Optimal Asset Allocation for Defined Benefit Plans under a Heavy-Tailed-Coupled Portfolio

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Abstract

Optimal Asset Allocation on pension funds has been widely studied during the last years; variables as mortality improvements, financial crisis, salary risks and unexpected inflation affect directly pension funds. Nowadays, plan sponsors are seeking alternative investment strategies and uniform costs for their pension plans under the ever changing conditions.

In the actual literature, most of optimal asset allocation strategies are based on stochastic differential equations, which relays on normality distribution of log returns, but in practice this assumption is replaced by heavy-tailed distributions and non-linear relationships. This paper addresses such issue improving heavy-tailed distributions and an extreme-value copula for the correlation structure to obtain the optimal strategy.

The aim of this paper is to compare under a heavy-tailed-coupled portfolio, in a traditional Defined Benefit Plan, several investment strategies depending on the funding level in terms of probabilities of financial ruin. The paper, estimates the probability via stochastic simulations for a closed group of employees until extinction and where the contributions to the fund are fixed at time zero. The simulations show that for an specified funding level, the probability of financial ruin is not significantly greater than the optimal strategy in a range of 10% of the proportion invested in stocks. The heavy tails of the investment returns affect strongly investment strategies near the optimal, making them almost equal in terms of probabilities. In addition, scenarios where the expected return in bonds are less than assumed in valuations, impact drastically the probabilities, and for reaching an optimal strategy, is needed assuming more risk, but with significant greater probabilities.

Keywords: Stochastic simulation, copulas, extreme value , probability of financial ruin, optimal asset allocation, heavy-tailed-coupled portfolio, pensions.

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1 Introduction

A global trend exists nowadays to shift defined benefit (hereinafter called DB) plans to defined contribution (hereinafter called DC) plans; In DB plans, the benefits are fixed in advance by the sponsor, whom assumes the financial risk of all participants; in DC plans the contributions are fixed from the beginning, and benefits depends of the returns of the assets on the fund, where the financial risk is now assumed by the participant. In this shifting process, most of DB plans still have a transition generation until extinction.

To fund those pension plans, an asset allocation strategy is selected which consist in investing commonly a proportion of the fund in bonds and the rest in stocks, such proportion can be either constant or variable through time. The solutions in finding the optimal asset allocation depend on the objective function. In the case of a DC pension fund, the objective of the shareholder is to maximize the expected utility obtained from fund accumulation at a fixed date, and in DB plans the objective should be related with risk minimization instead of the maximization of fund assets. Thus, due to the differences in both types of pension plans, the results from DC to DB plans shall not be transferred. In this sense, the main concern of the sponsor is the solvency risk, related to the security of the pension fund in attaining the comprised liabilities.

In the literature of optimal asset allocation, the log-normality assumption for the assets returns has been a constant factor. Since Merton (1969) who found a constant proportion solution, the usage of log-normality assumptions is a widely improved in continuous and discrete time. In the DC context, papers as Bouler *et al.* (2001); Deelstra *et al.* (2003), Battocchio & Menoncin (2004), Cairns *et al.*(2006), Zhang *et al.* (2013), and more recently, Chen and Delong (2015) improved stochastic differential equations to find an optimal solution without considering the random nature of the remaining time. The concept of *probability of lifetime ruin* as risk metric was introduced by Milevsky and Robinson (2000) an then studied by Huang *et al.* (2004), Young (2004), Bayraktar, Moore and Young (2008), Wang and Young (2012). In the case of DB plans see Boulier *et al.* (1995); Sundaresan and Zapatero (1997) and Josa-Fombellida and Rincón-Zapatero (2010) found analytic solutions that include a Merton type solution and proportions that are in function of the funding level or the actuarial liability. Similarly, an optimal management focused on optimal funding strategies in discrete time in Haberman and Sung (1994), Chang (1999) and Chang *et al.* (2003).

In practice, log-returns do not follow normality assumptions, instead of that assumptions follow heavy tailed distributions. Works that dealt with heavy tailed distributions in portfolios are Meerschaert and Scheffler, (2003), Ibragimov (2004) Rachev*et al.* (2004), Ortobelli *et al.* (2010), Agatonovic, M. (2010) and Qiu *et al.* (2014). In general, results that compared the joint normal distribution with the heavy tailed distributions are substantially different.

Copula functions in the financial framework have been used to capture and model non-linear relationships between the assets returns. The concept of copula was introduced in 1959 by Abe Sklar in the context of probabilistic metric spaces. Nevertheless, applications in financial and actuarial fields are revealed only in the end of the 90's where Frees and Valdez (1998) introduced the concept of copula to actuaries with insurance data and Embrechts for what concerns financial applications (Embrechts et al., 2002, 2003) and more recently in the actuarial field Hürlimann (2014) and Yang *et al.* (2015). In pensions, Miccoci and Masala (2003) with an application for pricing benefits of a defined contribution pension plan and with a similar approach in Melo (2007). With respect the problem of asset allocation using copulas in the financial framework are Hennessyand Lapan (2002), Patton (2004), Hatherley and Alcock (2007), Alcock and Hatherley (2009) and Zhu and Zeng (2014).

In our model, in order to obtain the optimal asset allocation depending on the funding level, we use copula functions with heavy tailed distributions into the calculation of the probability of ruin of a DB pension plan, considering the lifetime for an entire closed group and where the contributions to the fund are fixed at time zero. In the model, there are five sources of uncertainty, i) the risky asset returns, ii) the bond returns, iii) the evolution of pensions linked to inflation and also based on iv) the salary growth, and v) the demography of a population until extinction.

Our aim is to analyze in a realistic framework a fixed type investment strategy that minimize the probability of financial ruin in a classic DB plan, modeling the actual tendency of many DB plan sponsors that will maintain DB benefits for a closed group until extinction.

The paper is organized as follows: Section 2 describes the stochastic dynamics of the pension fund and all the variables involved into the model. In section 3, a copula model is fitted with statistical data. In section 4, we present the final model selected for the simulations and present the estimated probabilities of financial ruin. We also present a sensitivity analysis, where the expected return in bonds are less than expected in valuations and also, a sensitivity analysis where there is an underperformance in stocks. Finally, section 5 is dedicated to the conclusions and in the appendix is described, in more detail, the copula functions and some type of families with their explicit formulas.

2 The Stochastic Model

In this section, we introduce the market structure and define the stochastic dynamics of stock, bonds, salaries, inflation and the number of participants for an aggregated pension plan of the DB type for a closed group until extinction.

2.1 The financial, economic and demographic model

We consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Where $\mathbb{P}$ is a probability measure on Ω, and $\mathcal{F} = \{\mathcal{F}_n\}_{n\geq 0}$ is complete and discrete filtration generated by five-dimensional stochastic processes; $\mathcal{F}_n = \sigma\{(sp_j, b_j, s_j, \pi_j, N_i) : 0 \leq j \leq n\}$ denotes the information structure generated by the log-returns of the stock prices and bond indexes; the log-growth of salaries and inflation indexes, and a closed group of participants of the plan until time n. The total number of participants at time n can be segregated as $N_n = A_n + R_n$ by active employees and actual retires respectively. This closed group follows the mortality table $\{q_x\}$ and is assumed independent from the other variables.

The financial and economic sequence $\{(sp_n, b_n, s_n, \pi_n)\}_{n\geq 0}$ are independent and identically distributed random vectors, where each of the marginal distributions F_1, F_2, F_3, F_4 are heavy tailed type respectively and their dependence is modeled by a copula function (described in the appendix) as follows:

$$F\left(s_n, sp_n, b_n, \pi_n\right) = C\left(F_1(s_n), F_2(sp_n), F_3(b_n), F_4(\pi_n)\right) \tag{2.1}$$

In our paper, we assume that the heavy-tailed marginal distributions are Student's t type with location $\mu_1, \mu_2, \mu_3, \mu_4$; scale parameter $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ and degrees of freedom df_1, df_2, df_3, df_4 respectively. The use of these distributions is due to that they have been widely applied in theory and practice in the financial framework, remarking that in the previous formula any type of distributions can be used.

2.2 The stochastic salaries and pensions

Let S_n the total payroll at period n of the actives members A_n. The total payroll is the sum of all the salaries of each participant j.

$$S_n = \sum_{j=1}^{A_n} S_n^j \tag{2.2}$$

In this equation, in each period the salaries increase depending on the random variable s_n, thus, the equation can be rewritten as follows

$$S_n = \sum_{j=1}^{A_n} S_0^j \, e^{\sum_{i=1}^{n} s_i} \tag{2.3}$$

Analogously, let P_n the total pensions paid at period n as the sum of each pension of the retired j.

$$P_n = \sum_{j=1}^{R_n} P_n^j \tag{2.4}$$

As similar as salaries, in each period the pensions increase depending on the random variable π_n, that means that pensions are indexed to inflation and the formula can be rewritten as follows

$$P_n = \sum_{j=1}^{R_n} P_0^j \, e^{\sum_{i=r_j}^{n} \pi_i} \tag{2.5}$$

Where r_j is the period where participant j retires. Moreover, it is important to remark that the pension in most cases is in function of the final salary. Thus, the pension of an active member after retirement age can be expressed by the next formula

$$P_0^j = P(S_0^j e^{\sum_{i=1}^{r_j-1} s_i}) \, e^{\sum_{i=r_j}^{n} \pi_i} \tag{2.6}$$

Where the function $P(.)$ is a general benefit formula. Without loss of generality, we assume that the benefit function $P(.)$ brings in average a fixed constant replacement rate Δ_r.

2.3 Valuation

For the valuation of the liabilities of the DB plan, a the fixed retirement age r and $x_1, x_2..., x_{N_0}$ participants ages at time 0 following the mortality table $\{q_x\}$ are considered.

In practice, the actuarial present value of future benefits at time zero $APVFB_0$ can be computed as follows:

$$APVFB_0 = \sum_{j=1}^{A_0} \Delta_r \, S_0^j e^{\alpha(r-x_j)} \, {}_{r-x_j}p_{x_j}^{(\tau)} e^{\delta(r-x_j)} \, \ddot{a}_r + \sum_{j=1}^{R_0} P_0^j \, \ddot{a}_{x_j} \tag{2.7}$$

Where the parameter δ is the force of interest equivalent to the yield curve. In other words, both valuations using the force of interest δ and the yield curve are equal. The

force of interest δ can be interpreted by the expected return that the pension fund might achieve investing all the assets in bonds, i.e. $\delta = E[b_n]$. Similarly, the parameter α is the expected increase rate of salaries, i.e. $\alpha = E[s_n]$ which is used to project the salaries to retirement age; the real force of interest δ_r used in the annuity $\ddot{a}_r$ is the expected real return of the bonds, i.e. $\delta_r = E[b_n - \pi_n]$.

To finance liabilities of equation (2.7) a financial method is used. A common and practical one is a constant proportion type, which at each period the contribution to the fund is a constant percentage k of the total salary mass and fixed since time 0 and also depends on the initial fund F_0.

$$k = max\left(\frac{APVFB_0 - F_0}{APVFS_0}, 0\right) \tag{2.8}$$

Where $APVFS_0$ is the actuarial present value of future salaries computed as

$$APVFS_0 = \sum_{j=1}^{A_0} S_0^j \, \ddot{a}_{x_j:\overline{r-x_j}|} \tag{2.9}$$

For this case the force of interest δ_s used in the annuity $\ddot{a}_{x_j:\overline{r-x_j}|}$ takes into account the increasing salaries, i.e. $\delta_s = E[b_n - s_n]$. It is important to remark that we will maintain constant k through time, in order to model the cases where governments and corporates do not make adjustments in contributions.

2.4 Dynamic of the fund

Without loss of generality, the fund can only invest in bonds and a risky asset in positive proportions. The Fund is invested in x_n proportion at time n in the risky asset and the rest in bonds; at the same time the fund receives contributions as a percentage of the total payroll and it is consumed by the pensions paid at time n.

$$F_{n+1} = F_n\left(x_n e^{sp_n} + (1 - x_n)e^{b_n}\right) + kS_n - P_n \tag{2.10}$$

By substituting (2.3) and (2.6) in (2.10) we obtain:

$$F_{n+1} = F_n\left(x_n e^{sp_n} + (1 - x_n)e^{b_n}\right) + k\sum_{j=1}^{A_n} S_0^j \, e^{\sum_{i=1}^{n} s_i} - \sum_{j=1}^{R_n} \Delta_r S_0^j e^{\sum_{i=1}^{r_j} s_i + \sum_{i=r_j}^{n} \pi_i} \tag{2.11}$$

The equation (2.11) considers all the random variables described in the model, and it is important to note that in the dynamic of the fund have convolutions of Student's t random variables that have unknown closed forms, just for a few cases. Thus, the probability distribution of the fund at period n might be too complicated to found a closed form, even if an independent copula is considered.

2.5 Probability of Ruin

Let $T = max(T_1, T_2, ..., T_N)$ the maximum of the remaining life of the N participants. The random variable T represents the time until extinction f the closed group. We define the probability of financial ruin with funding level $u = F_0/APVFB_0$ and strategy $\{x_n\}$ as:

$$\psi(u, x_n) = \mathbb{P}\left[\inf_{0<n\leq T}\{F_n\} \leq 0 \middle| F_0 = u,\ x_n\right] \tag{2.12}$$

So, the optimal asset allocation $\{x_n^*\}$ minimizes the probability over all possible paths $\{x_n\} \in [0, 1]$.

$$\psi(u, x_n^*) = \inf_x \{\psi(u, x_n)\} \tag{2.13}$$

Noting that the definition of probability of financial ruin in equation (2.12) is similar to the concept of the *probability of lifetime ruin* used in the defined contribution framework introduced by Milevsky and Robinson (2000). The difference relays on that in equation (2.12) is considered the time until extinction for a closed group of N participants in the DB context. In the other side, the time until extinction of only one participant in the DC context is assumed.

Other consideration in this paper is not allowing short positions, thus, there is no possibility to have proportions neither in bonds nor stocks above the hundred per cent, as theoretical papers as Cairns *et al.*(2006) Josa-Fombellida and Rincón-Zapatero (2010) and Zhang *et al.* (2013) just to mention some works. We wanted to give more realistic situation to find an optimal allocation with these restrictions. Also, as many investment politics invest in fixed proportions during all pension plan, we analyze under this kind of strategies what are the best decisions to make under the constrains mentioned above in cases that original strategies remain.

Nevertheless, the estimation of the optimal strategy under heavy-tailed distribution cannot be derived for a closed form solution like under a log-normal assumption, even convolutions of heavy-tailed distributions have no closed forms, and even less with a copula playing an important role in the model. To solve the issue, we needed to evaluate several paths through stochastic simulations.

3 Copula model

In this section, the statistical data considered for the financial and economic vector (sp_n, b_n, s_n, π_n) and the final selected copula model for the stochastic simulations are presented.

3.1 Statistical data

For the financial and economic variables, we consider monthly historical data of the Mexican Stock Index (IPC) for the risky asset; for bonds the PIPG-Bonos index, which is a public index constructed of Mexican sovereign bonds published by FTSE TMX Global Debt Capital Markets Limited[1] and the Mexican company PIP LATAM, SA de C.V.; for the salary growth, historical data of the average salary of the Mexican Social Security Institution (IMSS for its acronym in spanish); and historical inflation by the Mexican Central Bank. Data is from January 2001 to October 2014 in order to compare all the available time series (see the list of references for the hyperlink).

In the [Table 1] is presented basic statistics of the monthly log-returns and log-growth rates of the stocks, bonds, salaries and inflation respectively. Noting that the bonds have reached low minimums and high maximums due to the fluctuation of interest rates over time. The kurtosis of the bond index is interesting, indicates that the points are closer to the median for being apparently a stable index for its structure of fixed income investments, but the large range of possible values in comparison of the standard deviation tells that there is a presence of heavy tails due to the sensitivity of interest rates changes induced by political, economic or market reasons.

Table 1: Basic statistics

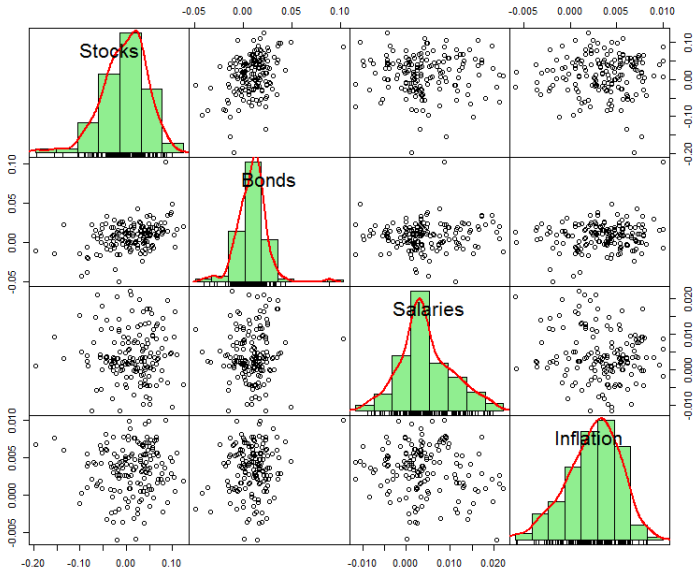
	Mean	Std. deviation	Median	Minimum	Maximum	Skewness	Kurtosis
Stocks	1.2%	5.2%	1.6%	-19.7%	12.4%	-0.7	4.2
Bonds	0.9%	1.7%	1.1%	-4.9%	10.3%	0.5	8.4
Salaries	0.4%	0.7%	0.3%	-1.2%	2.2%	0.3	3.0
Inflation	0.3%	0.3%	0.4%	-0.6%	1.0%	-0.5	2.9

Salaries and inflation are mor stable, being the salaries more volatile, doubling the standard deviation, and also doubling the minimum and the maximum over the history range. Both salary and inflation are closer to the kurtosis of a normal distribution than the kurtosis of stocks and bonds.

[1]FTSE TMX Global Debt Capital Markets is a trademark of the Group companies of London Stock Exchange Group.

In [Figure 1] a comparison pair by pair of the four indexes selected with the correspondent histogram and their smoothed kernel density estimation is shown.

Figure 1: Log-returns and log-growth rates



Graphically, data seems to be normal, but the normality test shown the opposite in the [Table 2]. The corresponding marginal distributions do not follow normality assumptions, the p-values of the Shaphiro-Wilk tests reject the hypothesis of normality distribution by cause of the presence of heavy tails in the four indexes.

Table 2: Shapiro-Wilk Normality Test

	p-values
Stocks	0.002
Bonds	6e-07
Salaries	0.005
Inflation	0.023

Based on the previous analysis of the [Table 1] the bond index reject obviously the normality test due its high kurtosis. For the case of salaries and inflation that were closer to normal kurtosis, the hypothesis testing is rejected induced by the tails.

Regarding the correlation structure, analyzing the [Table 3] a strong relationship can be seen only between bonds and stocks in three correlation measures. Hence, by parsimony stocks and bonds will be modeled independently from the salaries and inflation.

Table 3: Correlation measures and p-values

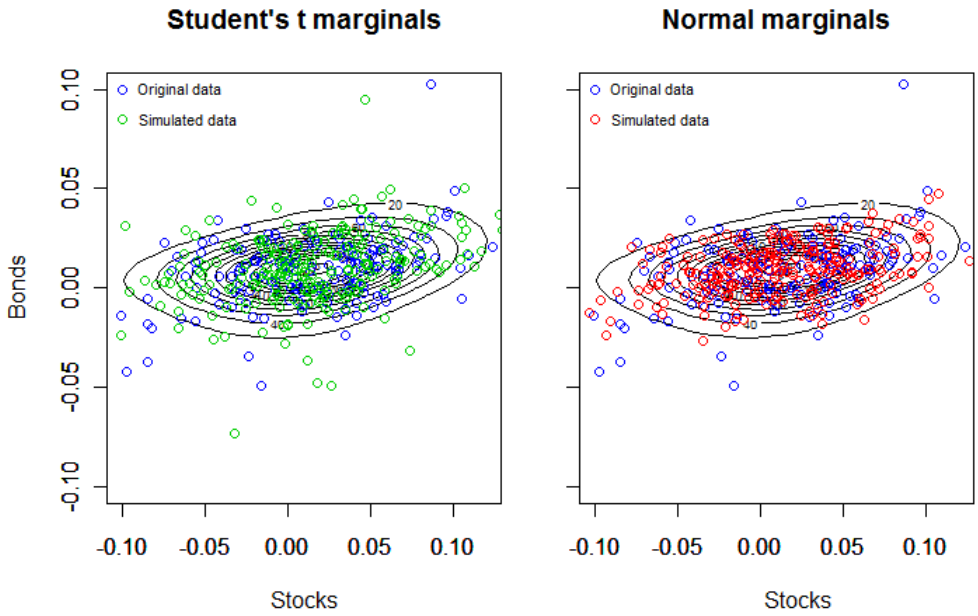
Index	Stocks	Bonds	Salaries	Inflation	Stocks	Bonds	Salaries	Inflation
	Pearson				$H_0 : \rho_{xy} = 0$			
Stocks	1.00	0.38	-0.003	0.03	-	5e-07	0.97	0.75
Bonds	0.38	1.00	0.05	0.18	5e-07	-	0.49	0.02
Salaries	-0.003	0.05	1.00	-0.16	0.97	0.49	-	0.05
Inflation	0.03	0.18	-0.16	1.00	0.75	0.02	0.05	-
	Kendall-tau				$H_0 : \tau = 0$			
Stocks	1.00	0.22	0.01	0.04	-	3e-05	0.92	0.49
Bonds	0.22	1.00	0.04	0.05	3e-05	-	0.51	0.30
Salaries	0.01	0.04	1.00	-0.05	0.92	0.51	-	0.32
Inflation	0.04	0.05	-0.05	1.00	0.49	0.30	0.32	-
	Spearman-rho				$H_0 : \rho = 0$			
Stocks	1.00	0.32	0.02	0.06	-	4e-05	0.82	0.45
Bonds	0.32	1.00	0.06	0.08	4e-05	-	0.51	0.30
Salaries	0.02	0.06	1.00	-0.09	0.82	0.51	-	0.46
Inflation	0.06	0.08	-0.09	1.00	0.45	0.30	0.46	-

In [Figure 2] the original data of Stocks and bonds with two copula models are compared, one with Student's-t marginals and other with Normal marginals. In the X-axis is plotted the log-returns of the Stocks and in the Y-axis the log-returns of the Bonds which are represented with the blue points. The green points were simulated from the model with Student's-t marginals and the red ones from the model with Normal marginals.

The model with Student's t marginals reflects much better the original data in both tails than the Normal model. The contour lines indicate that the probability distribution is not symmetrical and more opened in the lower tails than in the upper tails, indicating that is stronger the tail dependence in the upper tails than in the lower tails.

This is a graphical evidence that the selected copula might be a heavy tailed copula (See appendix for more details) that represents such dependence. An interpretation for this asymmetry model and its shape could be for instance, given a crisis scenario in stocks, bonds reflects also negative investment returns, but the negative range of possibilities are widen than the positive range in case of a positive scenario, in other words the bond market is more unpredictable when the stock market is going down in crisis than in positive months.

Figure 2: Student's-t v.s. Normal marginals



3.2 Selected model

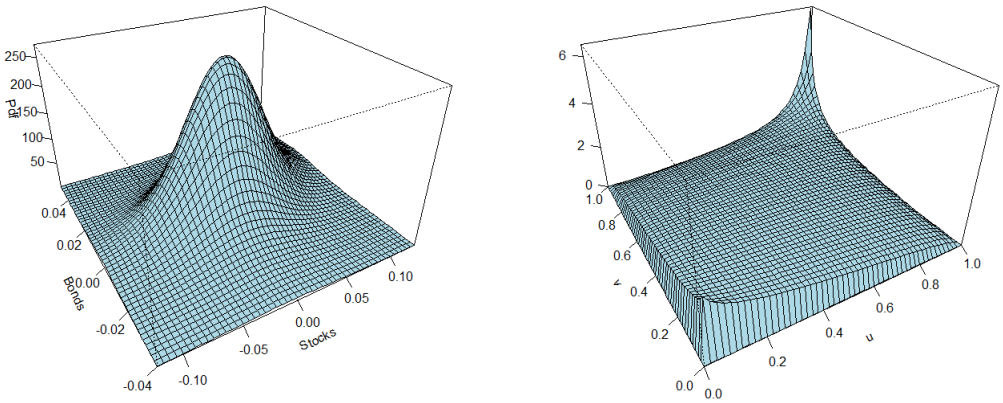
The fitted copula for our model was selected based on the Bayesian Information Criterion (BIC). All models were estimated with the maximum likelihood estimation method.

Table 4: Bayesian Information Criterion

Copula	BIC
Hüsler Reiss	-1,399.04
Galambo	-1,398.93
Gumbel	-1,398.67
Tawn	-1,396.27
Clayton	-1,395.38
Student's-t	-1,395.34
Normal	-1,394.52
Ali-Mikhail-Haq	-1,394.27
Frank	-1,392.52

The best model based on BIC is an extreme value copula, as same as the following three models with better BIC (the Galambo, Gumbel and Tawn copulas), following by the Elliptical (Student's-t and Normal copulas) and Archimedean copulas (Clayton, Ali-Mikhail-haq and frank copulas, for more details see Appendix). These results, are congruent with the graphical analysis of [Figure 2] where the shape of the dispersion plot indicates higher tail dependence in the upper tail than in the lower tail.

Figure 3: Left: The probability density with Student's-t marginals. Right: density of the Hüsler Reiss-Copula



In [Figure 3] the probability density surface of the Hüsler-Reiss copula fitted with Student's-t marginals and the density of the Hüsler-Reiss copula as defined in the Appendix are shown.

The model parameters of salaries and inflation were estimated by the maximum likelihood estimation method of two independent Student's-t distribution function is improved. Hence, the cumulative distribution function of the financial and economic vector is a nested copula with a Hüsler-Reiss copula of dimension two for the Stocks and Bonds, and an independent copula of dimension two for Salaries and Inflation, as follows:

$$C(F_1, F_2, F_3, F_4) = exp\left(-\hat{F}_1\Phi\left[\frac{1}{\theta} + \frac{\theta}{2}log\left(\frac{\hat{F}_1}{\hat{F}_2}\right)\right] - \hat{F}_2\Phi\left[\frac{1}{\theta} + \frac{\theta}{2}log\left(\frac{\hat{F}_2}{\hat{F}_1}\right)\right]\right) F_3\, F_4 \tag{3.1}$$

Where $\hat{F} = -log(F)$, $\Phi = [.]$ is the standard normal distribution function and $\theta = 0.97$. The marginal Student's-t distributions have the following parameters of location, scale and degrees of freedom respectively:

Table 5: Marginal Student's-t parameters

Index	μ	σ	df
Stocks	1.45%	4.6%	9.9
Bonds	0.95%	1.3%	4.0
Salaries	0.39%	0.7%	10.6
Inflation	0.36%	0.3%	11.4

4 Simulations and results

In this section, an explicit DB pension plan for a closed group until extinction simulated stochastically, changing the funding level u and the strategy $\{x_n\}$ in order to analyze which strategies are optimal and verify what is the sensitivity of the probability under the actual situation of low interest rates.

Pension model

For the pension model, a fixed replacement rate $\Delta_r = 60\%$ is supposed, starting with salaries of one unit for all participants, i.e. $S_0^j = 1$ for $j = 1, \dots, N$.

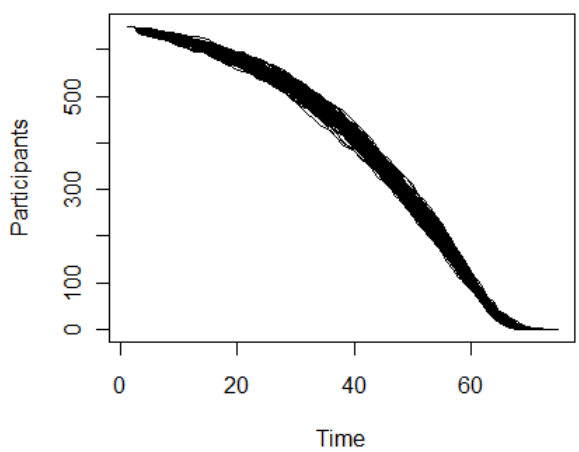
Demographic model

For the demographic variables, a closed group of six hundred participants with equal ages of thirty five years old is supposed. Regarding the mortality table, the Mexican mortality table EMSSA-2009 which has mortality improvements is considered. A recent research about mortality tables made in 2014 by the Organization for Economic Cooperation and Development (OECD) says that, the mortality table EMSSA-2009, seem to sufficiently provision for expected mortality improvements for now, mortality experience should be closely monitored for changing patterns to ensure that the table remains adequate. In [Figure 4], a simulation of a closed group assuming the mortality table EMSSA-2009 is shown.

Financial and economic vector

For the financial and economic vector, the copula model described in equation (3.1) is considered.

Figure 4: Group until extinction



4.1 Results

For each funding level u and a fixed strategy $\{x_n\}$ ten thousand simulations were made to estimate the probability of financial ruin for the closed group until extinction. For the simulations were used the statistical software *R-project* with the package 'copula'.

In the [Table 6] the estimated probabilities for a funding level from zero to 130%, and fixed proportion strategies are set from zero to 100% in the risky assets are shown.

Table 6: Probabilities of financial ruin

	Percentage invested in stocks										
Funding	0%	10%	20%	30%	40%	50%	60%	70%	80%	90%	100%
0%	27.0%	11.4%	5.0%	2.9%	1.9%	1.6%	1.5%	1.5%	1.6%	1.8%	2.0%
10%	25.6%	10.4%	4.4%	2.6%	1.8%	1.5%	1.3%	1.4%	1.4%	1.6%	1.8%
20%	24.4%	9.4%	4.0%	2.3%	1.5%	1.3%	1.2%	1.2%	1.3%	1.5%	1.7%
30%	23.1%	8.9%	3.7%	2.1%	1.4%	1.1%	1.1%	1.1%	1.2%	1.4%	1.6%
40%	22.2%	8.3%	3.4%	1.9%	1.3%	1.1%	1.0%	1.1%	1.2%	1.3%	1.5%
50%	21.1%	7.6%	3.2%	1.8%	1.2%	1.0%	1.0%	1.0%	1.1%	1.3%	1.4%
60%	20.4%	7.3%	3.1%	1.7%	1.2%	1.0%	0.9%	1.0%	1.1%	1.2%	1.4%
70%	19.6%	7.1%	3.0%	1.6%	1.1%	1.0%	0.9%	1.0%	1.1%	1.3%	1.5%
80%	19.0%	6.7%	2.9%	1.6%	1.1%	1.0%	0.9%	1.0%	1.1%	1.2%	1.5%
90%	18.3%	6.5%	2.8%	1.5%	1.1%	1.0%	0.9%	1.0%	1.1%	1.2%	1.5%
100%	17.8%	6.4%	2.8%	1.5%	1.1%	1.0%	0.9%	1.0%	1.2%	1.3%	1.5%
110%	12.0%	3.8%	1.7%	0.8%	0.6%	0.6%	0.6%	0.7%	0.8%	1.0%	1.2%
120%	7.9%	2.5%	0.9%	0.5%	0.4%	0.3%	0.4%	0.5%	0.6%	0.7%	0.9%
130%	5.2%	1.5%	0.5%	0.3%	0.2%	0.2%	0.3%	0.3%	0.5%	0.5%	0.7%

As the intuition says and, if the assumptions remain valid, the more money the fund has, the probability of financial ruin decreases for all fixed strategies. The optimal asset allocations remain in the same range of 60% of the fund invested in risky assets until the point where a 100% of funding is reached, which is the actuarial fair value. Funding levels over the complete actuarial present value, the optimal asset allocation decreases to 40%, this makes sense and could be interpreted as once the liabilities are fully funded is not necessary to assume more risk, this result is in line with Josa-Fambellida and Rincón-Zapatero (2010) where they found that the optimal asset allocation is in function of the unfunded liability and the actuarial liability. Nevertheless, differences in terms of probability of financial ruin between the optimal asset allocation and other strategies in a range of 10% of the proportion invested in stocks are not highly significant, no more than 0.1% of difference in almost all cases. This can be explained by the heavy tails of the investment returns of the stocks and bonds and their positive relationship, financial crisis affects strongly investment strategies making them almost equal in terms of probability of financial ruin. Thus, any strategy in a given range will not make a significant difference.

4.2 Sensitivity analysis

In this part, the probability of financial ruin is estimated under the scenario where the liabilities are under-valuated caused by the usage of a greater interest rate in the actuarial valuations, or equivalently, where the expected return of the bonds is lower than the interest rate assumed by the actuary, causing lower contributions than the necessary to fund the compromises.

This case scenario is in line with the actual global situation where the interest rates around the globe are less than expected in past valuations, inclusive with negative interest rates in some cases. Thus, past valuations considered higher interest rates in the yield curve, and in consequence lower contributions and higher probabilities of financial ruin.

To model this situation, we will valuate liabilities with 1% more in the annualized force of interest in the equation (2.7). In the [Table 7] the impact of contributions to the fund under the changing assumption in the interest rate used in valuations is shown.

Table 7: Impact in the contribution rate k

Funding	Original k	Shocked k
0%	6.6%	5.1%
10%	5.9%	4.6%
20%	5.3%	4.1%
30%	4.6%	3.6%
40%	4.0%	3.1%
50%	3.3%	2.5%
60%	2.6%	2.0%
70%	2.0%	1.5%
80%	1.3%	1.0%
90%	0.7%	0.5%

In the [Table 8] the estimated probabilities of financial ruin for the scenario where the interest rate used in valuations is over-estimated by 1% are presented. As same as in [Table 6] the probability decreases as the funding level increases in all investment strategies, and the optimal asset allocation decreases as the funding level does after being fully funded.

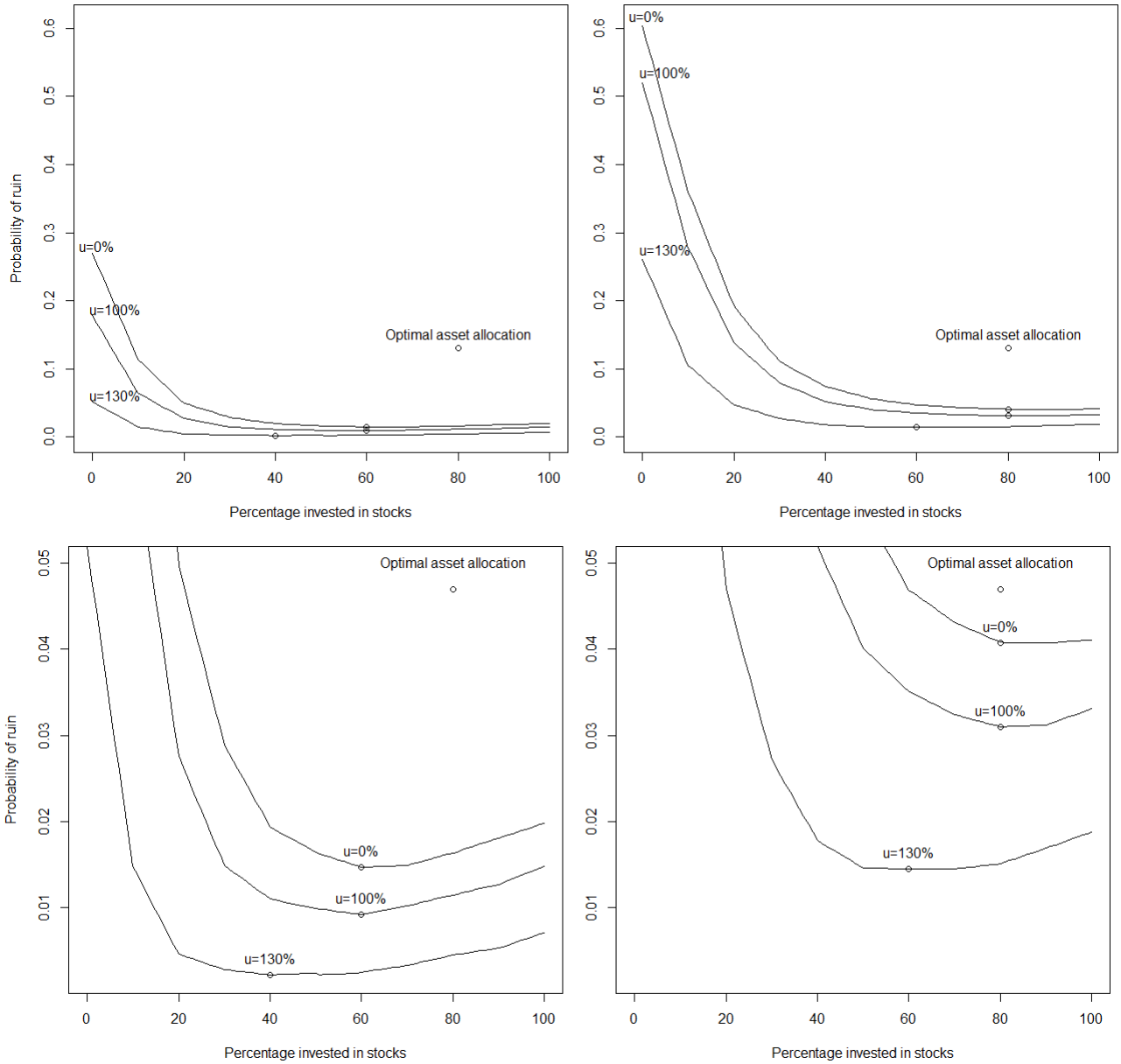
Table 8: Probabilities of financial ruin (under-valuated liabilities)

	Percentage invested in stocks										
Funding	0%	10%	20%	30%	40%	50%	60%	70%	80%	90%	100%
0%	60.3%	36.0%	19.2%	11.1%	7.4%	5.6%	4.7%	4.3%	4.1%	4.1%	4.1%
10%	59.4%	34.6%	18.3%	10.4%	6.9%	5.3%	4.4%	4.0%	3.8%	3.8%	3.9%
20%	58.4%	33.6%	17.5%	9.8%	6.8%	4.9%	4.2%	3.7%	3.6%	3.5%	3.7%
30%	57.5%	32.6%	16.7%	9.4%	6.5%	4.7%	4.0%	3.5%	3.4%	3.4%	3.6%
40%	56.6%	31.8%	16.0%	9.1%	6.1%	4.5%	3.8%	3.5%	3.4%	3.4%	3.5%
50%	55.7%	30.8%	15.4%	8.9%	5.7%	4.3%	3.7%	3.4%	3.2%	3.3%	3.4%
60%	54.9%	30.1%	15.0%	8.7%	5.5%	4.3%	3.6%	3.3%	3.1%	3.2%	3.4%
70%	54.4%	29.4%	14.6%	8.4%	5.4%	4.2%	3.5%	3.2%	3.1%	3.1%	3.3%
80%	53.3%	28.8%	14.4%	8.3%	5.3%	4.0%	3.5%	3.1%	3.1%	3.1%	3.2%
90%	52.6%	28.3%	14.0%	8.0%	5.2%	4.1%	3.5%	3.1%	3.1%	3.1%	3.3%
100%	52.0%	27.8%	13.8%	7.9%	5.2%	4.0%	3.5%	3.2%	3.1%	3.1%	3.3%
110%	42.1%	20.1%	9.6%	5.4%	3.7%	3.0%	2.6%	2.5%	2.4%	2.6%	2.8%
120%	33.2%	14.4%	6.8%	3.7%	2.7%	2.0%	1.9%	1.9%	2.0%	2.1%	2.3%
130%	26.1%	10.5%	4.7%	2.7%	1.8%	1.5%	1.5%	1.5%	1.5%	1.7%	1.9%

For a graphical representation, the probability of financial ruin for a three funding levels u and fixed strategies from 0% to 100% are plotted. In the superior charts of the [Figure 5] can be appreciated how changing 1% in the valuation affects more to the strategies that invest more money in bonds. The sensitivity of this change makes in all cases that the probability of financial ruin is more than the double of the original cases. In the worst case, where the funding level is zero and all is invested in bonds the probability of ruin increases from 27% to 60%, more than the double of the original probability. The actual low rates could affect drastically the provisions made in the past to fund the pensions, so it is clear the high sensitivity that the probability presents under the changing assumptions in valuations.

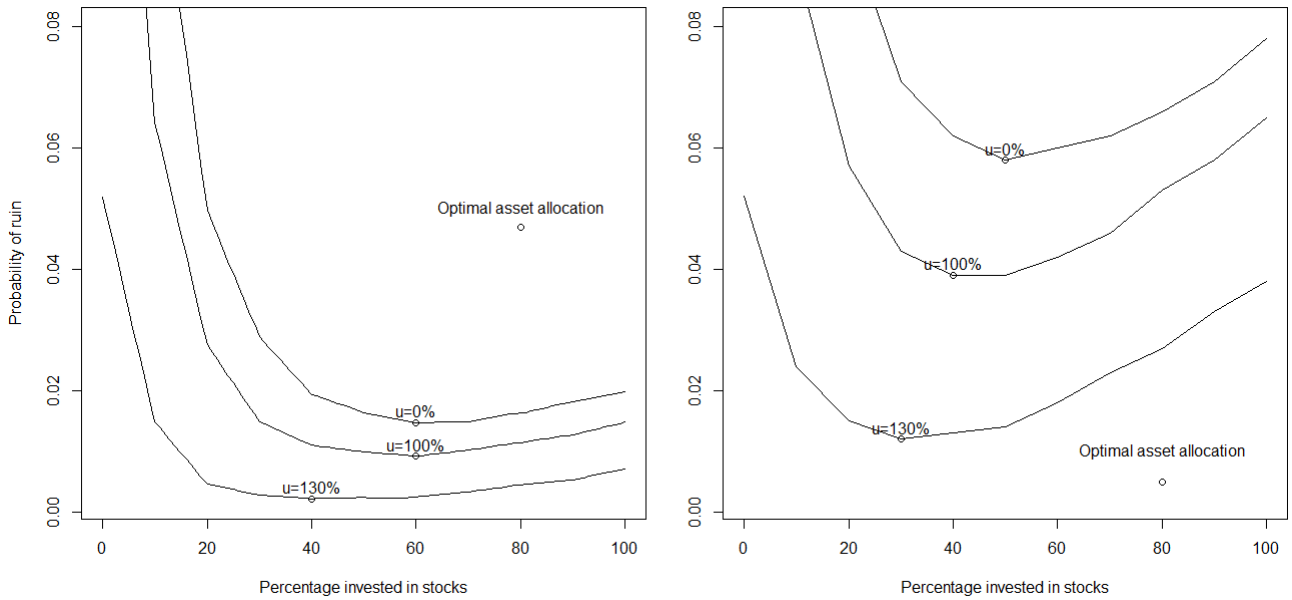
In the point of view of the sensitivity of the optimal strategy, in [Figure 5] the optimal proportion of the fund invested in stocks is represented by a black point on the correspondent probability curve. The lower two charts are re-scaled in order to have a better comparison between the original case and the shocked case. In the graphs can be seen that, to compensate the shock in valuations is required to assume more risk. The optimal proportion in risky stocks passes from a range of 60%-40% to 80%- 60% with higher probabilities. In the shocked scenario, the probabilities of the optimal strategy in all funding levels are bigger than all the probabilities of the optimal strategies at any funding level u of the original case. The optimal strategy is high sensitive to minimal changes in the interest rate of valuations.

Figure 5: Probabilities of financial ruin. Left: Original scenario. Right: Under-valuated liabilities



In other point of view, in the case where the spread between the asset and bonds returns start to be closer due to an underperformance of the stocks, the optimal proportion in stocks starts to decrease. In [Figure 6] the probabilities of financial ruin are plotted where the Sharpe ratio passes from 0.38% to 0.25%, or equivalently the stocks gives 2% less returns per year. The optimal asset allocation passes from 60%-40% to 50%-30% given a funding level u.

Figure 6: Probabilities of financial ruin. Left: Original scenario. Right: Underperforming stock returns



5 Conclusions

In this paper we analyzed the probability of financial ruin through stochastic simulations for a hypothetical DB pension plan for a closed group until extinction where contributions are fixed since time zero. For this purpose, we established a model where the financial and economic variables of log-returns of the stocks and bonds, the log-growth rates of salaries and inflation are related by a copula function with marginals that follow heavy tailed distributions independently of the mortality table.

For the simulations, a copula model was fitted with Student's-t marginals using statistical data of an stock index, a bond index of sovereign bonds, a salary index from a social security institution and an inflation index from the central bank database and a specific mortality table with mortality improvements for the simulation of the closed group was selected.

The simulations show that the probability of financial ruin decreases for all fixed strategies as the funding level increases. The optimal proportion of the fund invested in stocks remains more or less in the same range until a hundred percent of funding level is reached; once the fund reaches the actuarial present value, the optimal percentage of the fund invested in risky assets starts to decrease because there is no necessity to assume more risk. Furthermore, given a funding level, assuming more risk decreases the probability of financial ruin until one point where no further risk changes significantly the probability of ruin. The heavy tails of the investment returns of the stocks and bonds and their positive relationship, affect strongly all investment strategies near the optimal, making

them almost equal in terms of probability of financial ruin. Thus, any strategy in a given range will not make a significant difference if the assumptions remain until the pension plan extinction.

In the sensitivity analysis, the situation where interest rates of bonds are less than expected in the valuation was modeled. In this case, the probability is impacted drastically. In addition, to reach an optimal strategy, more risk has to be assumed, but with significant greater probabilities of ruin.

Other sensitivity analysis was to diminish the difference between the expected return of bonds and stocks, due to an underperformance in stocks. In this scenario, the optimal strategy are closer to bonds.

In conclusion, the optimal allocation in stocks and the probability of financial ruin are highly sensitive to changes in the assumptions under heavy tailed distributions and results are in line with the intuition. But to find an explicit solution, the use of stochastic simulations is a useful tool to find numerically the optimal strategy and to measure the impact in the probability of the financial ruin induced by changes in the assumptions. Thus, to enhance risk minimization, the use of mathematical and probabilistic tools as copulas and heavy tailed distributions can be applied in order to make a right and more informed investment decision.

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Data sources
Bond index PIPG-Bonos: https://www.piplatam.com
Inflation: http://www.banxico.org.mx
Mexican stock index (IPC): http://finance.yahoo.com
Salary growth: http://www.imss.gob.mx

6 Appendix

6.1 Copulas

Copula functions allow to modelling efficiently the dependence structure between variables, increasing in the last years as a tool for financial and actuarial applications.

If $X = (X_1, X_2, ..., X_n)$ is a random vector with continuous marginal cumulative distribution functions $F_1, F_2, ..., F_n$ then their joint cumulative distribution function $F_{X_1,...,X_n}(x_1, ..., x_n)$ can be described by:

$$F_{X_1,...,X_n}(x_1, ..., x_n) = C(F_1(x_1), \dots, F_n(x_n)) \tag{6.1}$$

Where C is a restriction to the unit square I^n, $I = [0, 1]$ of a n-dimensional density functions that concentrates all the probability mass on I^n and which has uniform marginals on I. For the copula C the following properties have to hold:

1. For $C(u_1, u_2, ..., u_n)$ is increasing in u_i.
2. $C(1, ..., 1, ui, 1, ..., 1) = u_i$ for all $i \in [1, ..., n]$ $u_i \in [0, 1]$.
3. For all $(a_1, ..., a_n)$, $(b_1, ..., b_n) \in I^n$ with $a_i \leq b_i$ it holds that

$$\sum_{i_1}^{2} \cdots \sum_{i_n}^{2} C(u_{i_1}, ..., u_{i_n}) \geq 0 \tag{6.2}$$

   Where $u_{j_1} = a_j$ and $u_{j_2} = b_j$ for all $j \in [1, ...n]$

Using these properties one can derive the Fréchet-Hoeffding bounds which in the bivariate case are $max(u + v - 1, 0) \leq C(u, v) \leq min(u, v)$, where $max(u + v - 1)$ is known as Fréchet-Hoeffding lower bound and $min(u, v)$ is known as Fréchet-Hoeffding upper bound.

For the implicit copula of an absolutely continuous joint distribution function F and density f with strictly continuous marginal distribution functions, the copula density c is given by

$$c(F_1(x_1), \dots, F_n(x_n)) = \frac{f(F_1^{-1}(x_1), \dots, F_n^{-1}(x_n))}{f_1(F_1^{-1}(x_1)) \cdots f_n(F_n^{-1}(x_n))} \tag{6.3}$$

Tail dependence

Talking about dependent measures, a useful one is the concept of *tail dependence* in the bivariate case, which is a measure for extreme co-movements in the lower and upper tail of $F_{XY}(x, y)$, respectively. The upper tail dependence coefficient (TDC) is usually defined by

$$\lambda_U = \lim_{u \to 1^-} P\left(Y > F_Y^{-1}(u) | X > F_X^{-1}(u)\right) = \lim_{u \to 1^-} \frac{1 - 2u + C(u, u)}{1 - u} \in [0, 1]. \tag{6.4}$$

noting that λ_U is solely depending on $C(u, u)$ and not on the marginal distributions. Analogously, the lower TDC is defined as

$$\lambda_U = \lim_{u \to 0^+} P\left(Y \leq F_Y^{-1}(u) | X \leq F_X^{-1}(u)\right) = \lim_{u \to 0^+} \frac{C(u,u)}{u} \in [0,1]. \tag{6.5}$$

In the following part, different class of copulas are described.

6.1.1 Archimedean copulas

Special classes of copulas are known as Archimedean copulas. An Archimedean copula can be written on the form:

$$C(u_1, ..., u_d) = \Phi^{-1}\left(\Phi(u_1) + ... + \Phi(u_d)\right) \tag{6.6}$$

Where Φ is a strictly decreasing function in $[0,1] \to [0,\infty)$ with pseudo-inverse Φ^{-1}. Φ is called the *generator*.

Clayton copula

The Clayton copula is an archimedean copula introduced by Clayton (1978), which is an asymmetric copula. This copula is given by.

$$C(u,v) = exp\left(u^\theta + v^\theta - 1\right)^{-1/\theta} \tag{6.7}$$

For $0 < \theta < \infty$.Perfect dependence is obtained if $\theta \to \infty$. While $\theta \to 0$ implies independence. The Clayton copula is mostly used to study correlated risks because of their ability to capture lower tail dependence and left skew.

Gumbel copula

The Gumbel copula is both an archimedean copula and an extreme value copula (explained further). In the bivariate case it is defined as:

$$C(u,v) = exp\left(-\left((-logu)^\theta + (-logv)^\theta\right)^{1/\theta}\right) \tag{6.8}$$

Where $\theta \in [1,\infty)$. When $\theta = 1$ the variables (u,v) are independent and when $\theta \to \infty$ we obtain perfect positive dependence between variables. For $\theta > 1$ the Gumbel copula exhibits upper tail dependence, i.e. if u is large then v is also expected to be large. This copula is famous for its ability to capture strong upper tail dependence and weak lower tail dependence.

Frank copula

The Frank copula is given by

$$C(u,v) = -\theta^{-1} log\left(1 + \frac{(e^{-\theta u} - 1)(e^{-\theta v} - 1)}{e^{-\theta} - 1}\right) \tag{6.9}$$

Where θ is the copula parameter that may take any real value. Unlike the Clayton and the Gumbel copula, the Frank copula allows the maximum range of dependence. This means that the dependence parameter of the Frank copula allows the approximation of the upper and the lower Fréchet-Hoeffding bounds and thus the Frank copula allows modeling positive as negative dependence in the data. When θ approaches to $-\infty$ and $+\infty$ the Fréchet-Hoeffding upper and lower bound will be attained. The independence case will be attained when θ approaches zero. However, the Frank copula has neither lower nor upper tail dependence. The Frank copula is thus suitable for modeling data characterized by weak tail dependence.

Ali-Mikhail-Haq copula

Archimedean copula with parametric generator

$$\Phi(t) = (1-\theta)/(exp(t)-\theta) \tag{6.10}$$

With $\theta \in [0,1)$ The range of admissible Kendall's tau is $[0, 1/3)$. Note that the lower and upper tail-dependence coefficients are both zero, that is, this copula family does not allow for tail dependence.

6.1.2 Elliptical copulas

Elliptical copulas are simply the copulas of elliptically contoured (or elliptical) distributions. The most commonly used elliptical distributions are the multivariate normal and Student-t distributions. The key advantage of elliptical copula is that one can specify different levels of correlation between the marginals and the key disadvantages are that elliptical copulas do not have closed form expressions and are restricted to have radial symmetry. For the copula of an elliptically symmetric distribution, the two tail dependence coefficients are equal.

Normal copula

The Normal copula is given by

$$C(u,v) = \int_{-\infty}^{\Phi^{-1}(u)} \int_{-\infty}^{\Phi^{-1}(v)} \frac{1}{2\pi(1-\rho)^{1/2}} exp\left(-\frac{x^2-2\rho xy+y^2}{2(1-\rho^2)}\right) dxdy \tag{6.11}$$

Where ρ is the parameter of the copula, and $\Phi^{-1}[.]$ s the inverse of the standard univariate Normal distribution function. The tail dependence coefficient for this copula is zero.

Student's t copula

The Student's t-copula allows for joint fat tails and an increased probability of joint extreme events compared with the Normal copula. This copula can be written as.

$$C(u,v) = \int_{-\infty}^{t_\nu^{-1}(u)} \int_{-\infty}^{t_\nu^{-1}(v)} \frac{1}{2\pi(1-\rho)^{1/2}} \left(-\frac{x^2 - 2\rho xy + y^2}{\nu(1-\rho^2)} \right)^{-\frac{\nu+2}{2}} dxdy \tag{6.12}$$

Where t_ν^{-1} is the inverse of the standard univariate student-t-distribution with ν degrees of freedom, expectation 0 and variance $\frac{\nu}{\nu+2}$. The Student's t-dependence structure introduces an additional parameter compared with the Normal copula, namely the degrees of freedom ν. Increasing the value of ν decreases the tendency to exhibit extreme co-movements, its tail dependence coefficient is positive.

6.1.3 Extreme value copulas

There also exists a class of copulas known as extreme value copulas. A copula C^* is called an extreme value copula if there exists a copula C such that

$$C^*(u_1, ..., u_d) = \lim_{n\to\infty} C^n \left(u_1^{1/n}, ..., u_d^{1/n} \right) \tag{6.13}$$

for all $i \in [1, ..., d]$, $u_i \in [0, 1]$. The copula C is said to be in the domain of attraction of C^*.

Galambos Copula

The Galambos copula is an extreme value copula which, for the bivariate case, is defined as

$$C(u,v) = u\, v\, exp\left(((-logu)^{-\theta} + (-logv)^{-\theta})^{-1/\theta} \right) \tag{6.14}$$

Where $\theta \in (0, 1)$. When $\theta \to \infty$ complete dependence is obtained and as $\theta \to \infty$ dependence is obtained. the Galambos copula is characterized by strong upper tail dependence and right skew.

Hüsler-Reiss Copula

The Hüsler-Reiss copula is an extreme value copula, for more details see Hüsler & Reiss (1989), and in the bivariate case it is defined as:

$$C(u,v) = exp\left(-\hat{u}\Phi\left[\frac{1}{\theta} + \frac{\theta}{2} log\left(\frac{\hat{u}}{\hat{v}} \right) \right] - \hat{v}\Phi\left[\frac{1}{\theta} + \frac{\theta}{2} log\left(\frac{\hat{v}}{\hat{u}} \right) \right] \right) \tag{6.15}$$

Where $\hat{u} = -log(u)$, $\hat{v} = -log(v)$, $\Phi = [.]$ is the standard normal distribution function and $\theta \in (0, \infty)$. As θ approaches zero independence is obtained and when θ goes to infinity complete independence is obtained as well. Hüsler–Reiss copula allows dependence in the upper tail while the lower tail dependence coefficient is zero. As the Galambos copula has strong upper tail dependence and right skew.

The Tawn Copula

The Tawn copula is an extreme value copula and is in the bivariate case defined as

$$C(u,v) = u\,v\,exp\left(-\theta\frac{log(u)log(v)}{log(uv)}\right) \tag{6.16}$$

Where $\theta \in (0,1)$. For $\theta = 0$ independence is achieved but is not possible to obtain complete dependence. This copula has strong upper tail dependence and zero lower tail dependence coefficient.