31 May - 03 June 2016
at
ISEG- Lisbon School of Economics
and Management

Risk estimation model on epidemic outbreaks for an insurer

Prepared by Naoyuki Ishimura
co-authored with Daniel Komadel and Yasukazu Yoshizawa

Presented to the ASTIN COLLOQUIUM
LISBOA 2016
31 MAY – 03 JUNE 2016

IAP: Instituto dos Atuários Portugueses
Campo Grande 28, 8 C 1700-093 Lisboa
Portugal
TEL: + 351 21 846 38 82
Email: astincolloquium2016@gmail.com
http://www.actuaries.org/lisbon2016/

Risk estimation model on epidemic outbreaks for an insurer

Naoyuki Ishimura (Chuo University, Tokyo, Japan)
Daniel Komadel (University of Economics in Bratislava, Slovakia)
Yasukazu Yoshizawa (Tokio Marine and Nichido Fire Insurance Co. Ltd., Tokyo, Japan)

Abstract. Epidemic outbreaks, which have been observed in recent years, have caused a big impact on modern society once if they are occurred. We recall, for examples, SARS (Severe Acute Respiratory Syndrome), Ebola virus diseases, MERS (Middle East Respiratory Syndrome), dengue and/or Zika fever and so on. Insurance companies have a potential to mitigate the effect of such events. Here we propose a simple stochastic model for an insurer to estimate the risk of epidemic outbreaks.

The construction of our model is based on several apparently separate aspects. The first one is a rough estimate on ultimate number of removals once the epidemic burst takes place. This is given through the threshold theorem, which is originally due to Kermack and McKendrick. The second point is the onset of epidemic outbreaks, which is modelled by the doubly stochastic Poisson process. This part is motivated by the catastrophe options as risk transfer instruments. Finally, the market risk is involved to derive a financial product.

Key words: Epidemic outbreaks, Kermack-McKendrick model, Threshold theorem, Doubly stochastic Poisson processes, Catastrophe options, Risk estimation model for an insurer

I. INTRODUCTION

It has been observed in recent years that epidemic outbreaks, such as SARS (Severe Acute Respiratory Syndrome), Ebola virus diseases, MERS (Middle East Respiratory Syndrome), dengue and/or Zika fever, and so on, sometimes made a large impact on the modern society. These phenomena are challenging issues for whole world.

Apart from the scientific research, the function of insurance companies will be important in order to mitigate and stabilize the effect due to epidemic bursts. The overall estimate of the risk from these tragic events is then necessarily investigated.

In this article, we introduce a simple stochastic model to estimate such risk of epidemic outbreaks for an insurer.

Our strategy is a combination of several apparently independent ingredients. The first aspect is of course a mathematical modeling of diseases, which has been studied over 300 years and much progress has been made so far. Here we just employ the classical deterministic model of Kermack and McKendrick, one of whose important outcomes is the so-called threshold theorem; we can estimate a rough number of ultimate removed individuals from the population suffering from an infectious disease.

The second point is the onset of epidemic outbreaks, which is regarded as a trigger variable and is described by a stochastic process. From the financial viewpoint of an insurer, the phenomenon of epidemic bursts is of a catastrophic nature because it may bring enormous financial expenditures within the short period of time. Such a feature makes it similar to the catastrophic events such as a huge earthquake, a big typhoon, and so on. We utilize a doubly stochastic Poisson process to model the onset.

Finally, in addition to above, the point to be involved is the market risk. Once an epidemic outbreak takes place, a depression will follow and we should be concerned with the depreciation of securities. We simply use a standard Black-Scholes-Merton type model to evaluate the depreciation process. Combining all together, we are then able to estimate the risk of the considered epidemics bursts by taking the expectation.

The present paper is a slight extension as well as modification with necessary corrections of our previous work [7] together with establishments made in [8]. The organization is as follows: In Section II, preliminary issues of our classical epidemic model as well as stochastic process are recalled. Our model for the estimation of risk is explained in Section III. Section IV is devoted to the proof and the computation of risk. Section V concludes with discussion.

II. PRELIMINARY

Here we recall several known basic concepts for the readers' convenience.

A. Structured risk management products

Compared to the traditional approach to risk management, the emerging approach has been developed and applied extensively since 1980s. The former approach is based on standardized insurance products and treated on a case-by-case basis. The latter approach, on the other hand, combines insurance risks and financial risks and is handled on a long term risk management program, which involves multiple long term risks and is implemented by the so-called structured risk management products. Examples of transactions include dual trigger instruments, some catastrophe options, and so on.

Our proposed model for estimating the risk for an insurer of epidemic outbreaks is on the line of this concept of structured risk management products; we observe common characteristics to the catastrophe options. The writer of these products may expect higher return if no epidemics bursts takes place. For further details and other information, we refer for instance to a nice paper by Cox, Fairchild, and Pedersen [1].

B. Epidemic Model

In the present article, for our epidemic modeling, we simply employ the classical deterministic Kermack-McKendric model, since it gives a benchmark result for later sophisticated researches. For background issues and other details, see for instance, nice monographs by Dalley and Gani [3], Murray [11].

Let $x(t)$ denote the number of susceptibles, $y(t)$ infectives, and $z(t)$ removals, which are counted according to the disease status. We note that removals mean dead, isolated, or immune individuals. The total size of the population $N = x(t) + y(t) + z(t)$ is assumed and turned out to be fixed for all $t \geq 0$.

The system of ordinary differential equations due to Kermack and Mckendrick is then expresses as follows:

$$\frac{dx(t)}{dt} = -\beta x(t)y(t) \tag{1}$$

$$\frac{dy(t)}{dt} = \beta x(t)y(t) - \gamma y(t) \tag{2}$$

$$\frac{dz(t)}{dt} = \gamma y(t). \tag{3}$$

The equations is analysed under the initial conditions $\big(x(0), y(0), z(0)\big) = (x_0, y_0, 0)$ with $x_0 + y_0 = N$. Here, β denotes the infection parameter representing the strength of epidemics, and γ is the removal parameter indicating the rate of infectives becoming immune. We define the critical parameter $\rho = \gamma/\beta$ as the relative removal rate. It is easy to see that $x(t)$ is monotone non-increasing, and $z(t)$ is non-decreasing. If $x_0 \leq \rho$, then $y(t)$ is monotone decreasing for all $t > 0$.

Now the well-known Kermack-Mckendrick threshold theorem is summarized as follows, whose way of formulation is essentially taken from [3].

Theorem (Kermack and Mckendrick). (i) *Let $x_\infty = \lim_{t\to\infty} x(t)$ and $z_\infty = \lim_{t\to\infty} z(t)$. Then, when infection ultimately ceases spreading, it follows that*

$$N - z_\infty = x_0 + y_0 - z_\infty = x_0 e^{-\frac{z_\infty}{\rho}},$$

where $y_0 < z_\infty < x_0 + y_0 = N$.
(ii) *A major outbreak occurs if and only if $x_0 > \rho$.*
(iii) *If $x_0 = \rho + \nu$ with small $\nu > 0$ and y_0 is small relative to ν, then the total number of susceptibles left in the population are approximately $\rho - \nu$, and we have*

$$z_\infty \approx 2\nu \frac{\rho}{\rho + \nu} \approx 2\nu\,. \tag{4}$$

Here we just exhibit a sketch of proof for the readers' convenience. First we note that

$$\frac{d}{dt}\big(x(t) + y(t) + z(t)\big) = 0\,,$$

and hence

$$x(t) + y(t) + z(t) = N$$

for all $t \geq 0$.

Next we divide the equation (1) by (3) to obtain

$$\frac{dx}{dz} = -\frac{\beta}{\gamma}x = -\frac{1}{\rho}x\,,$$

where $\rho = \gamma/\beta$ denotes the relative removal rate. Integration directly gives

$$x(t) = x_0 e^{-\frac{z(t)}{\rho}}.$$

In a similar way we find that

$$\frac{dy}{dx} = -1 + \frac{\rho}{x},$$

and so that

$$x(t) + y(t) - \rho \log x(t) = x_0 + y_0 - \rho \log x_0\,.$$

Within the region considered where x, y, z are positive, we easily deduce that $y_\infty = \lim_{t\to\infty} y(t) = 0$, which implies that the identity of (i) holds. The properties (ii) and (iii) of the theorem will be deduced by a direct and an approximation argument. We may safely omit the details.

It is known that a detailed analysis on (1)(2)(3) is possible. Indeed, Kendall [9] has succeeded in an exact handling and has deduced that $z_\infty = \zeta$, where ζ is a positive root of

$$N - z - (\rho + \nu)e^{-\frac{z}{\rho}} = 0 \tag{5}$$

See also § 2.3 of [3].

We will utilize its outcomes to estimate the risk of epidemics.

C. Doubly Stochastic Poisson Process

The onset of epidemic bursts is regarded as a trigger process; as a general model for such rare event, we here employ the so-called doubly stochastic Poisson process or a Cox process. We refer to Lando [10] or a book of Rolski, Schmidli, Schmidt, and Teugels [12] for other background issues.

The reason why we prefer a doubly-Poisson process to a usual homogeneous Poisson process is that the latter has deterministic intensity and hence it is rather inappropriate for modelling the resulting claims for epidemics outbreaks. On the other hand, a doubly stochastic Poisson process or a Cox process are known to provide flexibility in modelling, since the intensity process is allowed to be stochastic.

Now we turn our attention to our process. Let $\Lambda = \{\lambda(t)\}_{t\geq 0}$ be an intensity process; namely, a nonnegative, measurable, and locally integrable stochastic process. A counting process $\{N(t;\Lambda)\}_{t\geq 0}$ is called a Cox process or a doubly stochastic Poisson process with intensity Λ if for each sequence $\{k_j\}_{j=1,2,3,\cdots,n}$ of nonnegative integers, and for $0 < t_1 \le s_1 \le t_2 \le s_2 \le \cdots \le t_{n-1} \le s_{n-1} \le t_n \le s_n$, there holds

$$P\left(\bigcap_{j=1}^{n}\{N(s_j;\Lambda) - N(t_j;\Lambda) = k_j\}\right) = \prod_{j=1}^{n} E\left[\frac{1}{k_j!}\left(\int_{t_j}^{s_j}\lambda(u)du\right)^{k_j}\exp\left(-\int_{t_j}^{s_j}\lambda(u)du\right)\right].$$

One typical example, which is favorably used to measure the effect of catastrophic events, and therefore suitable to epidemic bursts also, is the shot noise process. See for example Dassios and Jang [4]. However, we will not treat this process in the sequel.

The next section is devoted to the introduction of a model for estimating the risk concerning epidemic outbreaks, which is performed by the use of a doubly Poisson process or a Cox process.

III. MODEL

Now we formulate a model to estimate the risk due to epidemic outbreaks for an insurer. Main ingredients are as follows.

(1) The onset of epidemic outbreaks is monitored by a doubly stochastic Poisson process $N(t) = N(t;\Lambda)$ with intensity Λ, which should be combined with the epidemic model; for example in the Kermack-Mckendrick model, the term like $1_{\{N(T)>\rho/\varepsilon\}}$ is included. Here $\rho > 0$ means the threshold value in the threshold theorem, $1/\varepsilon$ is the magnification parameter with small $\varepsilon > 0$, and 1_B stands for the indicator function; that is,

$$1_B(\omega) = \begin{cases} 1 & \omega \in B \\ 0 & \omega \notin B \end{cases}.$$

(2) Once the epidemic burst takes place, the total number of removals is presumed by the threshold theorem; with the original Kermack-McKndrick approximation, we have $z_\infty \approx 2\nu$, and with the Kendall's analysis, $z_\infty = \zeta$, where ζ is a positive root of (5).

(3) The occurrence of epidemic outbreaks will result in the market disturbance. Indeed, it was observed that South Korea economy experienced a slight depression after the occurrence of MERS. Here the market model we consider for the security $\{S(t)\}_{t\ge 0}$ is a standard Black-Scholes-Merton type, which is expressed as follows:

$$S(t) = S_0\exp\left[-\alpha N(t) + \left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W(t)\right],$$

where $S_0 > 0$, $\alpha > 0$, $\mu > 0$ and the volatility σ are given positive constants. The process $S(t)$ is accompanied by the bond process $B(t) = e^{rt}$ with constant interest rate r. This is just a well-known Black-Scholes-Merton model. $N(t) = N(t;\Lambda)$ is a doubly stochastic Poisson process as above and $W(t)$ denotes the standard one-dimensional Brownian motion, which is independent of $N(t)$. The assumption of this independence is acceptable in view of the irrelevant occurrence of epidemic bursts.

Furthermore, under this circumstance, insurance companies keep a policy for the insured. For example, if the stop-loss policy is held, the term $\max\{S(T)-K,0\}$ is included, where K denotes a retention level and $T > 0$ is the time of maturity, which is interpreted as the period of insurance. We note that although T does not directly correspond to the ultimate time when infection ceases spreading, we may consider that T is related to this critical time from the viewpoint of estimating the risk.

In summary, our proposed model for the considered risk $R(t)$ $(0 < t \le T)$ is described as the discounted expected value of the combination of above all risk factors. To be precise, we have, in view of (4)

$$R(t) := E^Q[e^{-r(T-t)} 1_{\left\{N(T)>\frac{\rho}{\varepsilon}\right\}} \cdot 2\nu \max\{S(T)-K,0\}]\,, \tag{6}$$

for the Kermack-McKendrick original approximation, and, with ζ being a positive root of (5),

$$R(t) := E^Q[e^{-r(T-t)} 1_{\{N(T)>\rho/\varepsilon\}} \cdot \zeta \max\{S(T)-K,0\}]\,, \tag{7}$$

for the analysis by Kendall. We remark that the difference is just that 2ν in (6) is replaced by ζ in (7). In both cases, the calculation of the expectation is implemented under the risk-neutral probability measure Q defined below in the next section.

Now our main analytical estimate of this article is read as follows.

Theorem. *We infer that the risk $R(t)$ for an insurer can be expressed as*

$$R(t) = \sum_{l=\nu+1}^{\infty} 2\nu\left(S(t)e^{-\alpha l + \log\left(M_{(\Lambda(T)-\Lambda(t))}(k)\right)} \Phi\left(d_l + \sigma\sqrt{T-t}\right) - Ke^{-r(T-t)}\Phi(d_l)\right) \cdot E\left[\frac{\left(\Lambda(T)-\Lambda(t)\right)^l}{l!} e^{-l(\Lambda(T)-\Lambda(t))}\right],$$

for (6), and for (7), 2ν in $R(t)$ above is replaced by ζ with ζ being a positive root of (5). Here we have put

$$d_l = \frac{\log\left(\frac{S(t)}{K}\right) + r(T-t) - \alpha l + \log(M_{(\Lambda(T)-\Lambda(t))}(k))}{\sigma\sqrt{T-t}} - \frac{1}{2}\sigma\sqrt{T-t}\,. \tag{8}$$

with $k := 1 - e^{-\alpha}$ and $\Phi(d)$ denotes the cumulative distribution function for the standardized normal distribution

$$\Phi(d) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d} e^{-\frac{x^2}{2}} dx\,.$$

In the next section, we compute $R(t)$ in the spirit of no arbitrage principle. In this respect, we refer to Cummins and Geman [2].

IV. Proof of Theorem

A. Key Lemma

In order to estimate the risk above, the next lemma is useful, which is taken from [5]. See also [1].

Lemma. *Let $\{N(t) = N(t;\Lambda)\}_{t\geq 0}$ be a doubly stochastic Poisson process with intensity $\Lambda = \{\lambda(t)\}_{t\geq 0}$ and $M_{\Lambda(t)}(k)$ denotes the moment generating function of the aggregated process $\Lambda(t) := \int_0^t \lambda(s)ds$; namely*

$$M_{\Lambda(t)}(k) = E[\exp(k\Lambda(t))]\,.$$

Then, for $k := 1 - e^{-\alpha}$, the process

$$\left\{\exp\{-\alpha N(t) + \log(M_{\Lambda(t)}(k))\}\right\}_{t\geq 0}$$

gives a martingale with respect to the minimum augmented filtration $\mathcal{F}_t^\Lambda$ of

$$\sigma\{N(s) \mid 0 \leq s \leq t\}\,.$$

Sketch of proof. We wish to show that for $0 \leq s \leq t$

$$E\left[\exp\left\{-\alpha N(t) + \log\left(M_{\Lambda(t)}(k)\right)\right\} \mid \mathcal{F}_s^\Lambda\right] = \exp\{-\alpha N(s) + \log(M_{\Lambda(s)}(k))\}\,.$$

To ensure this, we firstly learn that

$$E\left[e^{-\alpha N(t)}\right] = E\left[\sum_{l=0}^{\infty} e^{-\alpha l}\, P(N(t) = l)\right] = E\left[\sum_{l=0}^{\infty} e^{-\alpha l}\frac{\Lambda(t)^l}{l!}e^{-\Lambda_t}\right]$$
$$= E[\exp(\Lambda(t)(e^{-\alpha} - 1))]\,.$$

We then compute

$$\begin{aligned}
&E\left[\exp\left\{-\alpha N(t) + \log\left(M_{\Lambda(t)}(k)\right)\right\} \mid \mathcal{F}_s^\Lambda\right] \\
&= E\left[\exp\left\{-\alpha\big(N(t) - N(s)\big) + \log\left(M_{(\Lambda(t)-\Lambda(s))}(k)\right)\right\} \mid \mathcal{F}_s^\Lambda\right] \\
&\qquad\qquad \cdot \exp\left\{-\alpha N(s) + \log\left(M_{\Lambda(s)}(k)\right)\right\} \\
&= E\left[\exp\{-\alpha\big(N(t) - N(s)\big)\} \mid \mathcal{F}_s^\Lambda\right] M_{(\Lambda(t)-\Lambda(s))}(k) \cdot \exp\left\{-\alpha N(s) + \log\left(M_{\Lambda(s)}(k)\right)\right\} \\
&= M_{(\Lambda(t)-\Lambda(s))}(-k) M_{(\Lambda(t)-\Lambda(s))}(k) \cdot \exp\left\{-\alpha N(s) + \log\left(M_{\Lambda(s)}(k)\right)\right\} \\
&= \exp\{-\alpha N(s) + \log(M_{\Lambda(s)}(k))\}\,.
\end{aligned}$$

This is what we wanted to know and the proof is completed.

B. Pricing Formula

Now we turn our attention to evaluate (6) and (7). The case (6) is treated. First we learn that Q-mesaure is defined by

$$W^Q(t) := W^P(t) + \frac{(\mu - r)t + \log\left(M_{\Lambda(t)}(k)\right)}{\sigma},$$

where the notation $W^P(t) := W(t)$ is just used for the sake of avoiding a confusion. The Radon-Nikodym derivative process is

$$\frac{dQ}{dP} = \exp\left\{-\frac{1}{2}\int_0^t \left(\frac{\mu - r + \gamma_k(s)}{\sigma}\right)^2 ds - \int_0^t \frac{\mu - r + \gamma_k(s)}{\sigma} dW^P(s)\right\}.$$

Here $\gamma_k(t) := E[\lambda(t)\exp(k\Lambda(t))]/M_{\Lambda(t)}(k)$.

Consequently. in view of the key lemma, we see that the process $e^{-rt}S(t)$ defined by

$$e^{-rt}S(t) = S_0\exp\left[-\alpha N(t) + \log(M_{\Lambda(t)}(k))\right] \cdot \exp\left[\sigma W^Q(t) - \frac{1}{2}\sigma^2 t\right]$$

gives a Q-martingale. Thus we are able to compute the discounted value of (6) under Q-measure. See a relevant discussion in [5].

V. Discussions

The phenomena of epidemic outbreaks, once it takes place, brings a huge confusion to the stability of modern society. We recall an example of MERS, which occurred in South Korea in the spring of 2015 and has resulted in a severe damage in economy, public activity, social mind, and other aspects. The risk management on this issue, both from natural and social scientific standpoint, is certainly indispensable. This attempt is nothing but the structured risk management product.

In this paper, we have introduced and formulated a simple stochastic model for an insurer to estimate the risk which originates from epidemic bursts. The onset of pandemic is modelled by a doubly stochastic Poisson process or a Cox process. After the trigger, the ongoing of epidemics follows the story described by the Kermack-Mckendrick threshold theorem. We then estimate the risk of an insurer subject to above mentioned risk factors. The method is based on a similar argument as employed in the case of catastrophe options. The closed form solution is obtained.

As is well known, one of important functions of insurance includes the mitigation of tragic events. To conduct this role, however, quantitative estimate on the risk is necessary and for such purposes an effective mathematical model is surely needed. Today, so many models for the spread of infectious diseases have been analyzed and applied; it may be possible to utilize some of them for further studies. Concerning various epidemics models, we refer for instance to a review by Hethcote [6].

As to applications of our model to real world, we remark that various hedging parameters, which are needed for considering hedging strategies, can be computed within our proposed model. Indeed, the next proposition is computed. We treat only the case of (6) for simplicity.

Proposition. *Let the risk $R(t)$ for an insurer be given by (6). Then the Delta, Gamma, Rho, and Vega of $R(t)$ are computed as the next expressions, respectively.*

$$\Delta(t) = \frac{\partial R(t)}{\partial S} = \sum_{l=\nu+1}^{\infty} 2\nu\, e^{-\alpha l + \log\left(M_{(\Lambda(T)-\Lambda(t))}(k)\right)} \Phi\left(d_l + \sigma\sqrt{T-t}\right) \cdot E\left[\frac{\left(\Lambda(T)-\Lambda(t)\right)^l}{l!} e^{-l(\Lambda(T)-\Lambda(t))}\right],$$

$$\Gamma(t) = \frac{\partial^2 R(t)}{\partial S^2} = \sum_{l=\nu+1}^{\infty} 2\nu \frac{e^{-\alpha l + \log\left(M_{(\Lambda(T)-\Lambda(t))}(k)\right)}}{S(t)\sigma\sqrt{T-t}} \varphi\left(d_l + \sigma\sqrt{T-t}\right) \cdot E\left[\frac{\left(\Lambda(T)-\Lambda(t)\right)^l}{l!} e^{-l(\Lambda(T)-\Lambda(t))}\right],$$

$$\rho(t) = \frac{\partial R(t)}{\partial r} = \sum_{l=\nu+1}^{\infty} 2\nu\, K(T-t) e^{-r(T-t)} \Phi(d_l) \cdot E\left[\frac{\left(\Lambda(T)-\Lambda(t)\right)^l}{l!} e^{-l(\Lambda(T)-\Lambda(t))}\right],$$

$$\nu(t) = \frac{\partial R(t)}{\partial \sigma} = \sum_{l=\nu+1}^{\infty} 2\nu\, S(t)\sqrt{T-t}\, e^{-\alpha l + \log\left(M_{(\Lambda(T)-\Lambda(t))}(k)\right)} \varphi\left(d_l + \sigma\sqrt{T-t}\right) \cdot E\left[\frac{\left(\Lambda(T)-\Lambda(t)\right)^l}{l!} e^{-l(\Lambda(T)-\Lambda(t))}\right],$$

where d_l is defined by (8) and $\varphi = \Phi'$.

Despite its importance, the quantitative research on an insurer against epidemic outbreaks, to the authors' knowledge, is not so common in the literature. We believe that mathematical studies on this subject should be pursued much further. We do hope that our study will put something forward on this line. It is to be noted that some empirical investigations should be undertaken, which will be our future project of researches.

Many questions are still open to further investigations. One of important directions is that the character of the pattern of outbreaks should be taken into account. Indeed, the number of removals may be different from each epidemics; one is modeled by double peak graphs and the other by one peak graph with the tail of dumped oscillations. We will conduct a better management if such differences are involved in the models. The other is to discuss certain practical applications combined with real data and numerical computation; the evaluation of our model may be performed. These will be our another theme for continuing researches.

ACKNOWLEDGMENT

We are grateful to Professor Takahiko Fujita for helpful discussions. The first author is partially supported by Grant-in-Aid for Scientific Research (C) (Kakenhi) No. 15K04992 from Japan Society for the Promotion of Science (JSPS).

REFERENCES

[1] S.H. Cox, J.R. Fairchild, and H.W. Padersen, "Valuation of structured risk management products," *Insurance: Mathematics and Economics*, vol. 34, pp. 46-57, 2004.
[2] J.D. Cummins and H. Geman, "Pricing catastrophe insurance futures and call spreads: An arbitrage approach," *Journal of Fixed Income*, vol. 4, pp. 46-57, 1995.
[3] D.J. Dalley and J. Gani, *Epidemic Modelling*, Cambridge University Press, 1999, ch. 2.
[4] A. Dassios and J.-W. Jang, "Pricing of catastrophe reinsurance and derivatives using the Cox process with shot noise intensity," *Finance and Stochastics*, vol. 7, pp. 73-95, 2003.
[5] T. Fujita, N. Ishimura, and D. Tanaka, "An arbitrage approach to the pricing of catastrophe options involving the Cox process," *Hitotsubashi Journal of Economics*, vol. 49, pp. 67-74, 2008.
[6] H.W. Hethcote, The mathematics of infectious diseases, SIAM Rev. 42 (2001), 599{653.
[7] C. Ishii, N. Ishimura and T. Fujita, Estimate on the risk of epidemic outbreaks for an insurer: A simple stochastic model, International J. Model. Optim., **5** (2015), 161-165.
[8] C. Ishii, On the risk of an insurer due to epidemic outbreaks (Densenbyo ga hassei sitasai no hoken-gaisya no shiharai risk ni tuite), Thesis for the Master Course Degree, Graduate School of Economics, Hitotsubashi University, March, 2015 (in Japanese).
[9] D.G. Kendall, "Deterministic and stochastic epidemics in closed populations," in "*Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability*," vol. 4, Ed. J. Neyman, University of California Press, Berkeley, pp. 149-165, 1956.
[10] D. Lando, "On Cox processes and credit risky securities," *Review of Derivatives Research*, vol. 2, pp. 99-120, 1998.
[11] J.D. Murray, *Mathematical Biology, I. Introduction*, 3rd Edition, Springer, 2002.
[12] T. Rolski, H. Schmidli, V. Schmidt, and J. Teugels, *Stochastic Processes for Insurance and Finance*, Springer-Verlag, New York, 1998.