\pi,b}(s,s,x) = &\; \varepsilon(s)\,\mu(s)\left(x + b(s,x)\right)^{1-\gamma} \ ,
\end{aligned} \tag{29}$$

for all $x \in \mathbb{R}$ and $0 \le t \le s \le n$.

Using Itô's formula on $\Lambda_i^{c,\pi,b}\left(t,s,X^{c,\pi,b}(t)\right)$ (for fixed s), plugging in (29), and skipping most arguments that are $\left(t,s,X^{c,\pi,b}(t)\right)$, $\left(t,X^{c,\pi,b}(t)\right)$ or t, we get that[2]

$$\begin{aligned} d\Lambda_i^{c,\pi,b}\left(t,s,X^{c,\pi,b}(t)\right) &= \Lambda_{i,t}^{c,\pi,b}\mathrm{d}t + \Lambda_{i,x}^{c,\pi,b}\mathrm{d}X^{c,\pi,b}(t) \\ &\quad + \frac{1}{2}\Lambda_{i,xx}^{c,\pi,b}\mathrm{d}\left[X^{c,\pi,b},X^{c,\pi,b}\right]^c(t) \\ &= \Lambda_{i,x}^{c,\pi,b}X^{c,\pi,b}\pi\sigma\mathrm{d}W(t) \ , \quad i=1,2 \ , \quad t\le s \ . \end{aligned}$$

Hence, for $t \le s$, we can write

$$\begin{aligned} &\Lambda_1^{c,\pi,b}\left(t,s,X^{c,\pi,b}(t)\right) \\ &\quad = c^{1-\gamma}\left(s,X^{c,\pi,b}(s)\right) \qquad (30) \\ &\qquad - \int_t^s \Lambda_{1,x}^{c,\pi,b}\left(u,s,X^{c,\pi,b}(u)\right)X^{c,\pi,b}(u)\,\pi\left(u,X^{c,\pi,b}(u)\right)\sigma\,\mathrm{d}W(u) \ , \\ &\Lambda_2^{c,\pi,b}\left(t,s,X^{c,\pi,b}(t)\right) \\ &\quad = \varepsilon(s)\,\mu(s)\left(X^{c,\pi,b}(s)+b\left(s,X^{c,\pi,b}(s)\right)\right)^{1-\gamma} \qquad (31) \\ &\qquad - \int_t^s \Lambda_{2,x}^{c,\pi,b}\left(u,s,X^{c,\pi,b}(u)\right)X^{c,\pi,b}(u)\,\pi\left(u,X^{c,\pi,b}(u)\right)\sigma\,\mathrm{d}W(u) \ . \end{aligned}$$

We assume that the stochastic integrals in (30) and (31) are martingales. Taking conditional expectation given $X^{c,\pi,b}(t)=x$ on both sides yields

$$\begin{aligned} \Lambda_1^{c,\pi,b}(t,s,x) &= E_{t,x}\left[c^{1-\gamma}\left(s,X^{c,\pi,b}(s)\right)\right] \\ &= m^{c,\pi,b}(t,s,x) \ , \qquad (32) \\ \Lambda_2^{c,\pi,b}(t,s,x) &= E_{t,x}\left[\varepsilon(s)\,\mu(s)\left(X^{c,\pi,b}(s)+b\left(s,X^{c,\pi,b}(s)\right)\right)^{1-\gamma}\right] \\ &= n^{c,\pi,b}(t,s,x) \ . \qquad (33) \end{aligned}$$

[2] We use Itô's formula as presented in (Protter, 2005, Chapter II, Theorem 33). Several terms are left out or simplified since $X^{c,\pi,b}$ is continuous, the operator $(x,y)\mapsto[x,y]$ is bilinear (see [ibid., p. 66]), $\mathrm{d}[W,W]^c(t)=t$ (see [ibid., p. 67]), and $\mathrm{d}[Id,Id]^c(t)=\mathrm{d}[Id,W]^c(t)=0$ by [ibid., Theorem 26 and 28] where $Id(t)=t$. The theorems apply since Id is adapted, cadlag, and have path of finite variation on compacts, whereas W is a continuous martingale.

For strictly positive, sufficiently integrable functions $a, b \in \mathcal{C}^{0,0,1}\left([0,n]^2 \times \mathbb{R}\right)$, we define the functions $K^{a,b}, I^{a,b} : [0,n] \times \mathbb{R} \to \mathbb{R}$ by

$$K^{a,b}(t,x) = \frac{1}{1-\gamma}\left(\int_t^n \delta e^{-\delta(s-t)} e^{-\frac{1}{\theta}\int_t^s \mu(v)\,\mathrm{d}v}\left(a^{\frac{1}{\kappa}} + b^{\frac{1}{\kappa}}\right)^{\frac{\kappa}{\theta}} \mathrm{d}s\right)^{\theta},$$

$$\begin{aligned} I^{a,b}(t,x) = {} & \frac{1}{1-\gamma}\left((1-\gamma) K^{a,b}(t,x)\right)^{1-\frac{2}{\theta}} \\ & \times \left(1 - \frac{1}{\theta}\right)\left(\int_t^n \begin{pmatrix} \delta e^{-\delta(s-t)} e^{-\frac{1}{\theta}\int_t^s \mu(v)\,\mathrm{d}v} \times \\ \left(a^{\frac{1}{\kappa}} + b^{\frac{1}{\kappa}}\right)^{\frac{\kappa}{\theta}-1} \times \\ \left(a^{\frac{1}{\kappa}-1} a_x + b^{\frac{1}{\kappa}-1} b_x\right) \end{pmatrix} \mathrm{d}s\right)^2 \\ & + \frac{1}{1-\gamma}\left((1-\gamma) K^{a,b}(t,x)\right)^{1-\frac{1}{\theta}} \\ & \times \left\{\left(\frac{1}{\theta} - \frac{1}{\kappa}\right)\int_t^n \begin{pmatrix} \delta e^{-\delta(s-t)} e^{-\frac{1}{\theta}\int_t^s \mu(v)\,\mathrm{d}v} \times \\ \left(a^{\frac{1}{\kappa}} + b^{\frac{1}{\kappa}}\right)^{\frac{\kappa}{\theta}-2} \\ \times \left(a^{\frac{1}{\kappa}-1} a_x + b^{\frac{1}{\kappa}-1} b_x\right)^2 \end{pmatrix} \mathrm{d}s \right. \\ & \left. + \left(\frac{1}{\kappa} - 1\right)\int_t^n \begin{pmatrix} \delta e^{-\delta(s-t)} e^{-\frac{1}{\theta}\int_t^s \mu(v)\,\mathrm{d}v} \times \\ \left(a^{\frac{1}{\kappa}} + b^{\frac{1}{\kappa}}\right)^{\frac{\kappa}{\theta}-1} \\ \times \left(a^{\frac{1}{\kappa}-2} (a_x)^2 + b^{\frac{1}{\kappa}-2} (b_x)^2\right) \end{pmatrix} \mathrm{d}s \right\}. \end{aligned} \tag{34}$$

Here, we have skipped all arguments (t,s,x) inside the integrals. By (32)–(33), we can write $Z^{c,\pi,b}(t,x) = K^{\Lambda_1^{c,\pi,b}, \Lambda_2^{c,\pi,b}}(t,x)$. Hence, assuming sufficient integrability, applying (29), and skipping all arguments that are (t,x) or t, we get the following partial derivative

$$\begin{aligned} Z_t^{c,\pi,b} = {} & -f\left(t, c, x+b, K^{\Lambda_1^{c,\pi,b}, \Lambda_2^{c,\pi,b}}\right) - \left(x(r + \pi\lambda) - c - \hat{\mu} b + w\right) Z_x^{c,\pi,b} \\ & - \frac{1}{2}\pi^2\sigma^2 x^2 Z_{xx}^{c,\pi,b} + \frac{1}{2}\pi^2\sigma^2 x^2 I^{\Lambda_1^{c,\pi,b}, \Lambda_2^{c,\pi,b}}. \end{aligned} \tag{35}$$

Here, f and $I^{\Lambda_1^{c,\pi,b}, \Lambda_2^{c,\pi,b}}$ are defined in (16) and (34). Assuming that $Z^{c,\pi,b}$ is in $\mathcal{C}^{1,2}$, using Itô's formula on $Z^{c,\pi,b}\left(t, X^{c,\pi,b}(t)\right)$, and skipping most arguments that are t or $\left(t, X^{c,\pi,b}(t)\right)$, we get that

$$dZ^{c,\pi,b}\left(t, X^{c,\pi,b}(t)\right) = Z_t^{c,\pi,b}\,\mathrm{d}t + Z_x^{c,\pi,b}\left[X^{c,\pi,b}(r+\pi\lambda) - c - \hat{\mu}b + w\right]\mathrm{d}t + Z_x^{c,\pi,b}X^{c,\pi,b}\pi\sigma\,\mathrm{d}W(t) + \frac{1}{2}Z_{xx}^{c,\pi,b}\pi^2\sigma^2\left(X^{c,\pi,b}\right)^2\mathrm{d}t\ .$$

Hence, plugging in the partial derivatives of $Z^{c,\pi,b}$ and skipping most arguments that are $\left(u, X^{c,\pi,b}(u)\right)$ or u, we get that

$$Z^{c,\pi,b}\left(t, X^{c,\pi,b}(t)\right) = -\int_t^n Z_x^{c,\pi,b}X^{c,\pi,b}\pi\sigma\,\mathrm{d}W(u) + \int_t^n \left(\begin{array}{c} f\left(u, c, X^{c,\pi,b} + b, K^{\Lambda_1^{c,\pi,b},\Lambda_2^{c,\pi,b}}\right) \\ -\frac{1}{2}\pi^2\sigma^2\left(X^{c,\pi,b}\right)^2 I^{\Lambda_1^{c,\pi,b},\Lambda_2^{c,\pi,b}} \end{array}\right)\mathrm{d}u\ . \tag{36}$$

The actual proof

First, replace the pseudo-Bellman equation in (17) with the equation

$$U_t(t,x) = \inf_{(c,\pi,b)\in\Gamma(x,t)}\left[\begin{array}{c} -f\left(t, c, x+b, K^{l_1,l_2}(t,x)\right) \\ -\left((r+\pi\lambda)x - c - \hat{\mu}(t)b + w(t)\right)U_x \\ -\frac{1}{2}\sigma^2\pi^2x^2U_{xx}(t,x) + \frac{1}{2}\pi^2\sigma^2x^2I^{l_1,l_2}(t,x) \end{array}\right], \tag{37}$$
$$U(n,x) = 0\ .$$

Assume that the functions (U, l_1, l_2) from Theorem 2.1 exist, let (c^*, π^*, b^*) be the function of (t,x) that realizes the infimum in (37), and assume that (c^*, π^*, b^*) satisfies the assumptions of Theorem 2.1. Then (c^*, π^*, b^*) is easily seen to be a control in $\mathcal{U}^e$. In the next two subsections, we prove that $U = Z^{c^*,\pi^*,b^*}$, and that (c^*, π^*, b^*) is an equilibrium control for $Z^{c,\pi,b}$. In the third subsection, we work our way back to the pseudo-Bellman equation in (17) and derive closed form expressions for (c^*, π^*, b^*).

Proof: $U = Z^{c^*,\pi^*,b^*}$

By assumption, U is in $\mathcal{C}^{1,2}$, so using Itô's formula on $U\left(t, X^{c,\pi,b}(t)\right)$ for some $(c,\pi,b) \in \mathcal{U}^e$, plugging in (37), and skipping all arguments that are

$\left(u, X^{c,\pi,b}(u)\right)$ or u, we get that

$$U\left(t, X^{c,\pi,b}(t)\right) \geq -\int_t^n U_x X^{c,\pi,b} \pi\sigma \,\mathrm{d}W(u) + \int_t^n \begin{pmatrix} f\left(u, c, X^{c,\pi,b} + b, K^{l_1,l_2}\right) \\ -\frac{1}{2}\pi^2\sigma^2 \left(X^{c,\pi,b}\right)^2 I^{l_1,l_2} \end{pmatrix} \mathrm{d}u \; . \tag{38}$$

We write $Z^* = Z^{c^*,\pi^*,b^*}$, $X^* = X^{c^*,\pi^*,b^*}$, and $\Lambda_i^* = \Lambda_i^{c^*,\pi^*,b^*}$ to simplify notation. To establish the relation $U = Z^*$, we note that $\Lambda_i^* = l_i$, $i = 1, 2$. Plugging this into (36) with the control (c^*, π^*, b^*), we get that

$$Z^*(t, X^*(t)) = -\int_t^n Z_x^* X^* \pi^* \sigma \,\mathrm{d}W(u) + \int_t^n \begin{pmatrix} f\left(u, c^*, X^* + b^*, K^{l_1,l_2}\right) \\ -\frac{1}{2}(\pi^*)^2 \sigma^2 (X^*)^2 I^{l_1,l_2} \end{pmatrix} \mathrm{d}u \; . \tag{39}$$

Also, with the control (c^*, π^*, b^*), there is equality in (38) (because the infimum in (37) is realized), so we get that

$$U(t, X^*(t)) = -\int_t^n U_x X^* \pi^* \sigma \,\mathrm{d}W(u) + \int_t^n \begin{pmatrix} f\left(u, c^*, X^* + b^*, K^{l_1,l_2}\right) \\ -\frac{1}{2}(\pi^*)^2 \sigma^2 (X^*)^2 I^{l_1,l_2} \end{pmatrix} \mathrm{d}u \; . \tag{40}$$

We assume that the stochastic integrals in (39) and (40) are martingales. Fixing some $(s, y) \in [0, n] \times \mathbb{R}$, subtracting $U(s, X^*(s))$ from $Z^*(s, X^*(s))$, and taking conditional expectation given $X^*(s) = y$, we finally arrive at

$$U(s, y) - Z^*(s, y) = E_{s,y}\left[-\int_s^n (U_x - Z_x^*) X^* \pi^* \sigma \,\mathrm{d}W(u)\right] = 0 \; .$$

Since (s, y) were arbitrary, we have proven that $Z^* = U$, and consequently

$$U = K^{\Lambda_1^*, \Lambda_2^*} = K^{l_1, l_2} \, . \tag{41}$$

Proof: (c^*, π^*, b^*) **is an equilibrium control**

We fix a control $\left(\bar{c}, \bar{\pi}, \bar{b}\right)$ in $\mathcal{U}^e$, a (small) real number $h > 0$, and an initial point $(u, y) \in [0, n] \times \mathbb{R}$. We then define the control $\left(c^h, \pi^h, b^h\right)$ by

$$\left(c^h, \pi^h, b^h\right)(t, x) = \begin{cases} \left(\bar{c}, \bar{\pi}, \bar{b}\right)(t, x) \ , & u \le t < u + h, x \in \mathbb{R} \ , \\ \left(c^*, \pi^*, b^*\right)(t, x) \ , & u + h \le t \le n, x \in \mathbb{R} \ . \end{cases}$$

Below, we write $Z^h = Z^{c^h, \pi^h, b^h}$. To prove that (c^*, π^*, b^*) is an equilibrium control for $Z^{c,\pi,b}$, we introduce the following non-standard assumptions:

Assumptions A.1. *We assume that there exist functions Λ_1^h and Λ_2^h that satisfy* (29) *for the control $\left(c^h, \pi^h, b^h\right)$ for all $u \le t \le s \le n$ and $x \in \mathbb{R}$. We assume that the functions are sufficiently smooth such that for all $t \in [u, n]$ and $x \in \mathbb{R}$*

$$Z^h(t, x) = K^{\Lambda_1^h, \Lambda_2^h}(t, x) \ . \tag{42}$$

Also, we assume that Z^h is twice differentiable in the second argument and once differentiable in the first argument with the t-derivative from (35)*. Finally, we assume that the following convergences hold true:*

$$Z^h(u, y) \xrightarrow{h \to 0} U(u, y) \ , \qquad Z_x^h(u, y) \xrightarrow{h \to 0} U_x(u, y) \ , \tag{43}$$
$$Z_{xx}^h(u, y) \xrightarrow{h \to 0} U_{xx}(u, y) \ , \qquad I^{\Lambda_1^h, \Lambda_2^h}(u, y) \xrightarrow{h \to 0} I^{l_1, l_2}(u, y) \ .$$

To prove that (c^*, π^*, b^*) is an equilibrium control in the sense of Definition 2.1, we need to verify that the condition (15) is satisfied. We recall that $Z^* = U$. Hence, equation (15) reads

$$\liminf_{h \to 0} \frac{U(u, y) - Z^h(u, y)}{h} \ge 0 \ .$$

By construction, we have that $Z^h(t, x) = U(t, x)$ for $t \in [u + h, n]$ and $x \in \mathbb{R}$. Thus, applying Taylor's formula for fixed $x = y$, we get that

$$\begin{aligned} \frac{U(u, y) - Z^h(u, y)}{h} &= \frac{U(u, y) - U(u + h, y) - Z^h(u, y) + Z^h(u + h, y)}{h} \\ &= -U_t(u, y) + Z_t^h(u, y) + o(h) \ . \end{aligned}$$

Hence, what we need to show is that

$$\liminf_{h\to 0}\ \left[-U_t\left(u,y\right)+Z_t^h\left(u,y\right)\right]\geq 0\ . \tag{44}$$

Applying (35), (37), (41), and (42) and skipping most arguments that are (u,y) or u, we get that

$$\begin{aligned}-U_t+Z_t^h\geq\ & f\left(u,\bar{c},y+\bar{b},U\right)-f\left(u,\bar{c},y+\bar{b},Z^h\right)\\&+\left(y\left(r+\bar{\pi}\lambda\right)-\bar{c}-\hat{\mu}\bar{b}+w\right)\left(U_x-Z_x^h\right)\\&+\frac{1}{2}\bar{\pi}^2\sigma^2y^2\left(U_{xx}-Z_{xx}^h\right)+\frac{1}{2}\bar{\pi}^2\sigma^2y^2\left(I^{\Lambda_1^h,\Lambda_2^h}-I^{l_1,l_2}\right)\ .\end{aligned} \tag{45}$$

The function f is obviously continuous. Hence, plugging (43) into (45) as h tends to 0, we see that (44) is satisfied. This concludes the proof with the alternative pseudo-Bellman equation.

The original pseudo-Bellman equation

To get back to the pseudo-Bellman equation in Theorem 2.1, we assume that l_1 and l_2 are separable in the sense that there exist $\mathcal{C}^{1,0}$-functions $h_1,h_2:[0,n]^2\to\mathbb{R}$ such that

$$l_i\left(t,s,x\right)=h_i\left(t,s\right)\left(x+L\left(t\right)\right)^{1-\gamma}\quad,\quad i=1,2\ , \tag{46}$$

where L is the investor's human wealth defined in (4). Then, by (41),

$$U\left(t,x\right)=\frac{1}{1-\gamma}\left(x+L\left(t\right)\right)^{1-\gamma}g^{\theta}\left(t\right)\ \ ,$$

where the function $g:[0,n]\to\mathbb{R}$ is given by

$$g\left(t\right)=\int_t^n\delta e^{-\delta(s-t)}e^{-\frac{1}{\theta}\int_t^s\mu(v)\,\mathrm{d}v}\left(h_1^{\frac{1}{\kappa}}\left(t,s\right)+h_2^{\frac{1}{\kappa}}\left(t,s\right)\right)^{\frac{\kappa}{\theta}}\mathrm{d}s\ .$$

In the above, we assume that $x+L\left(t\right)>0$ and $t<n$. This can be done without loss of generality because if $x+L\left(t\right)=0$ or $t=n$ then $U\left(t,x\right)=0$.

Now, assuming sufficient integrability and skipping all arguments that are (t,s,x), (t,x), (t,s), or t, we get the partial derivatives

$$
\begin{aligned}
U_x &= (x+L)^{-\gamma} g^{\theta} \ , \\
U_{xx} &= -\gamma (x+L)^{-\gamma-1} g^{\theta} \ , \\
U_t &= \frac{1}{1-\gamma} (x+L)^{1-\gamma} \theta g^{\theta-1} g_t + L_t (x+L)^{-\gamma} g^{\theta} \ ,
\end{aligned} \tag{47}
$$

$$
\begin{aligned}
(l_i)_x &= (1-\gamma) h_i (x+L)^{-\gamma} \ , \\
(l_i)_{xx} &= -(1-\gamma) \gamma h_i (x+L)^{-\gamma-1} \ , \\
(l_i)_t &= (h_i)_t (x+L)^{1-\gamma} + (1-\gamma) h_i (x+L)^{-\gamma} L_t \ ,
\end{aligned} \tag{48}
$$

and we easily verify that

$$
I^{l_1,l_2}(t,x) = K^{a,b}(t,x) \times \begin{pmatrix} \left(1-\frac{1}{\theta}\right)\left(\frac{1-\gamma}{x+L(t)}\right)^2 + \\ \left(\frac{1}{\theta}-\frac{1}{\kappa}\right)\left(\frac{1-\gamma}{x+L(t)}\right)^2 + \\ \left(\frac{1}{\kappa}-1\right)\left(\frac{1-\gamma}{x+L(t)}\right)^2 \end{pmatrix} = 0 \ . \tag{49}
$$

Plugging (49) and (41) into (37) and skipping all arguments that are (t,x) or t, the differential equation for U reduces to

$$
U_t = \inf_{(c,\pi,b)\in\Gamma(x,t)} \begin{bmatrix} -f(t,c,x+b,U) \\ -((r+\pi\lambda)x - c - \hat{\mu} b + w) U_x \\ -\frac{1}{2}\sigma^2\pi^2x^2U_{xx} \end{bmatrix} \ . \tag{50}
$$

Also, plugging (48) into (18), skipping all arguments that are (t,s,x), (t,x), or t, dividing by $(x+L)^{1-\gamma}$, and subtracting $(1-\gamma) h_i (x+L)^{-1} L_t$, we get the following differential equations for h_1 and h_2:

$$
\begin{aligned}
(h_i)_t = &-\left(r + \hat{\mu} - \frac{c^*}{x+L} - \hat{\mu}\frac{b^*+x}{x+L} + \lambda\frac{\pi^* x}{x+L} - \frac{1}{2}\left(\frac{\pi^* x}{x+L}\right)^2 \sigma^2\gamma\right) \\
&\times (1-\gamma) h_i \ , \quad i = 1,2 \ ,
\end{aligned} \tag{51}
$$

$$h_1(s,s) = \left(\frac{c^*(s,x)}{x+L(s)}\right)^{1-\gamma} ,$$
$$h_2(s,s) = \varepsilon(s)\,\mu(s)\left(\frac{b^*(s,x)+x}{x+L(s)}\right)^{1-\gamma} .$$

We recognise (50) as the pseudo-Bellman equation from Theorem 2.1, but we need to verify the separability assumption in (46). From (51) we see that the differential equations for h_1 and h_2 become ordinary (and independent of x), when $\frac{\pi^* x}{x+L}$, $\frac{c^*}{x+L}$, and $\frac{b^*+x}{x+L}$ do not depend on x. Therefore, to verify the assumption (46), it suffices to verify that $\frac{\pi^* x}{x+L}$, $\frac{c^*}{x+L}$, and $\frac{b^*+x}{x+L}$ do not depend on x. For the verification, we recall that (c^*, π^*, b^*) solves the continuum of minimization problems in (50). Plugging (47) into (50) and innocently dividing by $(1-\gamma)U$, we face the problem

$$\frac{\theta}{1-\gamma}\frac{g_t}{g} + \frac{L_t}{x+L} \tag{52}$$
$$= \inf_{(c,\pi,b)\in\Gamma(x,t)} \left[\begin{array}{c} -\frac{1}{1-\gamma}\left(\theta\delta\left(\left(\frac{c}{x+L}\right)^{\frac{1-\gamma}{\kappa}} + \varepsilon^{\frac{1}{\kappa}}\mu^{\frac{1}{\kappa}}\left(\frac{b+x}{x+L}\right)^{\frac{1-\gamma}{\kappa}}\right)^{\frac{\kappa}{\theta}}\frac{1}{g}\right) \\ +\frac{1}{1-\gamma}(\mu+\theta\delta) - \left(\frac{(r+\pi\lambda)x}{x+L} - \frac{c+\hat{\mu}b-w}{x+L} - \frac{1}{2}\gamma\sigma^2\left(\frac{\pi x}{x+L}\right)^2\right) \end{array}\right].$$

To solve this minimization problem, we differentiate the objective function with respect to each of the (sub)controls and set the partial derivatives equal to zero. Note that we look for an interior solution because of the constraint $(c,\pi,b) \in \Gamma(x,t)$. We get the solution in (19), so we have the crucial independence of x, and it is easily seen that

$$(c^*(t,x), \pi^*(t,x), b^*(t,x)) \in \Gamma(t,x) .$$

Hence, we have verified the separability assumption, and we have derived expressions for the optimal control.

Next, we would like to derive closed-form expressions for the functions h_1, h_2, and g. Plugging the optimal control in (19) back into (52), subtracting $\frac{L_t}{x+L} = \frac{-w+(r+\hat{\mu})L}{x+L}$, and dividing by $\frac{\theta}{(1-\gamma)g}$, we get the following differential equation for g:

$$
\begin{aligned}
g_t = &-\phi\delta^{\frac{1}{\phi}}\left(1+\left(\frac{\varepsilon\mu}{\hat{\mu}^{1-\gamma}}\right)^{\frac{1}{\gamma+\kappa-1}}\right)^{\frac{(\kappa-1+\gamma)(1-\phi)}{(1-\gamma)\phi}} g^{1-\frac{1}{\phi}} \\
&-\left((1-\phi)\left(r+\frac{1}{2}\frac{1}{\gamma}\frac{\lambda^2}{\sigma^2}+\hat{\mu}-\frac{\mu}{1-\gamma}\right)-\delta\right) g \ , \qquad (53)\\
g(n) = &\ 0 \ .
\end{aligned}
$$

This differential equation has a well-known form, and the solution is given in Theorem 2.2. Moreover, plugging the optimal control in (19) back into (51), we get the following ordinary differential equations for h_1 and h_2:

$$
(h_i)_t = -(1-\gamma)\left(r+\hat{\mu}-\delta^{\frac{1}{\phi}}g^{-\frac{1}{\phi}}\left(1+\left(\frac{\varepsilon\mu}{\hat{\mu}^{1-\gamma}}\right)^{\frac{1}{\gamma+\kappa-1}}\right)^{\frac{(\kappa-1+\gamma)(1-\phi)}{(1-\gamma)\phi}}+\frac{1}{2}\frac{\lambda^2}{\gamma\sigma^2}\right) h_i \ ,
$$

$$
h_1(s,s) = \left(\delta^{\frac{1}{\phi}}g^{-\frac{1}{\phi}}(s)\right)^{1-\gamma}\left(1+\left(\frac{\varepsilon(s)\mu(s)}{\hat{\mu}^{1-\gamma}(s)}\right)^{\frac{1}{\gamma+\kappa-1}}\right)^{\frac{\kappa-\phi\kappa-1+\gamma}{\phi}} \ ,
$$

$$
h_2(s,s) = h_1(s,s)\left(\frac{\varepsilon(s)\mu(s)}{\hat{\mu}^{1-\gamma}(s)}\right)^{\frac{\kappa}{\gamma+\kappa-1}} \ .
$$

Again, these differential equations have a well-known form, and the solutions are given in Theorem 2.2. This concludes the proof with the original pseudo-Bellman equation. □

B Derivation of SDE for c^*

Define the function $v : [0,n] \to \mathbb{R}$ by

$$
v(t) = \left(1+\left(\frac{\varepsilon(t)\mu(t)}{\hat{\mu}^{1-\gamma}(t)}\right)^{\frac{1}{\gamma+\kappa-1}}\right)^{\frac{\kappa-\phi\kappa-1+\gamma}{(1-\gamma)\phi}} \ .
$$

Then v is in $\mathcal{C}^1([0,n])$ if the mortality intensities μ, $\hat{\mu}$, and the weight function ε are so, and the optimal consumption rate from Theorem 2.2 can be written as $c^*(t,x) = \delta^{\frac{1}{\phi}}g^{-\frac{1}{\phi}}(t)\,v(t)\,(x+L(t))$.

Assume that ε, μ, and $\hat{\mu}$ are $\mathcal{C}^1$-functions. Since also g and L are $\mathcal{C}^1$-functions, it holds that c^* is in $\mathcal{C}^{1\times 2}$, and we get the partial derivatives

$$c_t^*(t,x) = \left(-\frac{1}{\phi}\frac{g_t(t)}{g(t)} + \frac{v_t(t)}{v(t)} + \frac{-w(t) + (r + \hat{\mu}(t)) L(t)}{x + L(t)}\right) c^*(t,x) \ ,$$

$$c_x^*(t,x) = \delta^{\frac{1}{\phi}} g^{-\frac{1}{\phi}}(t)\, v(t) = \frac{1}{x + L(t)} c^*(t,x) \ ,$$

$$c_{xx}^*(t,x) = 0 \ .$$

Let X^* be the wealth process stemming from the optimal control (c^*, π^*, b^*). Using Itô's formula on $c^*(t, X^*(t))$ (see footnote page 36) , we get the SDE

$$\begin{aligned} \frac{\mathrm{d}c^*(t,X^*(t))}{c^*(t,X^*(t))} &= \frac{c_t^*(t,X^*(t))\,\mathrm{d}t + c_x^*(t,X^*(t))\,\mathrm{d}X^*(t)}{c^*(t,X^*(t))} \\ &= \frac{1}{\phi}\left(r + \hat{\mu}(t) - \delta - \frac{1}{\theta}\mu(t) + (1+\phi)\frac{1}{2}\frac{\lambda^2}{\gamma\sigma^2}\right)\mathrm{d}t \\ &\quad + \frac{v_t(t)}{v(t)}\mathrm{d}t + \frac{\lambda}{\gamma\sigma}\mathrm{d}W(t) \ , \\ c^*(0, X^*(0)) &= c^*(0, x_0) \ . \end{aligned} \tag{54}$$

In the calculations, we have used the expressions for the optimal control (c^*, π^*, b^*) from Theorem 2.2. Also, we have plugged in the derivative g_t from (53) in Appendix A. In (54), the entity $\frac{v_t(t)}{v(t)}$ is rather complicated, but it simplifies if we assume that $\hat{\mu} = \alpha\mu$ for some constant $\alpha > 0$ and that ε is constant. We then get that

$$\begin{aligned} \frac{v_t(t)}{v(t)} &= \frac{\kappa - \phi\kappa - 1 + \gamma}{(1-\gamma)\phi}\left(1 + \varepsilon^{\frac{1}{\gamma+\kappa-1}}\alpha^{\frac{\gamma-1}{\gamma+\kappa-1}}\mu(t)^{\frac{\gamma}{\gamma+\kappa-1}}\right)^{-1} \\ &\quad \times \frac{\gamma}{\gamma+\kappa-1}\mu_t(t)\,\varepsilon^{\frac{1}{\gamma+\kappa-1}}\alpha^{\frac{\gamma-1}{\gamma+\kappa-1}}\mu(t)^{\frac{\gamma}{\gamma+\kappa-1}-1} \ . \end{aligned}$$

Also, the SDE in (54) reduces to

$$\begin{aligned} \frac{\mathrm{d}c^*(t,X^*(t))}{c^*(t,X^*(t))} &= \frac{1}{\phi}\left(r - \delta + \left(\alpha - \frac{1}{\theta}\right)\mu(t) + (1+\phi)\frac{1}{2}\frac{\lambda^2}{\gamma\sigma^2}\right)\mathrm{d}t \\ &\quad + \frac{v_t(t)}{v(t)}\mathrm{d}t + \frac{\lambda}{\gamma\sigma}\mathrm{d}W(t) \ . \end{aligned}$$