Dividend payment with ruin constraint in the Lundberg model

Christian Hipp
Bergisch Gladbach, Germany

Abstract

We maximize the accumulated expected present value of dividends under the constraint that the with dividend risk process has a ruin probability not exceeding a small number. This problem is considered in the de Finetti model and in the Lundberg model. From the solution in the de Finetti model given in Hipp (2003) we learn that one can use a modified dynamic equation in two state variables: the current surplus and the running ruin probability. The dynamics of the second variable constitutes a martingale which – in the Lundberg model with constant barriers between claims – is a function of present surplus, number of claims and surplus after the last claim. We first maximize dividend payments up to the first claim, and derive an iteration scheme from this initial solution. There are cases with exponential claims in which the optimal barriers are constant; for these, the resulting value function dominates the solution computed with the heuristic improvement procedure presented in Hipp (2016); it is close to the value function in the unconstraint case: ruin constraints are cheap!

1 Introduction

Starting with de Finetti's famous article [5], many papers have been written on dividend maximization for insurers in which the stock holder's interest is seen as the only objective; the interests of policyholders are neglected. The corresponding optimal dividend strategies lead – in most actuarial models – to certain ruin for the with dividend process. Here, we consider dividend maximization under the constraint that the probability of the with dividend risk process is limited by some small number. Similar problems have been considered by Albrecher and Thonhauser [1], and Hernandez and Junca [10]. For dividend strategies without certain ruin see [4] and [7].

We shall first reconsider the de Finetti model in which time and space are discrete, for which earlier results can be found in [8] and [9] in which a modified Hamilton-Jacobi-Bellman equation is derived and used for numerical calculations. This modified HJB equation involves running ruin probabilities which form a martingale. In this note we study these martingales in the Lundberg model and arrive at a characterization of the optimal strategy of the problem which leads to an efficient numerical procedure.

This is work in progress; many problems, e. g., solutions for other risk processes, are still open.

2 The de Finetti model

Let $X_1, X_2, ...$ be independent identically distributed random variables with

$$\mathbb{P}\{X_i = 1\} = 1 - \mathbb{P}\{X_i = -1\} = p > 1/2,$$

and for an integer s define

$$S(t) = s + X_1 + ... + X_t, t = 0, 1, 2, ... \tag{1}$$

In this most simple model Bruno de Finetti has investigated the problem of optimal dividend payment in his fundamental 1957 paper (see [5]). We shall thus call the model the *de Finetti model*. The random variables $S(t)$ can be seen as the time t surplus of a company with initial surplus s, losing 1 or earning 1 in each period. It might be an insurer who insures claims of size 2 which occur with probability $q = 1 - p$ for a premium 1. Our assumption $p > 1/2$ implies that $S(t) \to \infty$ for $t \to \infty$; otherwise, ruin would be certain. The ruin probability for initial surplus $s \geq 0$ is

$$\psi(s) = \mathbb{P}\{S(t) < 0 \text{ for some } t \geq 0 | S(0) = s\} = (q/p)^{s+1}, s \geq -1.$$

It is a solution of the linear difference equation

$$f(s) = pf(s+1) + (1-p)f(s-1), \ \ s \geq 0, \tag{2}$$

which is considered for functions $f(s), \ s \geq -1$. Equation (2) is linear, its solution space is spanned by the two functions 1 and $\psi(s)$, and a solution can be identified by its values at two points.

2.1 Dividend problem

We allow for dividend payment in the de Finetti model, i.e. for non-decreasing $D(t), t \geq 0$, with $D(t)$ – depending on $X_1, ..., X_t$ – being the sum of dividends $d(n), n = 0, ..., t$, paid until time t, we consider

$$S^D(t) = S(t) - D(t), t \geq 0.$$

For a discount rate $0 < r < 1$ we define the expected present value of dividends

$$v^D(s) = E\left[\sum_{n=0}^{\tau^D - 1} r^n d(n) | S(0) = s\right],$$

where $\tau^D = \min\{t : S(t)^D < 0\}$ is the ruin time for S^D. The maximal possible present value

$$v(s) = \sup v^D(s), s \geq 0, \tag{3}$$

is often coined the *value of the company.* The supremum is taken over all dividend functions D with $D(t)$ non-decreasing and depending on $X_1, ..., X_t$. We may restrict the maximization over all dividend functions with $d(t)$ an integer for all $t \geq 0$, see [11], p. 11, Lemma 1.9.

The dynamic equation for the dividend problem reads

$$v(s) = \max\{r(pv(s+1) + (1-p)v(s-1)), 1 + v(s-1)\}, s \geq 1. \tag{4}$$

The second term in the brackets stands for a dividend of size 1 paid immediately, and the first for future dividend payments. For initial surplus $s = 0$ immediate dividend payment is forbidden, and for this case the dynamic equation is $v(0) = rpv(1)$.

The dynamic equation in (4)

$$v(s) = r(pv(s+1) + (1-p)v(s-1)) \tag{5}$$

is linear, its solution space has dimension 2, and its solutions can be identified by their values at one point $s > -1$.

The optimal dividend strategy for the problem is a barrier strategy, i.e. for some integer $M \geq 0$ we pay dividends as soon as we are above M. For a proof and a method for computation see [11], p. 16, Example 1.13. The value function will be denoted by $V_0(s)$. If $v(s)$ is the solution of the dynamic equation (5) with $v(-1) = 0$ and $v(0) = 1$, then the barrier M equals

$$M = \arg\min\{v(s+1) - v(s) : s \geq 0\},$$

and with

$$W_0(s) = \frac{v(s)}{v(M+1) - v(M)},\ s \geq 0$$

we have

$$V_0(s) = W_0(s), s \leq M, V_0(s) = V_0(M) + s - M,\ s > M.$$

2.2 Dividend payment with ruin constraint

When we maximize dividend payment, we generally generate certain ruin, and on the other hand maximizing survival probability leads to no dividend payment. One might instead try to maximize dividend payment under a ruin constraint: maximize

$$v^D(s) = E\left[\sum_{n=0}^{\tau^D - 1} r^n d_n | S(0) = s\right]$$

under the constraint $\mathbb{P}\{\tau^D < \infty | S(0) = s\} \leq \alpha$. To simplify the notation we shall assume that all dividend strategies $D(t) = d(1) + ... + d(t)$ satisfy $d(t) = 0$ whenever $t \geq \tau^D$. For the constraint we need that the no dividend ruin probability $\psi(s)$ satisfies $\psi(s) \leq \alpha$. In line with this we define

$$v(s, \alpha) = \sup_D v^D(s) = 0$$

whenever $\alpha \leq \psi(s)$. The dynamic equation for this problem reads

$$v(s, \alpha) = \max\{1 + v(s-1, \alpha), r[pv(s+1, \beta_1) + (1-p)v(s-1, \beta_2)]\}, \tag{6}$$

where the maximum is taken over all $0 \leq \beta_1 \leq \psi(s+1), 0 \leq \beta_2 \leq \psi(s-1)$ satisfying

$$p\beta_1 + (1-p)\beta_2 = \alpha. \tag{7}$$

Here the maximum with $1 + v(s-1, \alpha)$ is omitted whenever $\psi(s-1) > \alpha$. The maximum over the empty set is equal to zero. The maximizers β_1, β_2 define the process $b^D(t)$ of running ruin probabilities for the with dividend process $S^D(t)$ which is an $S^D(t)-$ martingale. At state α this process goes down to β_1 if $S^D(t)$ goes up, and it goes up to β_2 if $S^D(t)$ goes down. When dividends are paid, then the value α remains unchanged. The condition (7) implies that $b^D(t)$ is an $S^D(t)-$martingale.

From equation (6) we obtain an iteration procedure which converges monotonically to the value function of the problem:

$$V_{n+1}(s, \alpha) = \max\{1 + V_{n+1}(s-1, \alpha), r[pV_n(s+1, \beta_1) + (1-p)V_n(s-1, \beta_2)]\},$$

with initial function $V_1(s, \alpha) = 0$ or, more sophisticated,

$$V_1(s, \alpha) = V_0(s - s(\alpha)),$$

where $V_0(x)$ is the time value of dividends without ruin constraint, and $s(\alpha) = \min\{x : \psi(x) \leq \alpha\}$. This method, however, is not efficient, in each step we have to compute an array $V(s, \alpha), s \geq 0, \psi(s) \leq \alpha \leq 1$, for a fine grid of α's.

Numerical results for the example with $p = 0.7$ and $r = 1/1.03$ can be found in [8], section 3, p. 263-264; and in [9].

3 Lundberg model

Here we consider the classical model for non-life insurance claims, the Lundberg model, in which

$$S(t) = s + ct - \sum_{i=1}^{N(t)} Y_i, , t \geq 0, \tag{8}$$

where s is the initial surplus, c the constant premium per time unit, $N(t), t \geq 0$, a homogenous Poisson process with intensity λ, and independent claims

$Y, Y_1, Y_2, \ldots$ which are independent from the claims process $N(t)$. In this model, the ruin probability $\psi(s)$ for initial surplus s satisfies the dynamic equation

$$0 = \lambda E[v(s-Y) - v(s)] + cv'(s), s \geq 0. \tag{9}$$

For exponential claims with density $1/\mu \exp(-x/\mu), x \geq 0$, the function $g(s) = E[v(s-Y)]$ satisfies $g'(s) = (v(s)-g(s))/\mu$, so (9) leads to the second order linear differential equation with constant coefficients, and $\psi(s) = \lambda\mu/c \exp(-Rs)$, where $R = (c-\lambda\mu)/(c\mu)$. Notice that for all solutions $v(s)$ of (9) satisfying $v(s) = 1$ for $s < 0$ we have $\lambda(1-v(0)) = cv'(0)$, so the subspace of these functions has dimension 1. Similar linear differential equations for $g(s)$ are valid for claims having phase-type distributions.

3.1 Dividend problem

For optimal dividend payment without ruin constraint, we obtain the dynamic equation in the no action region

$$0 = -\delta v(s) + \lambda E[v(s-X) - v(s)] + cv'(s), s \geq 0. \tag{10}$$

The value function for the optimal dividend problem now is derived with the solution $v(s)$ of (10) satisfying $v(0) = 1$ and $v'(0) = 1$ and with barrier $M = \arg\min v'(s)$:

$$W_0(s) = v(s)/v'(M), s \geq 0,$$

and $V_0(s)$ with slope 1 for $s \geq M$.

3.2 Dividend payment with ruin constraint

We consider dividend payments $D(t)$ accumulated up to time t, with no payments at or after ruin. The problem to maximize the present value of future dividends under a ruin constraint has value function

$$V(s,\alpha) = \sup_D v^D(s),$$

where τ^D is the ruin time of the with dividend process,

$$v^D(s) = E\left[\int_0^{\tau^D} \exp(-\delta t) dD(t) | S(0) = s\right],$$

and the supremum is taken over all dividend payments satisfying

$$\mathbb{P}\{\tau^D < \infty | S(0) = s\} \leq \alpha.$$

Also here, we will consider a process $b(t)$ of running ruin probabilities which is a martingale with mean α. But before we study these processes in detail, we mention a heuristically defined iteration scheme which leads to good subsolutions of our problem.

3.3 Heuristic iteration scheme

The following iteration scheme for the computation of good suboptimal value functions is given in [9]. Assume we have a suboptimal dividend function $V_n(s,\alpha)$. For $U \geq s$ and $\alpha > \psi(s)$ define $a(B)$ as the solution to

$$\alpha = \frac{\psi(s)-\psi(U)}{1-\psi(U)} + a(U)\frac{1-\psi(s)}{1-\psi(U)}. \tag{11}$$

Then a better suboptimal value function $V_{n+1}(s,\alpha)$ is given by

$$V_{n+1}(s,\alpha) = \max\{\max_{U\geq s} v(s)V_n(U,a(U))/v(U), V_{n+1}(s-1,\alpha)+1\}, \tag{12}$$

where the outer maximum is taken only if $\psi(s-1) \leq \alpha$. The strategy behind this heuristic iteration is: start at s and wait without paying dividends until you reach U. When you reach U, then use an optimal dividend strategy starting at U and allowed ruin probability $a(U)$. The ruin probability for this strategy is the probability to be ruined before reaching U, plus the probability for reaching U before ruin, multiplied by $a(U)$. The dividend value of the strategy is the dividend value $V(U,a(U))$, discounted over the time τ until U is reached. The equation (12) results from the fact that $(\psi(s)-\psi(U))/(1-\psi(U))$ is the probability for ruin before reaching U, and

$$E[\exp(-\delta\tau)] = v(s)/v(U),$$

where $v(s)$ is the solution to (10) with $v(0) = v'(0) = 1$.

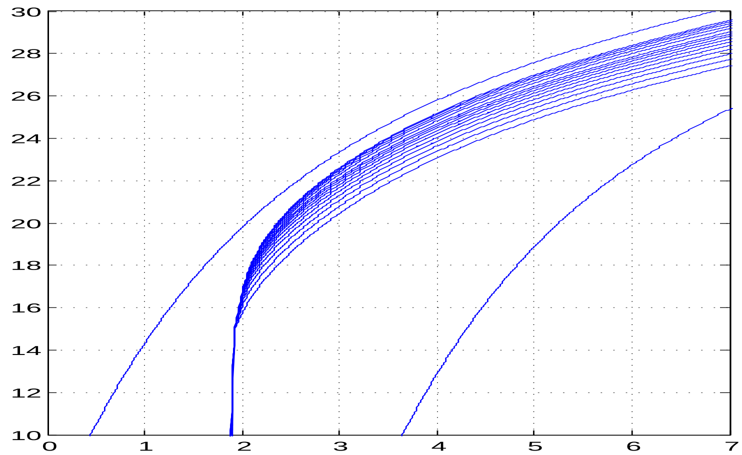
A possible initial function is $V_1(s,\alpha) = 0$, a more sophisticated starting point is

$$V_1(s,\alpha) = V_0(s - s(\alpha)), \tag{13}$$

where $E[\psi(s-s(\alpha))] = \alpha$. This iteration scheme does, however, not seem to converge to the value function, and actually the numerics is quite inefficient. Again, in each step we have to compute an array $V(s,\alpha), s \geq 0, \psi(s) \leq \alpha \leq 1$, for a fine grid of α's.

The following figure shows the heuristic improvement procedure for exponential claims with mean 1, for $\lambda = 1, c = 2$ and $\delta = 0.03$. The initial function (smallest values) is the one given in (13). The improvements $V_n(s,0.2)$ seem to converge, but the computation time for each step is too long for more than 200 iterations. Furthermore, a large number of iterations involves so many operations, that the results are no longer reliable.

We see that a first guess for the true value $V(2,0.2)$ would be approximately 17.



$V(s), V_n(s, 0.2)$ for $0 \leq s \leq 7$ and $n = 1, ..., 30$.

3.4 A promising classical approach

For fixed s and α we consider the dividend maximization problem in which we stop paying dividends after the $n-$th claim. This certainly leads to a sequence of value functions (for easier problems!) which converge to the value function of the given problem. We will first show that the optimal solution to these partial problems is easy: They can be solved using a sequence of barriers $M_i(Z), i = 1, ..., n$, which are constant between claims $i-1$ and i; they are affine linear in the state Z just after claim $i-1$. In an example with exponential claims which has optimal constant barriers we show numerically that this approach gives better results than the above heuristic iteration.

We first consider the case of dividend payment until the first claim, and then show in the case $n = 2$ that the optimal solution has constant barriers between claims $i-1$ and i which are of the form $M(Z) = \rho Z + H$, where H is a constant and Z the state at which claim $i-1$ happens.

3.5 Dividend payment until the first claim

This problem is solved as follows: let $B_0(x) = E[\psi(x-Y)]$; if $B_0(s) < \alpha$, then we can find $M_1 < s$ satisfying $B_0(M_1) = \alpha$, and we pay a lump sum $s - M_1$ immediately, and after that we pay out all incoming premia. If $B_0(s) \geq \alpha$ or $s < M_1$, then we wait until we reach M_1, and then we again pay out all premia. The present value of dividends for this strategy for $s \leq M_1$ is

$$E\left[\int_{(M_1-s)/c}^{\infty} \lambda \exp(-(\lambda+\delta)u) c du\right] = \frac{\lambda c \exp(-(\lambda+\delta)(M_1-s)/c)}{\lambda+\delta}. \tag{14}$$

The running ruin probabilities $b^D(t)$ for this optimal strategy are also easily computed: Since after the first claim no dividends are paid, $b^D(t) = B(S^D(t))$

will be given by the dynamic equation (martingale property)

$$0 = \lambda E[\psi(x - Y) - B(x)] + cB'(x),$$

with initial condition $B(s) = \alpha$ which has a solution of the form

$$B(x) = -\frac{\lambda}{c}\int_0^x E[\psi(u - Y)]\exp(-\lambda u/c)du\exp(-\lambda x/c) + C\exp(\lambda x/c), \quad (15)$$

where C is chosen such that the initial condition holds. Since $B(x)$ should be decreasing (and of course bounded by 1), we obtain that $S^D(t)$ should have an upper bound which defines the optimal barrier M_1 which is given by the smallest solution of

$$B(x) = E[\psi(x - Y)].$$

This simple structure looks promising; its properties can be used generally (for all n).

3.6 Dividend payment after the $n-$th claim

The optimal dividend value up to the $n+1-$th claim is obtained using the results for $n = 1$, in particular the above optimal strategy and formulas (15) and (14). Assume that claim n happens at time T and the position after this claim equals Z. Then we should maximize the dividends paid after time T and stop paying dividends at the next claim. For this we will choose a barrier $M(Z)$ and obtain a function $B(x)$ from equation (15) and a value $V(Z)$ form equation (14) in which M is replaced by $M(Z)$. In order to simplify the notation we will restrict in the following to exponential claims with mean $\mu = 1$. Notice that $V(Z)$ does not depend on the claim size distribution, but $B(x)$ in the exponential case simplifies to

$$B(x, M) = \psi(x) + R\exp(-R(2M - x)),$$

where $R = \lambda/c - 1$ is the adjustment coefficient for which we have $\psi(x) = (1 - R)\exp(-Rx)$. To identify the dependence of $M(Z)$ from Z we use the Lagrange multiplier approach, i.e. we look at $V(Z) - LB(Z, M)$ which is

$$K\exp(-\gamma(M - Z)) - L[(1 - R)\exp(-RZ) + R\exp(RZ)\exp(-2RM)]$$

and maximize it with respect to M. Here, $K = \lambda c/(\lambda + \delta)$ and $\gamma = (\lambda + \delta)/c$. We obtain for $\gamma < 1$ (which is true in most cases) that $M(Z) - \rho Z$ with $\rho = (R - \gamma)/(1 - \gamma)$ is a constant H_n which might depend on n. The form of the barrier

$$M(Z) = \rho Z + H_n$$

is intuitive since in most cases $\rho < 0$; this means that for small initial surplus Z we will choose a barrier which is larger that in the case of a large surplus. And this form of barrier we find in each time period of claims. This can be used to approximate the value function of the problem *dividend payment until the $n-$th claim* with an appropriate choice of the constants $H_i, i = 1, ..., n$.

3.7 Special case of constant barriers

We have $\rho = 0$ for exponential claims with mean 1 under the condition $R = \rho$ which is equivalent to $c = 2\lambda + \delta$. For given barriers $M_1 \geq ... \geq M_n$ we sequentially compute the running ruin probabilities $B_i(x)$ and the dividend values $V_i(x)$ with the dynamic equations

$$\begin{aligned} 0 &= \lambda E[B_{i-1}(x-Y) - B_i(x)] + cB_i'(x), \\ 0 &= -\delta V_i(x) + \lambda E[V_{i-1}(x-Y) - V_i(x)] + cV_i'(x), \end{aligned}$$

with the boundary values $B_i'(x) = 0$ for $x \geq M_i$, and $V_i'(x) = 1$ for $x \geq M_i$. Here, M_n is the barrier until the first claim, M_{n-1} is the barrier for the time between the first and the second claim, and so on. $B_1(x)$ is the running ruin probability before the $n-$th claim, so

$$0 = \lambda E[\psi(x-Y) - B_1(x)] + cB_1'(x),$$

and $B_2(x)$ is the running ruin probability between claim number $n-1$ and n. $V_1(x)$ is the present time T_{n-1} value of dividend payments over the last period, where T_i is the time for claim i. $V_1(x)$ can be obtained with formula (14), where the barrier is M_1.

For exponentially distributed claims with mean 1 the functions

$$G_i(x) = E[B_i(x-Y)] \text{ and } H_i(x) = E[V_i(x-Y)]$$

are computed with the useful differential equations

$$\begin{aligned} G_i'(x) &= (B_i(x) - G_i(x), \\ H_i'(x) &= (V_i(x) - H_i(x)), \end{aligned}$$

together with the boundary values

$$G_i(0) = 1, H_i(0) = 0.$$

It is surprising that the optimal strategies are all barrier strategies, while in the case of unconstraint optimal dividend payment band strategies show up (see [6] and [3]).

3.8 Numerical results

Our numerical work is done for exponential claims with mean $\mu = 1$, with $\lambda = 1, \delta = 0.03$ and $c = 2.03$. The first test shows that the number n of claims considered should be large, but not too large. You see the computed values for various n and for barriers $M_i, i = 1, ..., n$ which are all equal to M. The computations are done for $s = 2$ and $\alpha = 0.2$, which is a region in which the true value function of the problem is very steep (we have $\psi(2) = 0.1839$ and hence $V(2, 0.1839) = 0$ while $V(2, 0.2)$ is seen to be approximately 17 or larger in our figure above.

In the table below you see some suboptimal values for $V(2, 0.2)$ with barriers $M_i = M$ which are all equal, and for various n. You can see that n should be large, but not too large.

n	$V_n(2, 0.2)$	M
20	7.6025	11.77
50	15.2775	13.56
120	19.6333	15.27
150	19.8397	15.71
180	19.8263	16.07

Better results are possible when we take $n = 150$ and M_i which vary with i :

- $M_1 = 17, M_{i+1} = M_i - \varepsilon, \varepsilon = 0.0157 :\ V(2, 0.2) = 20.0594.$
- $M_1 = 19, M_{i+1} = M_i - \varepsilon, \varepsilon = 0.03615 :\ V(2, 0.2) = 20.1750.$
- $M_1 = 20, M_{i+1} = M_i/\varepsilon, \varepsilon = 1.00265 :\ V(2, 0.2) = 20.1816.$
- without ruin constraint: $V_0(2) = 20.6058$.
- ruin constraints are cheap, they lead to small reductions of the company value.

4 Conclusions

Above we wrote that this note is work in progress, many problems are still open, e. g. the solution for diffusion models for which heuristic improvements can be defined and calculated (see [9]). Others are:

- numerical computations for non-constant barriers;
- fast iteration schemes for other than exponential claims (phase-type);
- efficient methods for the calculation of optimal barriers;

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