LTCI: a multi-state semi-Markov model to describe the dependency process for elderly people

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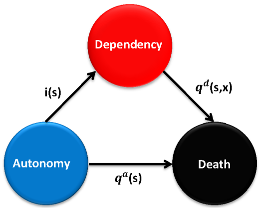
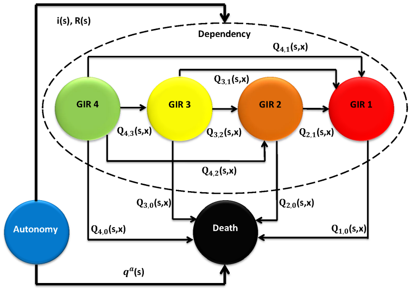
Same old people, brand new model

A simple 3 state model:



The new model:



- s: age.
- x: time spent in the current state.
- GIR 4 to 1: levels of dependency used for the french public aid. GIR 1 is the most severe state.

Properties of the new model

- 4 states of dependency.
- continuous time scale.
- semi-Markov model.

A few definitions

Definition (Markov process)

The future of the process **only** depends on its past through *the current state*.

Definition (semi-Markov process)

The future of the process depends on its past through **both** *the current state* **and** *the time spent in the current state*.

Definition (semi-Markov kernel)

A semi-Markov process is entirely determined by its semi-Markov kernel $Q_{i,j}(s,x)$ with:

- i: departure state.
- j: arrival state.
- s: age at entry in state i.
- x: duration.

Fundamental relation

$$Q_{i,j}(s,x) = \underbrace{p_{i,j}(s)}_{\text{probability}} \times \underbrace{F_{i,j}(s,x)}_{\text{duration law}} .$$

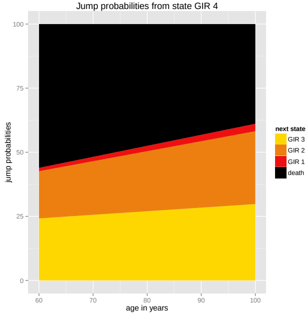
Jump probabilities and duration laws

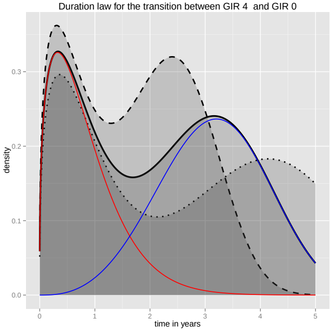
Jump probabilities

$$p_{i,j}(s) \;=\; a_{i,j} \;\times\; s \;+\; b_{i,j}.$$

Duration laws (cdf, index not displayed)

$$F(s,x) = (1-\lambda)W_1(s,x) + \lambda W_2(s,x)$$
$$W(s,x) = 1 - e^{-\sigma x^{\nu} e^{\beta s}}.$$





Estimation of parameters

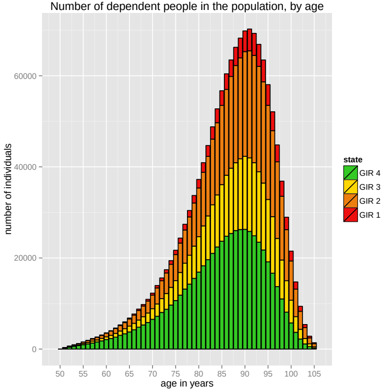
The likelihood function has the following expression:

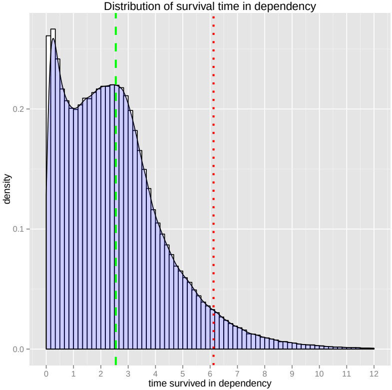
$$L = \prod_{p=1}^{N} \left[\underbrace{\left(\prod_{k=1}^{n_p - 1} c_{X_k^p, X_{k+1}^p}(t_k^p, t_{k+1}^p) \right)}_{\text{observed transitions}} \times \underbrace{c^1_{X_{n_p}^p}(t_{n_p}^p, T_2^p)^{\delta_1^p} \times c^2_{X_{n_p}^p}(t_{n_p}^p, T_1^p, T_2^p)^{\delta_2^p}}_{\text{specific censoring terms}} \right]$$

- $c_{X_k^p, X_{k+1}^p}(t_k^p, t_{k+1}^p) = \underbrace{p_{X_k^p, X_{k+1}^p}(t_k^p)}_{\text{jump probability}} \times \underbrace{f_{X_k^p, X_{k+1}^p}(t_k^p, t_{k+1}^p - t_k^p)}_{\text{density of duration law}}.$
- $c^1_{X_{n_p}^p}(t_{n_p}^p, T_2^p) = \underbrace{S_{X_{n_p}^p}(t_{n_p}^p, T_2^p - t_{n_p}^p)}_{\text{marginal survival function}}.$
- $c^2_{X_{n_p}^p}(t_{n_p}^p, T_1^p, T_2^p) = \underbrace{p_{X_{n_p}^p, 0}(t_{n_p}^p)}_{\text{jump probability}} \times \underbrace{F_{X_{n_p}^p, 0}(t_{n_p}^p, T_1^p - t_{n_p}^p)}_{\text{cds of duration law}} + \underbrace{S_{X_{n_p}^p}(t_{n_p}^p, T_2^p - t_{n_p}^p)}_{\text{marginal survival function}}.$

- Optimization performed using the Nelder-Mead algorithm.
- Selection of the best sub-model according to the BIC.

Statistics on simulated trajectories





Graphs generated using 1 million trajectories with initial age of 50.

Pricing methodology for pure level premium

The required pure level premium is the value p^* such that:

$$E\left[\text{NPV}(p^* \times \mathbb{I}_P)\right] = E\left[\text{NPV}(B)\right].$$

- NPV: Net Present Value.
- B: **benefit** cash flows.
- $\mathbb{I}_P$: **premium unit** cash flows.

Estimator of the premium: $\widehat{p}_n = \dfrac{\widehat{\mu_B}(n)}{\widehat{\mu_P}(n)} \underset{n \to +\infty}{\longrightarrow} p^*$ a. s. by the law of large numbers.

- $\widehat{\mu_B}(n)$ (resp. $\widehat{\mu_P}(n)$): estimator of empirical mean of NPV(B) (resp. NPV$(\mathbb{I}_P)$).
- $\widehat{\sigma_B}(n)$ (resp. $\widehat{\sigma_P}(n)$): estimator of empirical mean of NPV(B) (resp. NPV$(\mathbb{I}_P)$).

We show that with n large enough:

$$|p^* - \widehat{p}_n| \leq \left(\widehat{\sigma_B}(n) + \widehat{p}_n \widehat{\sigma_P}(n)\right) \frac{\Phi^{-1}(1 - \frac{\alpha}{2})}{\widehat{\mu_P}(n)\sqrt{n}}$$

with a level of confidence of $1 - \alpha$

- Φ: cumulative distribution function of the standard normal law.

Conclusion

Summary:

- A multi-state continuous time model based on semi-Markov process.
- We define it through jump probabilities and duration laws.
- Calibration uses the Maximum Likelihood method.
- Pricing relies on Monte Carlo simulations.

Long-term objectives:

- Assess the sampling error (Bootstrap methods).
- Study causes of dependency.
- Take into account other covariates (using e.g., Cox model).
- Find trends for model parameters.

Thank you for your attention !