RISK-SHIFTING AND OPTIMAL ASSET ALLOCATION IN LIFE INSURANCE: THE IMPACT OF REGULATION

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Abstract. In a typical participating life insurance contract, the insurance company is entitled to a share of the return surplus as compensation for the return guarantee granted to policyholders. This call-option-like stake gives the insurance company an incentive to increase the riskiness of its investments at the expense of the policyholders. The conflict of interests can partially be solved by regulation deterring the insurance company from taking excessive risk. In a utility-based framework where default is modeled continuously by a structural approach, we show that a flexible design of regulatory supervision can be beneficial for both the policyholder and the insurance company.

Keywords: regulation, life insurance, credit risk, barrier options, utility maximization, risk shifting
JEL: G11, G23

1. Introduction

Participating life insurance contracts usually provide a yearly or maturity guarantee for the policyholders. The surplus above this guaranteed amount is shared between policyholders and the owners (shareholders) of the insurance company. In return, the policyholders pay insurance premiums that are invested by the insurance company. The call-option-like stake of shareholders gives the insurance company incentives to invest the premiums as riskily as possible at the expense of the policyholder. A very similar conflict arises between debt and equity holder of corporations: Especially if the corporation is in distress, the equity holders tend to take on as much risk as possible, a line of action called "risk shifting". One possibility to solve this conflict of interests is the introduction of a regulator that restricts excessive risk taking of the insurance company (see, e.g., Filipović et al. [2015]). The aim of this paper is

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to propose ways on how such a regulatory scheme can be implemented in order to solve or at least alleviate this conflict of interests.

Regulatory supervision is necessary and justified, as insurance markets are still rather intransparent: An information asymmetry between insurance company and policyholders further increases the described conflict of interests (see, e.g., Rees et al. [1999]). Regulatory supervision in Europe has transformed from relatively simple methods to a comprehensive and very detailed line of action adequately reflecting all the risks inherent in life insurance companies. Thereby, a risk-based supervision based on Value-at-Risk or default probability gains more and more importance (see, e.g., Bauer et al. [2005]).

There are many different designs of a regulatory supervision: The insurance company may be forced to provide some risk-based capital, as, for example, specified in the Solvency II accord. The amount of risk capital needed is usually defined to impose an upper bound on default probability (see, e.g., McCabe and Witt [1980], Gatzert and Schmeiser [2008]). The regulator may also include price constraints by introducing restrictions on premium calculation (see, e.g., MacMinn and Witt [1987]). As pointed out by Schlütter [2014], however, price constraints are a less efficient instrument of solvency regulation than risk-based capital requirements. Third, being the main approach in this paper, the regulator may also impose constraints on the riskiness of the insurance company's investment decisions.[1] For distressed companies, the last is easier to implement than the risk-based provision of capital, because a distressed company might face problems acquiring new capital necessary to fulfill solvency requirements.

Due to the current regulatory practice, the provision of risk-based capital or solvency requirements is based on (1) a static, one-period model where default is only possible at the maturity of the contract and (2) a short time horizon to calculate the underlying risk measures. This current practice is also reflected in the academic literature, see, recently and among many others, Gatzert and Schmeiser [2008], Schmeiser and Wagner [2013], Dong and Schlütter [2014], Schlütter [2014], Filipović et al. [2015]. We want to depart from those two

[1]This might include a restriction of the share of stocks and other risky investments or some minimum diversification requirement. In Germany this is, for example, regulated by §3 Anlageverordnung (AnlV). An overview of regulations in other European countries is given in Davis [2001].

assumptions: First, we allow for a more realistic continuous default possibility of the insurance company (see, e.g., Grosen and Jørgensen [2002]). Second, life insurance contracts are mostly long-term contracts. To really capture inherent risks, regulation has to take into account risk measures on longer time horizons.

We first analyze the effect of a default constraint on the optimal asset allocation and assess whether this helps to (at least partially) solve the conflict of interests regarding the investment decision between insurance company and policyholders. In this first step, we assume that the insurance company commits to an investment strategy at contract initiation and leaves this strategy unchanged until contract termination.[2] In a second step, we analyze a more flexible regulatory scheme: The regulator introduces a "traffic light system" that indicates whether the life insurance company is in danger of facing solvency problems ("yellow bulb") or even has severe and immediate problems ("red bulb"). This traffic light solvency stress test is implemented in Denmark and Sweden, see, e.g., Jørgensen [2007]. Similar ideas have been introduced in other European countries and in the Solvency II regulations. A flexible regulatory system could force distressed insurance companies ("yellow bulb") to change their investment strategy in order to fulfill solvency requirements.

We investigate the effect of this more flexible regulatory scheme on the benefits of both the policyholders and the insurance company. If the regulator gets the possibility to force distressed insurance companies to decrease the riskiness of their investment strategy, this allows to significantly decrease solvency risk[3], only marginally changing the benefits of policyholders or the insurance company. If target default probabilities are the same under both the standard and the flexible regulatory framework, we exemplarily show that the regulator

[2] It is commonly accepted that the riskiness of the asset strategy should take account of its funding ratio (=assets divided by liabilities). An insurance company might adapt its asset allocation dependent on the possibility that it is (un)able to meet its obligations. Bohnert et al. [2015] suggest a CPPI-based strategy. Graf et al. [2011] and Hieber et al. [2014] change the asset allocation dependent on risk measures, i.e. the expected shortfall below the company's investment guarantees. Empirically, it is not obvious whether life insurance companies should increase or decrease risk in case of distress: Mohan and Zhang [2014] find that US public funds increase risk if they are underfunded, while Rauh [2009] shows that the asset allocation is less risky if the company's financial condition is weaker.

[3] In our numerical example, we reduce default risk by 2/3 while the certainty equivalent for policyholders and insurance company decreases by less than 3 %, see Table II.

might increase the benefits of policyholders and insurance company.

The remainder of the paper is organized as follows. In Section 2, we describe the model setup and introduce the payoffs of the policy- and shareholder. We set up their optimal investment problem, taking account of the possible default of the insurance company. More importantly, the flexible regulatory intervention (traffic light system) is presented. In the subsequent Section 3, the expected utility of the policy- and shareholder are computed analytically. In Section 4, some numerical examples are illustrated to examine the effect of the default constraint on the optimal investment strategies, and particularly the goodness of the flexible regulatory framework. Finally, we provide some concluding remarks in Section 5 and detailed proofs in Section 6.

2. Notations and Model Setup

Our model contains three parties: an insurance regulator, a representative shareholder (also equity holder) and a representative policyholder (also liability holder). The latter two constitute a mutual life insurance company. We assume that the representative policyholder invests in a participating life insurance contract with a maturity of T years, $T < \infty$. At the initiation of the contract, the policyholder invests a lump sum L_0 in a single premium contract; the shareholder provides initial equity $E_0 > 0$. Consequently, the initial asset value A_0 of the insurance company is given by the sum of both contributions, i.e. $A_0 := L_0 + E_0$. We denote the share of the policyholder's contribution (or equivalently the debt ratio of our insurance company) by $\alpha := L_0/A_0$, where obviously $\alpha \in (0, 1)$.

Asset model. Let us define a financial market consisting of one risk-free bond B with risk-free interest rate r, i.e. $\mathrm{d}B_t = rB_t \,\mathrm{d}t$ and $B_0 = 1$. Furthermore, there is the possibility to invest in a risky investment

$$\mathrm{d}S_t = \mu\, S_t \,\mathrm{d}t + \sigma\, S_t \,\mathrm{d}W_t, \quad S_0 = 1, \tag{1}$$

where $\mu > r$, $\sigma > 0$, and W is a standard Brownian motion under the real world measure $\mathbb{P}$. To start with, we assume that the insurance company invests the total proceeds A_0 in a diversified portfolio of risky and non-risky assets. Assume, a constant share θ_1 is invested in the risky asset S and the remainder in the risk-free asset B. With the initial asset investment

$A_0 > 0$, this yields the following asset dynamics:

$$\mathrm{d}A_t = \big(r + \theta_1(\mu - r)\big)\, A_t\, \mathrm{d}t + \sigma\theta_1\, A_t\, \mathrm{d}W_t\,. \tag{2}$$

Default of the insurance company. We want to explicitly take the default risk of the insurance company into account. Therefore, we make use of a structural approach and assume that the insurance company defaults as soon as its assets A_t hit or drop below a specified percentage η of the guaranteed amount $L_t = L_0\, e^{gt}$, where $g \le r$. Thus, we introduce a default barrier $D_t := \eta L_0\, e^{gt}$ whose accrual rate g is the same as for the guaranteed amount.[4] The time of default is then given by

$$\tau := \inf\left\{t \ge 0 \,\middle|\, A_t \le D_t\right\}, \tag{3}$$

where we set $\inf\{\emptyset\} = \infty$.

Terminal payoff to liability and equity holder. The insurance payoff to the policyholder is contingent on whether the insurance company survives the maturity date T. If there is no premature default of the insurance company, the policyholder receives the following terminal payoff:

$$\begin{aligned}\Psi_L(A_T) :=& \begin{cases} A_T & \text{if } A_T \le L_T \\ L_T + \delta\left[\alpha A_T - L_T\right]^+ & \text{else}\,, \end{cases} \\ =& \, L_T + \delta\left[\alpha A_T - L_T\right]^+ - [L_T - A_T]^+\,, \end{aligned} \tag{4}$$

where we denote by $[\,\cdot\,]^+$ the maximum $\max\{\,\cdot\,, 0\}$. The participation rate δ is the percentage of surpluses that is credited to the liability holder. If there is no premature default, the terminal contract payoff is a combination of a fixed payment L_T, a bonus call and a shorted put option on the insurance company's assets. The shorted put option refers to losses of the liability holder if the company is not defaulted prematurely but assets at maturity are insufficient to cover the guaranteed amount.

In the case of default, a rebate payment is provided to the policyholder at time τ. This rebate payment consists of the minimum of the current asset value $A_\tau = D_\tau$ and the current liabilities L_τ. If we – for time consistency reasons – assume that the rebate payment is until T accumulated at the risk-free rate r, the policyholder receives the following contract payoff

[4] We need to assume $\eta < A_0/L_0$, otherwise the insurance company is instantly defaulted.

at time T:

$$V_L(A_T) := \mathbb{1}_{\{\tau > T\}}\, \Psi_L(A_T) + \mathbb{1}_{\{\tau \leq T\}}\, e^{r(T-\tau)}\, \min(L_\tau, D_\tau)\,, \tag{5}$$

where $\mathbb{1}_B$ is an indicator function which gives 1 if B occurs and 0 otherwise. The equity holder always obtains the residual asset value. If there is no premature default of the insurance company, the payoff to the equity holder is

$$\begin{aligned}\Psi_E(A_T) &:= \begin{cases} 0 & \text{if } A_T \leq L_T \\ A_T - L_T & \text{if } L_T < A_T \leq A_0 e^{gT} \\ A_T - L_T - \delta\left[\alpha A_T - L_T\right]^+ & \text{else} \end{cases} \\ &= \left[A_T - L_T\right]^+ - \delta\left[\alpha A_T - L_T\right]^+ . \end{aligned} \tag{6}$$

If there is premature default, a rebate payoff $D_\tau - \min(L_\tau, D_\tau)$ is provided to the equity holder. More compactly, the total payoff of the equity holder at maturity T is thus given by

$$V_E(A_T) := \mathbb{1}_{\{\tau > T\}}\, \Psi_E(A_T) + \mathbb{1}_{\{\tau \leq T\}}\, e^{r(T-\tau)}\, \max(D_\tau - L_\tau, 0)\,. \tag{7}$$

Risk-neutrality of the equity holder. We assume that the insurance company is able to fully diversify its investments and thus wants to maximize its expected payoff at maturity T. Its investment at contract initiation is given by $E_0 = (1-\alpha)A_0$. The insurance company can decide on its share of risky investment θ_1 using the goal function

$$\max_{\theta_1}\, \mathbb{E}_{\mathbb{P}}\big[V_E(A_T)\big]\,. \tag{8}$$

Liability holder. The liability holder – in contrast – cannot fully diversify the investment and is assumed to be risk-averse. She optimizes with respect to a utility function $u_L(\,\cdot\,)$ (see Definition 2.1) and thus evaluates her payments according to the goal function

$$\max_{\theta_1}\, \mathbb{E}_{\mathbb{P}}\big[u_L\big(V_L(A_T)\big)\big]\,. \tag{9}$$

The resulting θ_1 would be chosen if she were allowed to decide on the investment portfolio. In reality, however, the liability holder cannot directly influence the insurance company's investment decision θ_1.

Definition 2.1 (Utility function). $u_L(\,\cdot\,)$ is increasing, concave, and twice differentiable on $\mathbb{R}$ with $u_L' > 0$, $u_L'' < 0$ $\lim_{x\to\infty} u_L(x) = -\infty$ and $\lim_{x\to\infty} u_L'(x) = \infty$.

Later on, we are exemplarily using power utility, see Example 2.2.

EXAMPLE 2.2 (Power utility). *Let $\gamma_1 > 0$, $\gamma_1 \neq 1$ be the relative risk aversion parameter of the policyholder, i.e. $u_L(V_L) := V_L^{1-\gamma_1}/(1-\gamma_1)$.*

Competitive market. In our setting, we assume that the underlying financial market is complete, frictionless, and competitive. Thus, arbitrage-free prices for any claim in this market are obtained via arbitrage-free pricing under the risk-neutral measure $\mathbb{Q}$. Under $\mathbb{Q}$, we still have a risk-free bond $dB_t/B_t = r\,dt$. The risky asset evolves as

$$\mathrm{d}S_t = r\,S_t\,\mathrm{d}t + \sigma\,S_t\,\mathrm{d}W_t^{\mathbb{Q}}\,, \tag{10}$$

where still $B_0 = S_0 = 1$ and $W^{\mathbb{Q}}$ is a standard Brownian motion under $\mathbb{Q}$.

If an insurance contract is fairly priced, the initial investment of the shareholder equals its arbitrage-free initial stake $E_0 = (1-\alpha)A_0$, i.e.

$$(1-\alpha)A_0 = \mathbb{E}_{\mathbb{Q}}\big[e^{-rT}\,V_E(A_T)\big]\,. \tag{11}$$

Similarly (and equivalently), from the policyholder's viewpoint, one has to ensure that

$$\alpha A_0 = \mathbb{E}_{\mathbb{Q}}\big[e^{-rT}\,V_L(A_T)\big], \tag{12}$$

with $V_E(A_T)$ and $V_L(A_T)$ as defined in Equation (7), respectively (5).

Regulatory intervention. The aim of this paper is to examine the impact of regulation on the optimal asset allocation determined by the insurance company (see the optimization problem (8)). In a first step, the regulator may force the insurance company to limit its default probability $\mathbb{P}(\tau \leq T)$ to an upper limit ϵ. In a second step, we equip the regulator with more flexibility by allowing it to restrict the insurance company's share θ_1 invested in the risky asset. The resulting regulatory scheme is more flexible and adapts to the evolution of the insurance company's assets. The concept is in analogy to Solvency II regulations in Europe where the regulator has the possibility to intervene, as soon as the assets drop below some critical level $\{K_t\}_{t\geq 0}$ ("yellow bulb") to avoid a default event. If the company's assets nevertheless drop below the default barrier $\{D_t\}_{t\geq 0}$ ("red bulb"), the insurance company defaults. The possible interaction in case of the "yellow bulb" gives the regulator more freedom to act in the interests of both liability and equity holder. The second (upper) threshold K is set as

$$K_t := K_0\,e^{gt}\,, \tag{13}$$

where $D_0 = \eta L_0 < K_0 < A_0$. The hitting time of this barrier is denoted

$$\hat{\tau} := \inf \left\{ t \geq 0 \,\middle|\, A_t \leq K_t \right\}, \tag{14}$$

where we again set $\inf\{\emptyset\} = \infty$. In case this barrier is hit, the regulator may once force the insurance company to change its investment strategy from θ_1 to θ_2. Then, the asset value process is – for $t \geq 0$ – given by

$$\mathrm{d}A_t = \big(r + \theta_{Z_t}(\mu - r)\big)\, A_t \,\mathrm{d}t + \theta_{Z_t}\sigma\, A_t \,\mathrm{d}W_t, \quad A_0 > 0, \tag{15}$$

where $Z_t = 1$ for $t \leq \hat{\tau}$ and $Z_t = 2$ for $t > \hat{\tau}$. The effect of the design of a regulatory policy on the benefits of equity and liability holder is analyzed in the remainder of this paper. For reasons of analytical tractability, we do not consider a strategy recovery of the insurance company, i.e. it is not possible to return to the original asset strategy θ_1.

Under this more flexible regulation, the default-triggering event remains unchanged. A default occurs when the asset process A_t hits the lower threshold D_t (i.e. if $\{\tau \leq T\}$). Since the asset process is continuous and the regulatory barrier K_t by definition greater than D_t, the event $\{\tau \leq T\}$ implies that $\{\hat{\tau} \leq T\}$, i.e. the upper threshold is hit before the lower one.[5]

3. Theoretical results

In order to determine the optimal investment strategy and examine the regulatory effects on it, we need to compute the expected payoff of the equity holder and the expected utility of the policyholder.

3.A. **No regulatory intervention prior to liquidation.** In this first case, we assume that there is no regulatory barrier $\{K_t\}_{t\geq 0}$ and thus the investment strategy stays constant at θ_1. Theorem 3.1 gives analytical expressions for the expected payoff to the equity holder and the expected utility of terminal payoffs to the policyholder.

Theorem 3.1 (Expected utility: No intervention). *Assume the model setup as described in Section 2 with asset process (2). Then, the desired expectations are given by*

$$\mathbb{E}_{\mathbb{P}}\big[u_L\big(V_L(A_T)\big)\big] =: \kappa_L^{(1)}(A_0, D_0, L_0, 0, T), \quad \mathbb{E}_{\mathbb{P}}\big[V_E(A_T)\big] =: \kappa_E^{(1)}(A_0, D_0, L_0, 0, T),$$

[5]The event $\{\hat{\tau} > T\}$ delineates the situation that the assets perform well until maturity T and all the time exceed the upper regulatory threshold. The event $\{\hat{\tau} \leq T, \tau > T\}$ describes the situation that the assets perform moderately until maturity T. The assets have hit the regulatory barrier but the insurance company has not defaulted prematurely. The event $\{\tau \leq T\}$ describes the situation that the company has defaulted.

where $\kappa_L^{(1)}(\,\cdot\,)$ *and* $\kappa_E^{(1)}(\,\cdot\,)$ *can be computed via*

$$\kappa_L^{(i)}(A_t, D_t, L_t, t, T) = \int_t^T u_L\Big(e^{r(T-t)+(g-r)(\tau-t)}\min(L_t, D_t)\Big)\, f^{(i)}(t,\tau,A_t,D_t)\,d\tau$$
$$+ \int_{\ln(D_t/A_t)}^{\infty} u_L\big(L_T + \delta\big[\alpha A_t e^{y+g(T-t)} - L_T\big]^+ - \big[L_T - A_t e^{y+g(T-t)}\big]^+\big)\, g^{(i)}(y,t,T,A_t,D_t)\,dy\,,$$

$$\kappa_E^{(i)}(A_t, D_t, L_t, t, T) = \int_t^T e^{r(T-t)+(g-r)(\tau-t)}\max(D_t - L_t, 0)\, f^{(i)}(t,\tau,A_t,D_t)\,d\tau$$
$$+ \int_{\ln(D_t/A_t)}^{\infty} \big(\big[A_t e^{y+g(T-t)} - L_T\big]^+ - \delta\big[\alpha A_t e^{y+g(T-t)} - L_T\big]^+\big)\, g^{(i)}(y,t,T,A_t,D_t)\,dy\,.$$

The densities g*, respectively* f*, are defined as*

$$g^{(i)}(y,t,T,A_t,D_t) := \frac{1}{\sigma\theta_i\sqrt{T-t}}\,\varphi\left(\frac{y-\tilde{\mu}_i(T-t)}{\sigma\theta_i\sqrt{T-t}}\right)\left(1 - e^{-2\,\frac{\ln(D_t/A_t)^2 - y\ln(D_t/A_t)}{\sigma^2\theta_i^2(T-t)}}\right),$$

$$f^{(i)}(t,\tau,A_t,D_t) := \frac{-\ln(D_t/A_t)}{\sigma\theta_i(\tau-t)^{\frac{3}{2}}}\,\varphi\left(\frac{\ln(D_t/A_t)-\tilde{\mu}_i(\tau-t)}{\sigma\theta_i\sqrt{\tau-t}}\right),\ \tilde{\mu}_i := r + \theta_i(\mu - r) - g - \sigma^2\theta_i^2/2\,,$$

where $\varphi(\,\cdot\,)$ *denotes the density of the standard normal distribution.*

Proof: See the Appendix.

In the case of power utilities, most of the integrals in Theorem 3.1 can be derived analytically. The default probability on the time interval $[0,T]$ is given by

$$\mathbb{P}(\tau \le T) = \Phi\left(\frac{\ln(D_0/A_0) - \tilde{\mu}_1 T}{\sigma\theta_1\sqrt{T}}\right) + \left(\frac{D_0}{A_0}\right)^{\frac{2\tilde{\mu}_1}{\sigma^2\theta_1^2}}\,\Phi\left(\frac{\ln(D_0/A_0) + \tilde{\mu}_1 T}{\sigma\theta_1\sqrt{T}}\right), \tag{16}$$

where $\tilde{\mu_1}$ is defined as in Theorem 3.1, see also the Appendix.

3.B. **Regulatory intervention prior to default.** Now, we are going to derive the same results as in Theorem 3.1 under the assumption that the investment strategy is changed from θ_1 to θ_2 as soon as the regulatory barrier $\{K_t\}_{t\ge 0}$ is hit. This leads to the asset process given by (15).

Technically, this setup is still analytically tractable: Until first hitting the regulatory threshold K at time $\hat{\tau}$, the asset process behaves as a geometric Brownian motion – one of the rare cases where the first-hitting time density is known analytically (the hitting time is distributed according to an inverse Gaussian law, see, for example, Folks and Chhikara [1978]). At time $\hat{\tau}$, the assets equal the barrier $K_{\hat{\tau}}$. After this hitting time, the assets are again a geometric

Brownian motion now with a different mean and volatility parameter due to the changed investment strategy θ_2. Thus, the time to default follows again an inverse Gaussian law. To sum up, the default time τ is given by the convolution of two inverse Gaussian random variables. The default probability can be evaluated via

$$\begin{aligned}\mathbb{P}(\tau \leq T) &= \int_0^T \mathbb{P}\big(\tau \leq T \,\big|\, A_{\hat{\tau}} = K_{\hat{\tau}}\big) \cdot f^{(1)}(0, \hat{\tau}, A_0, K_0)\, d\hat{\tau} \\ &= \int_0^T \int_0^{T-\tau} f^{(2)}(\hat{\tau}, \tau, K_{\hat{\tau}}, D_{\hat{\tau}}) \cdot f^{(1)}(0, \hat{\tau}, A_0, K_0)\; d\hat{\tau}\, d\tau\,, \end{aligned} \tag{17}$$

with f as defined in Theorem 3.1. Equation (16) results as the special case $\theta_1 = \theta_2$.

Similarly to Theorem 3.1, one can derive the expected terminal payoff to the equity holder $\mathbb{E}_{\mathbb{P}}[V_E(A_T)]$ and the expected utility $\mathbb{E}_{\mathbb{P}}[u_L(V_L(A_T))]$ of the liability holder, see Theorem 3.2.

THEOREM 3.2 (Expected utility: Regulatory intervention). *Assume the model setup as described in Section 2 with asset process (15). The regulator may intervene at time $\hat{\tau}$ – the first hitting time of the insurance company's assets A breaching the regulatory barrier K. At time $\hat{\tau}$, the insurance company is forced to change its investment strategy from θ_1 to θ_2. Then, the desired expectations are given by*

$$\mathbb{E}_{\mathbb{P}}\big[u_L\big(V_L(A_T)\big)\big] =: \zeta_L(A_0, D_0, K_0, L_0, 0, T), \quad \mathbb{E}_{\mathbb{P}}\big[V_E(A_T)\big] =: \zeta_E(A_0, D_0, K_0, L_0, 0, T)\,,$$

where

$$\begin{aligned}\zeta_L(A_t, D_t, K_t, L_t, t, T) = &\int_t^T \int_{\ln(D_t/K_t)}^{\infty} u_L\big(L_T + \delta\big[\alpha K_{\hat{\tau}} e^{y+g(T-\hat{\tau})} - L_T\big]^+ - \big[L_T - K_{\hat{\tau}} e^{y+g(T-\hat{\tau})}\big]^+\big) \\ &\cdot f^{(1)}(t, \hat{\tau}, A_t, K_t) \cdot g^{(2)}(y, \hat{\tau}, T, K_{\hat{\tau}}, D_{\hat{\tau}})\; dy\, d\hat{\tau} \\ &+ \int_t^T \int_{\hat{\tau}}^T u_L\Big(e^{r(T-t)+(g-r)(\hat{\tau}-t)} \min(L_t, D_t)\Big) \cdot f^{(1)}(t, \hat{\tau}, A_t, K_t) \cdot f^{(2)}(\hat{\tau}, \tau, K_{\hat{\tau}}, D_{\hat{\tau}})\; d\tau\, d\hat{\tau} \\ &+ \int_{\ln(K_t/A_t)}^{\infty} u_L\big(L_T + \delta\big[\alpha A_t e^{y+g(T-t)} - L_T\big]^+ - \big[L_T - A_t e^{y+g(T-t)}\big]^+\big)\, g^{(1)}(y, t, T, A_t, K_t)\, dy \end{aligned}$$

$$\begin{aligned}\zeta_E(A_t, D_t, K_t, L_t, t, T) = &\int_t^T \int_{\ln(D_t/K_t)}^{\infty} \big(\big[K_{\hat{\tau}} e^{y+g(T-\hat{\tau})} - L_T\big]^+ - \delta\big[\alpha K_{\hat{\tau}} e^{y+g(T-\hat{\tau})} - L_T\big]^+\big) \\ &\cdot f^{(1)}(t, \hat{\tau}, A_t, K_t) \cdot g^{(2)}(y, \hat{\tau}, T, K_{\hat{\tau}}, D_{\hat{\tau}})\; dy\, d\hat{\tau}\end{aligned}$$

$$+\int_t^T \int_{\hat\tau}^T e^{r(T-t)+(g-r)(\hat\tau-t)} \max(D_t - L_t, 0)\, f^{(1)}(t,\hat\tau,A_t,K_t)\cdot f^{(2)}(\hat\tau,\tau,K_{\hat\tau},D_{\hat\tau})\; d\tau\, d\hat\tau$$

$$+\int_{\ln(K_t/A_t)}^{\infty} \left(\left[A_t e^{y+g(T-t)} - L_T\right]^+ - \delta\left[\alpha A_t e^{y+g(T-t)} - L_T\right]^+\right) g^{(1)}(y,t,T,A_t,K_t)\, dy\,,$$

with f and g as defined in Theorem 3.1.

PROOF: See the Appendix.

REMARK 3.3 (Implementation of Theorems 3.1 and 3.2). The expectations presented in Theorems 3.1 and 3.2 are integrals over normal densities. That is why they can easily be implemented at high precision. Computation time is within fractions of seconds.
That is why it does not make sense to further simplify the given expressions and solve the integrals analytically, although it is, for example, possible to present $\kappa_E^{(i)}(A_t, D_t, L_t, t, T)$ in Theorem 3.1 in a (lengthy) closed-form expression.

4. Numerical example

The theoretical results from Section 3 are now used to assess the effect of regulation on the optimal asset allocation. Therefore, we choose a set of reasonable parameters for our asset-liability model. The initial asset value is $A_0 = 1$ and the initial guaranteed amount is given by $L_0 = \alpha A_0 = 0.8$. The company is assumed to be in default if the assets drop below $\eta = D_0/L_0 = 106.25\,\% > 1$ of its guaranteed amount. We additionally consider the case where $\eta = D_0/L_0 = 94.44\,\% < 1$. The accrual rate of the guaranteed amount is $g = 1.75\,\%$ and the time to maturity $T = 10$. The parameters of the financial market model are $\mu = 6\,\%$, $r = 2.5\,\%$ and $\sigma = 0.2$. The participation rate δ is set such that the contracts are initially fairly priced, i.e. such that Equation (11) or (12) is valid. For the policyholder, we use power utility with relative risk aversion parameter $\gamma_1 = 3$.

4.A. **No regulatory intervention prior to default.** If the company survives maturity T and if there is no regulatory intervention prior to default, the payoff to the liability and equityholder are given in Equation (4), respectively (6). Figure 1 presents the return of liability and equity holder dependent on the asset return $A_T/A_0 - 1$ in case that the insurance company survives maturity T. The dashed line corresponds to the guaranteed return $\exp(gT) - 1$

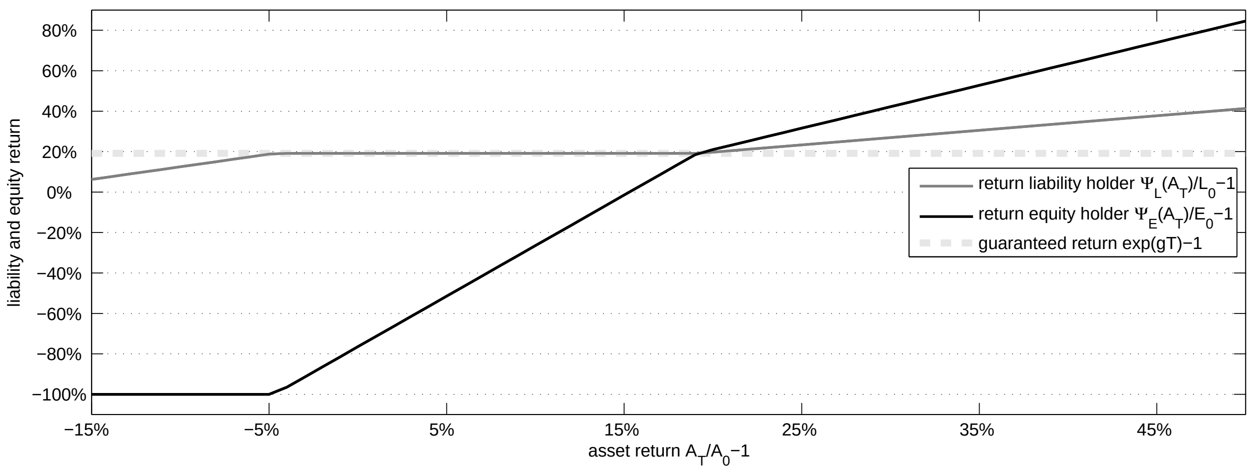

FIGURE 1. Return of liability and equity holder dependent on the asset return $A_T/A_0 - 1$ in case that the insurance company survives until maturity T, i.e. $\tau > T$. The parameters are set as $A_0 = 1$, $L_0 = \alpha A_0 = 0.8$, $D_0 = 0.85$, $\delta = 0.72$, $g = 1.75\,\%$, $T = 10$, $\mu = 6\,\%$, $r = 2.5\,\%$, $\theta_1 = 0.2155$, and $\sigma = 0.2$.

(dasehd line). If the asset return is greater than the guaranteed return, the liability holder (grey line) receives a bonus payment. If the insurance company survives the maturity, but the assets at that time are insufficient to cover the liabilities, the liability holder receives a return less than the guaranteed one. From Figure 1 one can observe that the equity holder's return (black line) is much more volatile than the liability holder's return.

To evaluate the payoffs to liability and equity holder, we look at the goal functions (8) and (9) of the equity, respectively liability holder (see Theorem 3.1). The results for different shares θ_1 of the risky asset are given in Figure 2. For the equity holder (black line), its call-option-like stake with a high participation in case the insurance company survives leads to an optimal investment decision of a $100\,\%$ stock investment, i.e. $\theta_1 = 1.0000$. The upside potential outweighs the losses due to a higher default probability. The liability holder is risk averse and judges its payoff according to a power utility function. Its expected utility (grey line) does not increase monotonically in θ_1 and displays a slightly humped shape. As the guaranteed rate is assumed to be lower than the risk-free interest rate, a $100\,\%$ investment in the risk-free asset leads to a zero default probability and a fixed payoff to equity and liability holder. This fixed payoff is given by $E_0 e^{rT}$, respectively $L_0 e^{rT}$. In case the liability holder evaluates her payoff by a power utility function with $\gamma_1 = 3$, this constant payoff does not lead to the highest utility. As – of course – high fluctuations in the payoff to the liability holder are punished by the (risk-averse) goal function, the liability holder would like to also

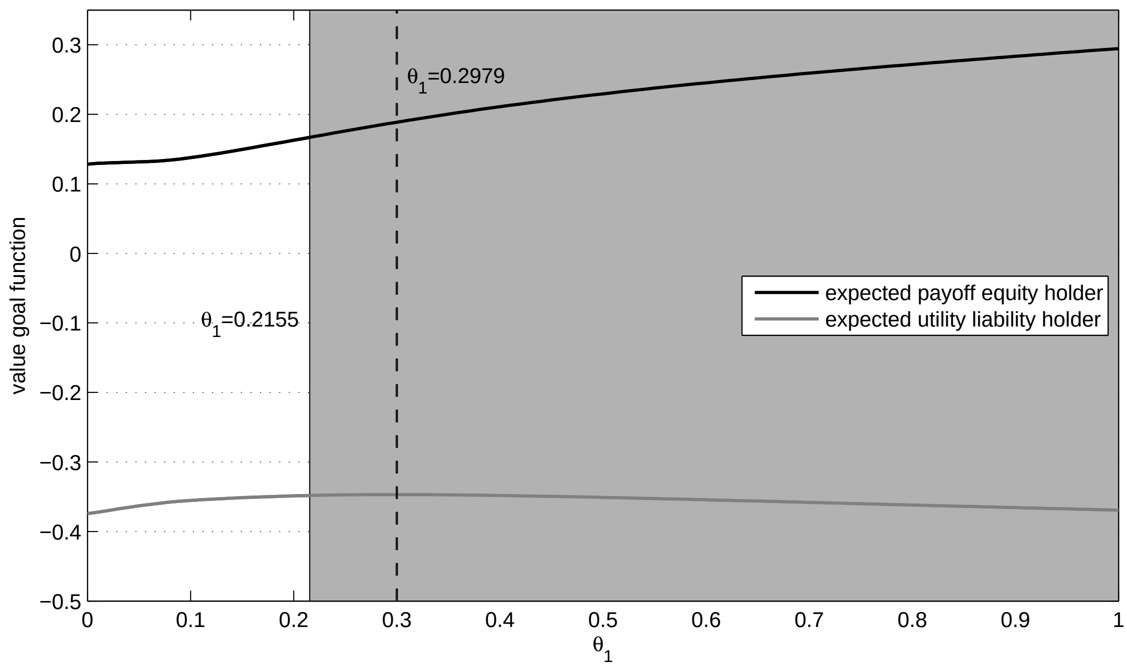

FIGURE 2. **Constant risk strategy**: Goal function of liability holder and insurance company for different shares of risky investment θ_1. The optimal investment decision of the liability holder is given by the dashed line. The equity holder would choose the optimal investment decision $\theta_1 = 1$. Investments in the grey area lead to a default probability greater than $\epsilon = 5\,\%$.

avoid high stock portions θ_1. This leads to the humped shape of the goal function in Figure 2 and to an optimal investment decision for the liability holder of a 29.79 % stock proportion, i.e. $\theta_1 = 0.2979$. To sum it up, the contract design implies quite different optimal investment decisions of equity and liability holder and consequently leads to a conflict of interests between the insurance company willing to take as much risk as possible and the liability holder wanting to limit (default) risks. A natural question arises whether this conflict can partly be solved if a Value-at-Risk-type regulatory constraint is added to the original unconstrained problem. In Europe, for example, the Solvency II accord asks the insurance company to provide equity such that the one-year default probability is less than 0.5 %. That is why, in a second step, we consider optimization (8) subject to a default constraint, i.e.

$$\max_{\theta_1} \mathbb{E}_{\mathbb{P}}\big[V_E(A_T)\big] \tag{18}$$

$$s.t.\ \mathbb{P}(\tau \leq T) \leq \epsilon.$$

On a horizon of $T = 10$ years, we set $\epsilon = 5\,\%$[6] and obtain as a result of the restricted optimization problem (18) a share of risky investment of 21.55 % ($\theta_1^* = 0.2155$), consequently an expected utility of the liability holder of $\mathbb{E}_{\mathbb{P}}\big[u_L\big(V_L(A_T)\big)\big] = -0.3482$, an expected payoff to the equity holder of $\mathbb{E}_{\mathbb{P}}\big[V_E(A_T)\big] = 0.1669$ and a fair participation rate $\delta = 0.79$. We immediately observe from Figure 2 that the regulatory restriction does not lead to an effective risk sharing between the policyholder and the insurance company. Both the policyholder and the insurance company are in this example willing to take more risks than a 21.55 % stock proportion ($\theta_1^* = 0.2155$). A risky share of 29.79 % ($\theta_1 = 0.2979$) would increase the goal function of both parties to $\mathbb{E}_{\mathbb{P}}\big[u_L\big(V_L(A_T)\big)\big] = -0.3469$ and $\mathbb{E}_{\mathbb{P}}\big[V_E(A_T)\big] = 0.1882$.

The above analysis shows that a Value-at-Risk-type regulation alone does not resolve the conflict in a satisfying way. It poses the question, whether the flexible design of the regulatory intervention through the second barrier K improves the situation and can be beneficial to both the policy- and the shareholder, while still keeping the default probability below ϵ.

4.B. **Regulatory intervention prior to default.** We exemplarily analyze this setup using the parameters from above and a regulatory barrier with $K_0 = 0.9$. We distinguish the following three cases:

- **Strategy A (risk reduction in distress)**: As soon as the regulatory barrier is hit, the regulator forces the insurance company to lessen its investments in the risky investment by 50 %, i.e. $\theta_2 = \theta_1/2 < \theta_1$. In this case, the regulator expects that the default probability of the company diminishes, when it takes on a less risky asset.
- **Strategy B (constant risk strategy)**: The regulator does not intervene if the regulatory barrier is hit, see Section 4.A.
- **Strategy C (risk increase in distress)**: As soon as the regulatory barrier is hit, the regulator allows the insurance company to extend its investments in the risky investment by 50 %, i.e. $\theta_2 = 3\theta_1/2 > \theta_1$. Note that the maximum share in the risky investment is assumed to be 100 %, i.e. practically $\theta_2 = \min\{1, 3\theta_1/2\}$. In this case, the regulator expects that the higher default risk will be well compensated from riskier

[6]Solvency II regulation requires that the insurance company holds enough captial to ensure that one-year default probability is below 0.5 %. On a 10-year horizon, this corresponds to a default probability of $1 - (0.995)^10 \approx 4.89\,\%$.

investments.[7]

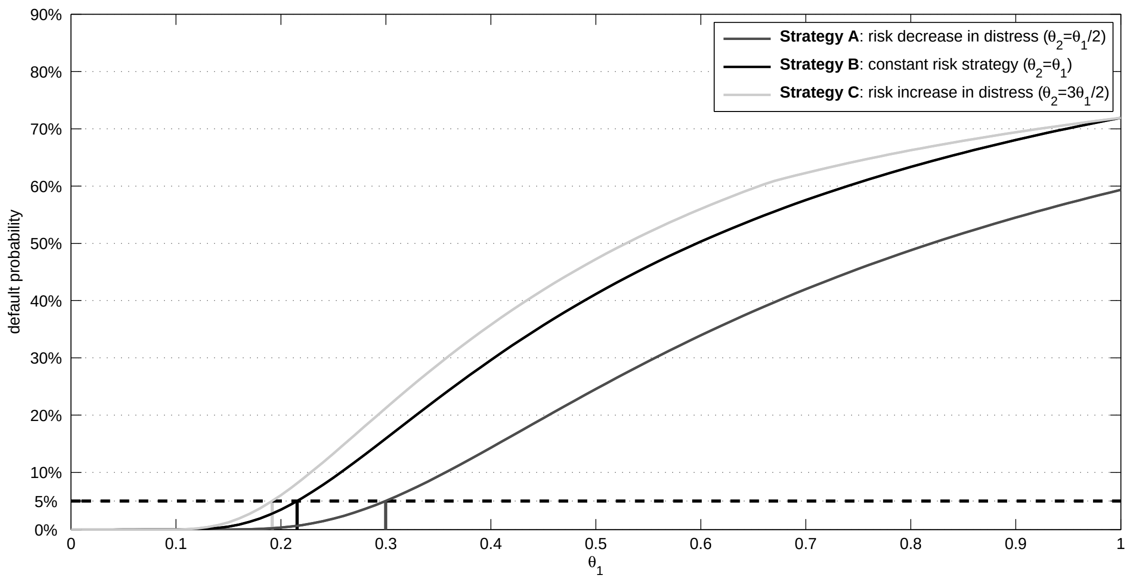

FIGURE 3. Default probabilities $\mathbb{P}(\tau \leq T)$ in the three strategies for different initial shares in risky investments θ_1. The dashed line gives a default probability constraint of $\epsilon = 5\,\%$. The intercept points give the critical portions in the risky asset θ_1 for the three strategies, i.e. $\theta_1 = 0.2998$ (Strategy A), $\theta_1 = 0.2155$ (Strategy B), and $\theta_1 = 0.1917$ (Strategy C), see als Table I.

Figure 3 presents the default probabilities for the different strategies contingent on the initial share θ_1 in the risky investment. Naturally the default probability increases in θ_1.[8] As the initial asset share θ_1 is the same for all strategies, a risk increase (respectively risk

[7]Empirically, all three considered cases are observed: Mohan and Zhang [2014] state that US public funds, unlike private funds, increase risk if they are underfunded. Rauh [2009] detects that the asset allocation of life insurance companies is less risky if the company's financial condition is weaker.

[8]Intuitively, a higher equity holding (leading to a higher volatility) results in a higher probability of hitting the default barrier D_t. If there is no regulatory barrier K_t, this can easily be proved analytically, i.e. from Equation (16) we obtain

$$\frac{\partial\,\mathbb{P}(\tau \leq T)}{\partial \theta_1} = \frac{-2\ln(D_0/A_0)}{\sigma\theta_1^2\sqrt{T}}\,\varphi\left(\frac{\ln(D_0/A_0) - \tilde{\mu}_1 T}{\sigma\theta_1\sqrt{T}}\right) - \ln(D_0/A_0)\left(\frac{4(r-g)T}{\sigma^2\theta_1^3} + \frac{\mu - r}{\sigma^2\theta_1^2}\right)\left(\frac{D_0}{A_0}\right)^{\frac{2\tilde{\mu}_1}{\sigma^2\theta_1^2}}\,\Phi\left(\frac{\ln(D_0/A_0) + \tilde{\mu}_1 T}{\sigma\theta_1\sqrt{T}}\right) > 0\,.$$

since $D_0 < A_0$ and $g \leq r < \mu$

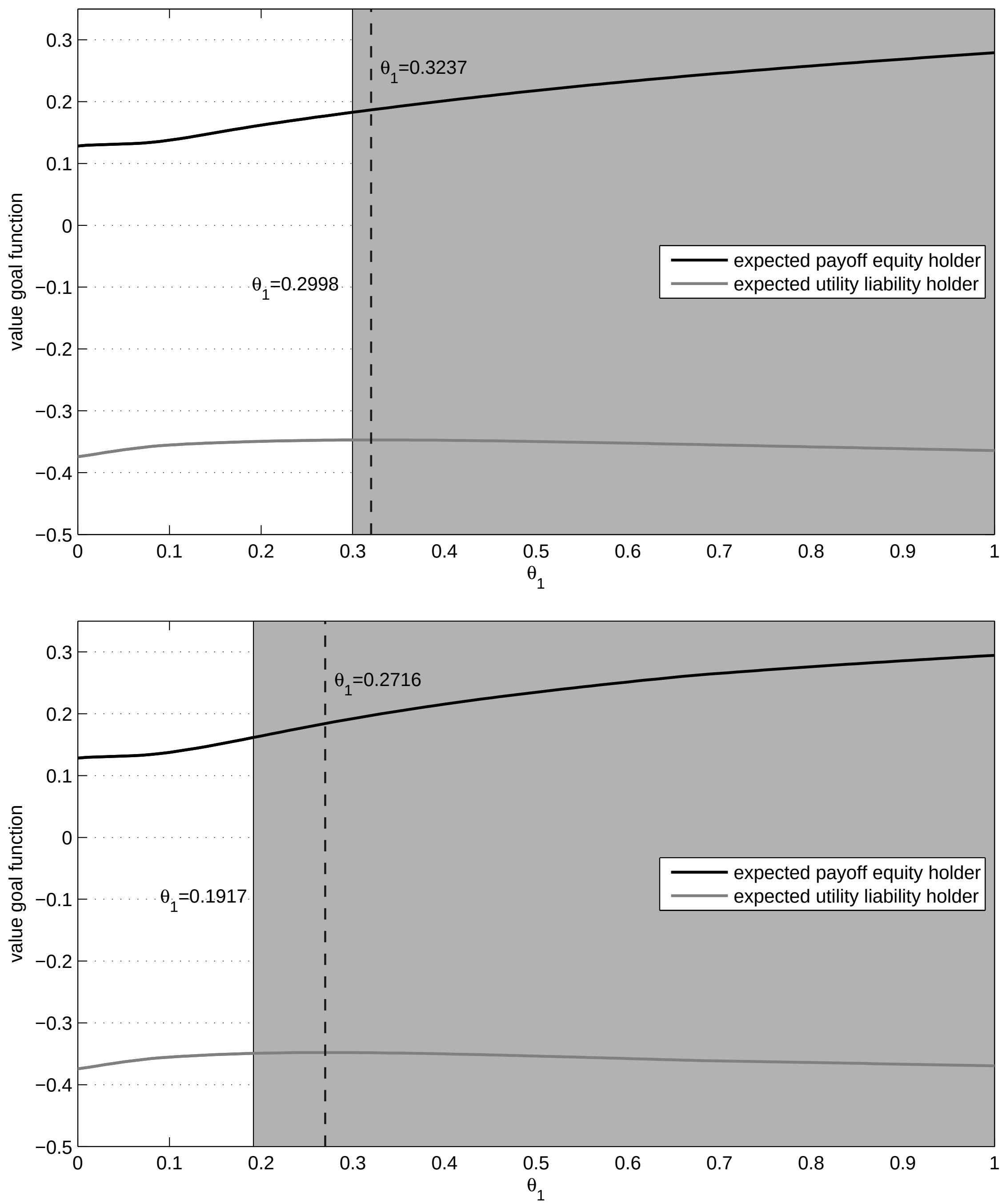

Figure 4. **Risk reduction in distress** (top, Strategy A) and **risk increase in distress** (bottom, Strategy C): Goal function of liability holder and the equity holder for different initial shares of risky investment θ_1. The optimal investment decision of the liability holder is given by the dashed line. The equity holder would choose $\theta_1 = 1$. Investments in the grey area lead to a default probability greater than $\epsilon = 5\,\%$.

decrease) in distress leads to a higher (respectively lower) default probability. Due to the fact that θ_2 is capped by 100 %, the default probability for $\theta_1 = 1$ is the same for Strategies A and B.

Strategy ($L_0 = 0.9 > D_0$)	θ_1	θ_2	$\mathbb{E}_{\mathbb{P}}\big[u_L\big(V_L(A_T)\big)\big]$ (CE)	$\mathbb{E}_{\mathbb{P}}\big[V_E(A_T)\big]$	$\mathbb{P}(\tau \leq T)$	δ
A (**risk reduction**)	0.2998	$\theta_1/2$	−0.3472 (1.2000)	0.1828	5.00 %	0.73
B (**no intervention**)	0.2155	θ_1	−0.3482 (1.1983)	0.1669	5.00 %	0.79
C (**risk increase**)	0.1917	$3\theta_1/2$	−0.3493 (1.1964)	0.1616	5.00 %	0.81
Strategy ($L_0 = 0.8 < D_0$)						
A (**risk reduction**)	0.2998	$\theta_1/2$	−0.4438 (1.0614)	0.3297	5.00 %	0.66
B (**no intervention**)	0.2155	θ_1	−0.4452 (1.0598)	0.3109	5.00 %	0.72
C (**risk increase**)	0.1917	$3\theta_1/2$	−0.4465 (1.0582)	0.3046	5.00 %	0.74

TABLE I. Goal function of liability holder (second column) and the equity holder (third column) for $\epsilon = 5\,\%$. The two cases $L_0 > D_0$ (top table) and $L_0 < D_0$ (bottom table) are considered. The utility of the liability holder is transformed into its certainty equivalent CE $= \big((1-\gamma_1)\,\mathbb{E}_{\mathbb{P}}\big[u_L\big(V_L(A_T)\big)\big]\big)^{1/(1-\gamma_1)}$. Furthermore, the default probability in $[0, T]$ and the fair participation rate δ are displayed.

The most interesting results of the paper are displayed in Table I. Table I distinguishes between the case where the liabilities are initially worth more (top table), respectively less (bottom table), than the default barrier. Qualitiatively, the following results are the same in both considered cases. Under the same default probability constraint ($\epsilon = 5\,\%$), we exhibit the optimal initial strategy θ_1's and the corresponding expected value/utility of the equity and liability holder for Strategy A, B and C. First, the same target default probability $\epsilon = 5\,\%$ has the consequence that the resulting optimal θ_1 is the highest under Strategy A, followed by Strategy B and C. In other words, the regulator can allow for a higher initial share θ_1 of risky assets while still keeping the initial restriction $\mathbb{P}(\tau \leq T) \leq \epsilon$. With a risky share of 29.98 % ($\theta_1 = 0.2998$), the default probability is still 5 % under Strategy A. Second, when we compare the resulting benefits for the equity and liability holder under the optimal stock holdings, we observe that both the equity and liability holder are better off under Strategy A than Strategy B. In other words, moving from a constant strategy to a risk-reducing strategy, we have achieved an improvement for both equity and liability holder. This result is very useful since this means that our new **regulatory design is more beneficial – for both the the equity holder and the policyholder**. Carefully choosing a regulatory policy might thus be beneficial for both parties and increase total benefits. Third, moving from Strategy B to C does not bring added value to the equity or liability holder. The resulting default constrained optimum ($\theta_1 = 0.1917$) under Strategy C is – for both equity and liability holder – even worse than Strategy B that does not change the equity share in distress.

Strategy ($L_0 = 0.9 > D_0$)	θ_1	θ_2	$\mathbb{E}_{\mathbb{P}}\big[u_L\big(V_L(A_T)\big)\big]$ (CE)	$\mathbb{E}_{\mathbb{P}}\big[V_E(A_T)\big]$	$\mathbb{P}(\tau \leq T)$	δ
A (**risk reduction**)	0.2998	$\theta_1/2$	-0.3472 (1.2000)	0.1828	5.00 %	0.68
B* (**no intervention**)	0.2979	θ_1	-0.3469 (1.2006)	0.1882	15.92 %	0.70

Table II. Goal function of liability holder (second column) and equity holder (third column) comparing risk reduction in distress to the case where the utility of the liability holder is maximized without default constraint. The utility of the liability holder is transformed into its certainty equivalent CE $= \big((1-\gamma_1)\,\mathbb{E}_{\mathbb{P}}\big[u_L\big(V_L(A_T)\big)\big]\big)^{1/(1-\gamma_1)}$. Furthermore, the default probability in $[0,T]$ and the fair participation rate δ are displayed.

Table II further stresses the advantages of the flexible regulatory system: The optimal investment strategy from the viewpoint of the liability holder ($\theta_1 = 0.2998$, see Figure 4, top) leads to a rather high default probability of 15.92 % (Strategy B*). The flexible Strategy A with a risk reduction to $\theta_2 = \theta_1/2$, where $\theta_1 = 0.2998$, leads to a signficantly lower default probability of 5 % changing the benefits for liability and equity holder only marginally.

5. Conclusion

The present paper discusses flexible regulatory supervision to partly solves the conflict of interests that arises by the option-like stakes of the insurance company and to improve benefits of both the policy- and the shareholder of a participating life insurance company. A Value-at-Risk-type constraint (default probability constraint) does not provide an optimal risk sharing between the two parties. Hence, it does not resolve the conflicts of interests. We show that an earlier intervention of the regulator and a more flexible regulatory framework forcing the insurance company to decrease risk in distress might mitigate the problem and improve the benefit of both the policy- and the shareholder.

In further reserach, the regulator's line of action can be further refined, for example by allowing for more regulatory barriers or the possibility to return to the original investment strategy after a system recovery.

6. Appendix

Proof of Theorem 3.1. First, we recall results on the first-hitting time τ of a geometric Brownian motion, i.e. the process A as defined in (2). The law of τ is known to be inverse Gaussian (see, e.g., Folks and Chhikara [1978]). Lemma 6.1 recalls some results on the first-hitting time in this special case.

Lemma 6.1 (First-hitting time distribution). *Consider the process A from (2). Then, the survival probability within the interval* $(t,T]$ *is given by*

$$\mathbb{P}\big(\tau > T \,|\, \tau > t\big) = \Phi\left(\frac{\tilde{\mu}_1(T-t) - \ln(D_t/A_t)}{\sigma\theta_1\sqrt{T-t}}\right) - \left(\frac{D_t}{A_t}\right)^{\frac{2\tilde{\mu}_1}{\sigma^2\theta_1^2}} \Phi\left(\frac{\tilde{\mu}_1(T-t) + \ln(D_t/A_t)}{\sigma\theta_1\sqrt{T-t}}\right),$$

where $D_t < A_t$, $\tilde{\mu}_1 := r + \theta_1(\mu - r) - g - \sigma^2\theta_1^2/2$, *and* $\Phi(\,\cdot\,)$ *denotes the standard normal cumulative distribution function. The density of* τ *can be obtained from*

$$f^{(i)}(t,\tau,A_t,D_t) := \frac{-\ln(D_t/A_t)}{\sigma\theta_i(\tau-t)^{\frac{3}{2}}}\,\varphi\left(\frac{\ln(D_t/A_t) - \tilde{\mu}_i(\tau-t)}{\sigma\theta_i\sqrt{\tau-t}}\right). \tag{19}$$

For $y := \ln(e^{-gT}\,A_T/A_t)$, *we define*

$$g^{(1)}(y,t,T,A_t,D_t) := \mathbb{P}\big(y \in dy,\, \tau > T\big)\,, \tag{20}$$

which is known to be

$$g^{(1)}(y,t,T,A_t,D_t) = \begin{cases} 0 & \text{for } y \le \ln(D_t/A_t) \\ \dfrac{\varphi\left(\frac{y-\tilde{\mu}_i(T-t)}{\sigma\theta_1\sqrt{T-t}}\right)}{\sigma\theta_1\sqrt{T-t}}\left(1 - e^{-2\,\frac{\ln(D_t/A_t)^2 - y\ln(D_t/A_t)}{\sigma^2\theta_1^2(T-t)}}\right) & \text{else} \end{cases} \tag{21}$$

where $\varphi(\,\cdot\,)$ *denotes the density of the standard normal distribution.*

Proof: See, e.g., Folks and Chhikara [1978], He et al. [1998], and Shreve [2004].

Note that the same results hold, if we replace the asset strategy θ_1 by θ_2 (and similarly $\tilde{\mu}_1$ by $\tilde{\mu}_2 := r + \theta_2(\mu - r) - g - \sigma^2\theta_2^2/2$). We denote the densities that result from this parameter change by $f^{(2)}(t,\tau,A_\tau,D_\tau)$, respectively $g^{(2)}(y,t,T,A_t,D_t)$.

We are now using Lemma 6.1 to prove Theorem 3.1. Note first that if the barrier D is not hit in the interval $(t, T]$, (21) helps us to obtain the distribution of the assets A at maturity T. To compute the expected utility of the terminal payoff $\Psi_L(A_T)$ from (4), one simply has to integrate its utility over (21) on the set $(\ln(e^{-g(T-t)}\, D_T/A_t), \infty) = (\ln(D_t/A_t), \infty)$. If the barrier is hit, i.e. $\tau \leq T$, the terminal payoff depends solely on the default time τ whose distribution can be obtained from (19). This then leads to

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}}\big[u_L(V_L)\big] &= \mathbb{E}_{\mathbb{P}}\big[u_L\big(\mathbb{1}_{\{\tau>T\}}\,\Psi_L(A_T) + \mathbb{1}_{\{\tau\leq T\}}\,e^{r(T-\tau)}\,\min(L_\tau, D_\tau)\big)\big] \\
&= \mathbb{E}_{\mathbb{P}}\big[\mathbb{1}_{\{\tau>T\}}\,u_L\big(\Psi_L(A_T)\big)\big] + \mathbb{E}_{\mathbb{P}}\big[\mathbb{1}_{\{\tau\leq T\}}\,u_L\big(e^{r(T-\tau)}\,\min(L_\tau, D_\tau)\big)\big] \\
&= \int_{\ln(D_t/A_t)}^{\infty} u_L\big(L_T + \delta\big[\alpha A_t e^{y+g(T-t)} - L_T\big]^+ - \big[L_T - A_t e^{y+g(T-t)}\big]^+\big)\, g^{(1)}(y, t, T, A_t, D_t)\, dy \\
&\qquad + \int_t^T u_L\Big(e^{r(T-t)+(g-r)(\tau-t)}\,\min(L_t, D_t)\Big)\, f^{(1)}(t, \tau, A_t, D_t)\, d\tau\,. \qquad (22)
\end{aligned}$$

In the case of power utility (see Example 2.2), the latter integrals can be further simplified.

Proof of Theorem 3.2. Theorem 3.2 can rather straightforwardly be derived using the previous results. Note that the regulatory barrier K is always hit prior to default due to the continuity of the process A. Up to time $\hat{\tau}$ the process A is a geometric Brownian motion with strategy θ_1 allowing us to use the density $f^{(1)}(t, \hat{\tau}, A_t, K_t)$ from Lemma 6.1 for $\hat{\tau}$. At time $\hat{\tau}$, we are back in the situation that is already solved in Theorem 3.1: One has to adapt the initial values for A, D, and L. Furthermore, the time to maturity is now $T - \hat{\tau}$ instead of T and the investment strategy is now θ_2. If the regulatory threshold K is never hit, we can in analogy to the proof of Theorem 3.1 compute the expected utility of the terminal payoffs to get the first terms of $\zeta_L(A_t, D_t, K_t, L_t, T)$, respectively $\zeta_E(A_t, D_t, K_t, L_t, T)$:

$$\begin{aligned}
\zeta_L(A_t, D_t, K_t, L_t, t, T) &= \int_t^T \kappa_L^{(2)}(K_{\hat{\tau}}, D_{\hat{\tau}}, L_{\hat{\tau}}, \hat{\tau}, T)\cdot f^{(1)}(t, \hat{\tau}, A_t, K_t)\, d\hat{\tau} \\
&\quad + \int_{\ln(K_t/A_t)}^{\infty} u_L\big(L_T + \delta\big[\alpha A_t e^{y+g(T-t)} - L_T\big]^+ - \big[L_T - A_t e^{y+g(T-t)}\big]^+\big)\, g^{(1)}(y, t, T, A_t, K_t)\, dy, \\
&= \int_t^T \int_{\ln(D_t/K_t)}^{\infty} u_L\big(L_T + \delta\big[\alpha K_{\hat{\tau}} e^{y+g(T-\hat{\tau})} - L_T\big]^+ - \big[L_T - K_{\hat{\tau}} e^{y+g(T-\hat{\tau})}\big]^+\big)
\end{aligned}$$

$$\cdot\, f^{(1)}(t,\hat\tau,A_0,K_0)\cdot g^{(2)}(y,\hat\tau,T,K_{\hat\tau},D_{\hat\tau})\;dy\,d\hat\tau$$

$$+\int_t^T\int_{\hat\tau}^T u_L\Big(e^{r(T-t)+(g-r)(\hat\tau-t)}\min(L_t,D_t)\Big)\cdot f^{(1)}(t,\hat\tau,A_t,K_t)\cdot f^{(2)}(\hat\tau,\tau,K_{\hat\tau},D_{\hat\tau})\;d\tau\,d\hat\tau$$

$$+\int_{\ln(K_t/A_t)}^{\infty} u_L\big(L_T+\delta\big[\alpha A_t e^{y+g(T-t)}-L_T\big]^+-\big[L_T-A_t e^{y+g(T-t)}\big]^+\big)\,g^{(1)}(y,t,T,A_t,K_t)\,dy$$

$$\zeta_E(A_t,D_t,K_t,L_t,t,T)=\int_t^T \kappa_E^{(2)}(K_{\hat\tau},D_{\hat\tau},L_{\hat\tau},\hat\tau,T)\cdot f^{(1)}(t,\hat\tau,A_t,K_t)\;d\hat\tau$$

$$+\int_{\ln(K_t/A_t)}^{\infty}\big(\big[A_t e^{y+g(T-t)}-L_T\big]^+-\delta\big[\alpha A_t e^{y+g(T-t)}-L_T\big]^+\big)\,g^{(1)}(y,t,T,A_t,K_t)\,dy$$

$$=\int_t^T\int_{\ln(D_t/K_t)}^{\infty}(\big[K_{\hat\tau}e^{y+g(T-\hat\tau)}-L_T\big]^+-\delta\big[\alpha K_{\hat\tau}e^{y+g(T-\hat\tau)}-L_T\big]^+)$$

$$\cdot\, f^{(1)}(t,\hat\tau,A_t,K_t)\cdot g^{(2)}(y,\hat\tau,T,K_{\hat\tau},D_{\hat\tau})\;dy\,d\hat\tau$$

$$+\int_t^T\int_{\hat\tau}^T e^{r(T-t)+(g-r)(\hat\tau-t)}\max(D_t-L_t,0)$$

$$\cdot\, f^{(1)}(t,\hat\tau,A_t,K_t)\cdot f^{(2)}(\hat\tau,\tau,K_{\hat\tau},D_{\hat\tau})\;d\tau\,d\hat\tau$$

$$+\int_{\ln(K_t/A_t)}^{\infty}\big(\big[A_t e^{y+g(T-t)}-L_T\big]^+-\delta\big[\alpha A_t e^{y+g(T-t)}-L_T\big]^+\big)\,g^{(1)}(y,t,T,A_t,K_t)\,dy\,,$$

with $\kappa_L^{(i)}(\,\cdot\,)$, $\kappa_E^{(i)}(\,\cdot\,)$, f, and g as defined in Theorem 3.1. Again, power utility simplifies the given expressions.

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