Non-Parametric Inference of Transition Probabilities Based on Aalen-Johansen Integral Estimators for Semi-Competing Risks Data

Application to LTC Insurance

Quentin Guibert

Autorité de Contrôle Prudentiel et de Résolution

ISFA, Université de Lyon, Université Claude-Bernard Lyon-1

Email: quentin.guibert@acpr.banque-france.fr

IAA Colloquia, Oslo, 7-10 June 2015.
Joint work with F. Planchet (ISFA and Prim'Act).

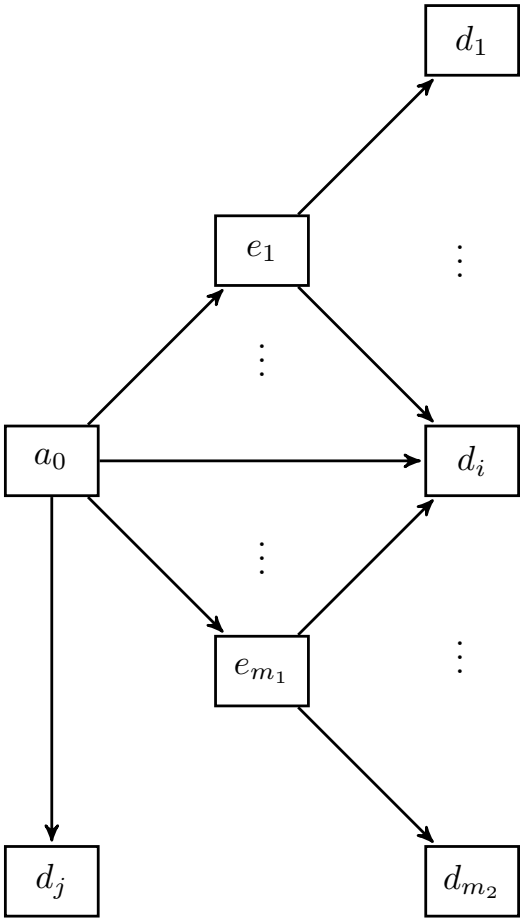
Outline

Insurance context

- Multi-state models are the suitable framework for modeling health and life insurance contracts (Haberman and Pitacco, 1998; Christiansen, 2012).
- For a LTC insurance model, transition probabilities are generally fitted assuming the Markov assumption holds. These quantities are the main inputs for pricing or reserving models.
- Need for realistic (best estimate) assumptions for the Solvency II purpose. Academics and practitioners generally use parametric models with the Markov assumption. Markov assumption is too strong.
- Goodness of fit checks are complicated to implement as non-parametric estimators are not available for multi-state models when this assumption does not hold.

Acyclic multi-state model

Consider an acyclic multi-state model which refers to a situation where both terminal and non-terminal events can occur during the lifetime of an individual.



Formally, two lifetimes are identified:

- S, the lifetime in healthy state

$$S = \inf\{t : X_t \neq a_0\},$$

- T, the overall lifetime

$$T = \inf\{t : X_t \in \{d_1, \ldots, d_{m_2}\}\},$$

where $(X_t)_{t\geq 0}$ is the current state of the individual.

Motivation

In the French insurance framework, we have longitudinal data with independent right-censoring (administrative censoring).

Right-censoring data

Let C the unique right censoring variable. The following variables are available

$$\begin{cases} Y = \min(S, C) \text{ and } \gamma = \mathbb{1}_{\{S \leq C\}}, \\ Z = \min(T, C) \text{ and } \delta = \mathbb{1}_{\{T \leq C\}}. \end{cases}$$

No Markov assumption.

Main goals

- Non-parametric estimation of transition probabilities for a such a right censoring acyclic multi-state model
- Non-parametric association measure between the failure time in healthy state and the overall failure time when non-terminal event occurs

Existing Estimators for Competing Risks Data

- Non-parametric estimation framework for Markov multi-state model (Andersen *et al.*, 1993)
- Let V be the indicator of the type of failure. The Aalen-Johansen (AJ) estimator for the cumulative incidence function (CIF) which is the joint distribution of (T, V) is

$$F^{(v)}(t) = \mathbb{P}(T \leq t, V = v).$$

Non-parametric estimator for CIF

- i.i.d. observations are composed of $(Z_i, \delta_i, \delta_i V_i,)_{1 \leq i \leq n}$
- Estimator can be expressed as a sum considering the ordered Z-values

$$\widehat{F}_n^{(v)}(z) = \sum_{i=1}^{n} \widetilde{W}_{in} J_{[i:n]}^{(v)} \mathbb{1}_{\{Z_{i:n} \leq z\}}, \quad \widetilde{W}_{in} = \frac{\delta_{[i:n]}}{n-i+1} \prod_{j=1}^{i-1} \left(\frac{n-j}{n-j+1} \right)^{\delta_{[j:n]}}$$

- $\widetilde{W}_{in}$ is the Kaplan-Meier (KM) weights and $J_i^{(v)} = \mathbb{1}_{\{V_i = v\}}$
- $\widehat{F}_n^{(v)}(\cdot)$ converges w.p.1 to $F^{(v)}(\cdot)$ and is asymptotic normal

Existing Estimators for other multi-state models

- No general framework for non-parametric estimation of multi-state models when Markov assumption does not hold
- Particular models for:
  - state occupation probabilities (Datta and Satten, 2002)
  - transition probabilities for illness-death model (Meira-Machado *et al.*, 2006)
- Classical approaches for semi-competing risks data use competing risks techniques and focus on estimating the survival function from the latent failure time to the non-terminal event:
  - non-parametric estimation with left-truncation and right-censoring (Peng and Fine, 2006)
  - semi-parametric model using copula-graphic estimators (e.g. Lakhal *et al.*, 2008)

Bivariate Competing Risks Data

- Idea: there is recent literature on estimating bivariate competing risks models (Cheng *et al.*, 2007). Our acyclical model can be viewed as a particular case with a unique right censoring process.
- Let (S, V_1) and (T, V) be 2 competing risks processes where:
  - V_1 is indicator taking its values in the set of arrival states by direct transition from a_0
  - $V = (V_1, V_2)$ with is V_2 indicator taken its values in the set of arrival states from non-terminal events

Bivariate CIF estimator

$$\widehat{F}_{0n}^{(v)}(y,z) = \sum_{i=1}^{n} \widetilde{W}_{in} J_{[i:n]}^{(v)} \mathbb{1}_{\left\{Y_{[i:n]} \leq y, Z_{i:n} \leq z,\right\}}$$

- Simple form for the weights as (S, V_1) is observed whether T is observed
- $\widehat{F}_{0n}$ is weakly convergent under independent censoring

Aalen-Johansen Integrals Estimators

- Consider an integral of the form $S^{(v)}(\varphi) = \int \varphi \, dF_0^{(v)}$ with φ a generic function
- S can be considered as a covariate

AJ integrals

$$\widehat{S}_n^{(v)}(\varphi) = \int \varphi(s,t) \; \widehat{F}_{0n}^{(v)}(ds,dt) = \sum_{i=1}^{n} \widetilde{W}_{in}^{(v)} \varphi\left(Y_{[i:n]}, Z_{i:n}\right), \; 0 \leq s \leq t \leq \tau_Z.$$

- $W_{in}^{(v)} = W_{in} J_{[i:n]}^{(v)}$, AJ weights (Suzukawa, 2002) for competing risks data
- Possibility to take account for left-truncation L considering

$$\widetilde{W}_{in}^{(v)} = \frac{\delta_{[i:n]} J_{[i:n]}^{(v)}}{nC_n(Z_{i:n})} \prod_{j=1}^{i-1} \left(1 - \frac{1}{nC_n(Z_{i:n})}\right)^{\delta_{[j:n]}},$$

where $C_n(x) = n^{-1} \sum_{i=1}^{n} \mathbb{1}_{L_i \leq x \leq Z_i}$

Transition Probabilities Estimators

Application for estimating key probabilities in actuarial science i.e.

$$p_{0e}(s,t,\eta) = \frac{\mathbb{P}\left(s < S \leq \min(t, t-\eta), T > t, V_1 = e\right)}{\mathbb{P}(S > s)},$$

$$p_{ee}(s,t) = \frac{\mathbb{P}\left(S \leq s, T > t, V_1 = e\right)}{\mathbb{P}\left(S \leq s, T > s, V_1 = e\right)},$$

$$p_{ed}(s,t,\eta,\zeta) = \frac{\mathbb{P}\left(\eta < T - S \leq \zeta, s < S \leq t, V = (e,d)\right)}{\mathbb{P}\left(T - S > \eta, s < S \leq t, V_1 = e\right)}.$$

Remarking that $\{V_1 = e\} = \{V_1 = e, V_2 \in \mathcal{C}_e\}$ where $\mathcal{C}_e$ is the set of children (i.e transition states from e) related to the state e, we can refer to our AJ integrals estimators

Transition Probabilities Estimators

Our estimators enlarge those of Meira-Machado *et al.* (2006).

$$\widehat{p}_{0e}(s,t,\eta) = \frac{\widehat{S}_n^{(e,\mathcal{C}_e)}\left(\varphi_{s,t,\eta}^{(1)}\right)}{1-\widehat{H}_n(s)}, \text{ with } \varphi_{s,t,\eta}^{(1)}(x,y) = \mathbb{1}_{\{s<x\leq \min(t,t-\eta),y>t\}},$$

$$\widehat{p}_{ee}(s,t) = \frac{\widehat{S}_n^{(e,\mathcal{C}_e)}\left(\varphi_{s,t}^{(2)}\right)}{\widehat{S}_n^{(e,\mathcal{C}_e)}\left(\varphi_{s,s}^{(2)}\right)}, \text{ with } \varphi_{s,t}^{(2)}(x,y) = \mathbb{1}_{\{x\leq s,y>t\}},$$

$$\widehat{p}_{ed}(s,\eta,\zeta) = \frac{\widehat{S}_n^{(e,d)}\left(\varphi_{s,\zeta}^{(3)}\right)}{\widehat{S}_n^{(e,\mathcal{C}_e)}\left(\varphi_{s,\eta}^{(4)}\right)}, \text{ with } \varphi_{s,\zeta}^{(3)}(x,y) = \mathbb{1}_{\{s<x\leq t,\eta<y-x\leq\zeta\}},$$

$\varphi_{s,\eta}^{(4)}(x,y) = \mathbb{1}_{\{s<x\leq t,\eta<y-x\}}$ and $\widehat{H}_n$ is the KM estimator of the distribution function of S.

Association measures

As Scheike and Sun (2012) for multivariate competing risks model, we regard local association measures based on cross-odds ratio.

$$\pi_0^{(e,d)}(s,t) = \frac{\text{odds}\left(T \leq t, V_2 = d \mid S \leq s, V_1 = e\right)}{\text{odds}\left(T \leq t, V_2 = d \mid V_1 = e\right)},$$

where $\text{odds}(A) = \dfrac{\mathbb{P}(A)}{1 - \mathbb{P}(A)}$.

For a couple (e,d), this non-parametric-estimator measures the effect of the duration spent in healthy state on the total lifetime.

$$\widehat{\pi}_{0n}^{(e,d)}(s,t) = \frac{\dfrac{\widehat{F}_{0n}^{(e,d)}(s,t)}{\widehat{H}_{0n}^{(e)}(s) - \widehat{F}_{0n}^{(e,d)}(s,t)}}{\dfrac{\widehat{F}_{n}^{(e,d)}(t)}{\widehat{H}_{0n}^{(e)}(\infty) - \widehat{F}_{n}^{(e,d)}(t)}},$$

where $\widehat{H}_{0n}^{(e)}$ is the estimator of the CIF of S for cause $V_1 = e$ and $\widehat{F}_n^{(e,d)}$ is that of T for cause $V = (e,d)$.

AJ integrals estimators

Theorem (Consistency)

Assume that

- φ is an F_0-integrable function,
- F_0 and censoring distribution function G are continuous,
- C is independent of the vector (S, T, V).

Then, we have

$$\widehat{S}_n^{(v)}(\varphi) \longrightarrow S_\infty^{(v)}(\varphi) = \int \mathbb{1}_{\{t<\tau_Z\}} \varphi(s,t) \; F_0^{(v)}(ds, dt) \; , \; v \in \mathcal{V} \;\; w.p.1.$$

Sketch of the proof

- We apply the strategy followed by Stute (1993) by considering S as a covariate and show that $\left(\widehat{S}_n^{(v)}(\varphi), \mathcal{F}_n^{(v)}, n \geq 0\right)$ is a reverse-time supermartingale where

$$\mathcal{F}_n^{(v)} = \sigma\left(Z_{i:n}, D_{[i:n]}^{(v)}, 1 \leq i \leq n, Z_{n+1}, D_{n+1}^{(v)}, \ldots\right), \; D_i^{(v)} = \left(Y_i, \delta_i, J_i^{(v)}\right).$$

- We compute the limit $\lim_{n\to\infty} \mathbb{E}\left[\widehat{S}_n^{(v)}(\varphi)\right] = S_\infty^{(v)}(\varphi)$ and use the independence assumption to obtain in particular

$$\mathbb{P}(T \leq C \mid S, T, V) = \mathbb{P}(T \leq C \mid T, V) = 1 - G(T).$$

AJ integrals estimators

Theorem (Weak convergence)

Assume that:

- $\displaystyle\int \frac{\varphi(S,T)^2\,\delta}{(1-G(T))^2}\,d\mathbb{P} < \infty,$
- $\displaystyle\int |\varphi(S,T)|\sqrt{C_0(T)}\mathbb{1}_{\{T<\tau_Z\}}\,d\mathbb{P} < \infty,$

  where $C_0(x) = \int_0^{x-} \dfrac{G(dy)}{(1-M(y))(1-G(y))}$ and $M(z) = \mathbb{P}(Z \leq z)$.

With the previous assumptions and assuming the support of Z is included in that of C, we have

$$\sqrt{n}\left\{\widehat{\boldsymbol{S}}_n(\varphi) - \boldsymbol{S}(\varphi)\right\} \xrightarrow{d} \mathcal{N}(0, \boldsymbol{\Sigma}(\varphi)).$$

- These results can be extended considering additional covariates $\boldsymbol{U} = (U_1, \ldots, U_p)$ and assuming

$$\mathbb{P}(T \leq C \mid S, T, \boldsymbol{U}, V) = \mathbb{P}(T \leq C \mid T, \boldsymbol{U}, V).$$

- But, it is difficult to use them directly for continuous covariate without developing smoothing techniques (Meira-Machado *et al.*, 2014).

Sketch of the proof

- Directly based on Stute (1995) proof, our strategy is in 2 steps: prove CLT when φ vanishes to the right of some $\nu < \tau_Z$ and then extend it on $[0, \tau_Z]$.
- For the first step, we show with similar arguments that $\widehat{S}_n^{(v)}$ admit the following representation for $t < \nu$

$$\widehat{S}_n^{(v)}(\varphi) = \frac{1}{n}\sum_{i=1}^{n} \frac{\varphi(Y_i, Z_i)\,\delta_i J_i^{(v)}}{1 - G(Z_i-)} + \frac{1}{n}\sum_{i=1}^{n}\left[\lambda_1^{(v)}(Z_i)(1-\delta_i) - \lambda_2^{(v)}(Z_i)\right] + R_n^{(v)},$$

where $|R_n^{(v)}| = O\left(n^{-1}\ln n\right)$ w.p.1,

$$\lambda_1^{(v)}(x) = \frac{1}{1 - M(x)} \int \frac{\varphi(s,t)\,\mathbb{1}_{\{x<t<\tau_Z\}}}{(1 - G(t-))} M^{(v)}(ds, dt),$$

$$M^{(v)}(y,z) = \mathbb{P}(Y \le y, Z \le z, \delta = 1, V = v),$$

and

$$\lambda_2^{(v)}(x) = \int \frac{\lambda_1^{(v)}(\tau)\,\mathbb{1}_{\{\tau<x\}}}{1 - M(\tau)} M_0(d\tau),\ M_0(z) = \mathbb{P}(Z \le z, \delta = 0).$$

Transition probabilities and association measures

Proposition (Asymptotic results for transition probabilities)

$\widehat{p}_{0e}(s,t,\eta)$, $\widehat{p}_{ee}(s,t)$ and $\widehat{p}_{ed}(s,t,\eta,\zeta)$ are consistent w.p.1 if the support of Z is included in that of C. These estimators admit a weak convergence result.

- Provide estimators when the Markov assumption is released.
- Application to goodness-of-fit testing. Practitioners often use simple multi-state Markov model or Cox semi-Markov model. Misspecification may lead to important errors.

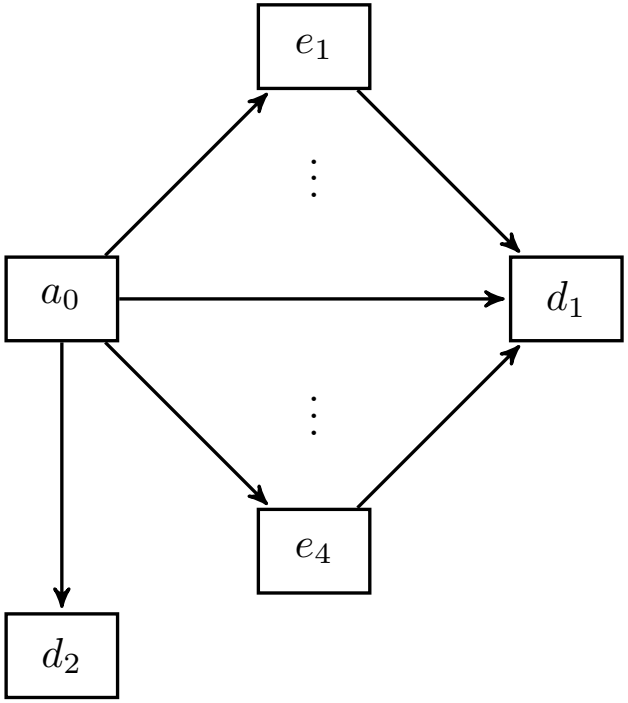
Proposition (Asymptotic results for assoaciation measures)

$\widehat{\pi}_{0n}^{(e,d)}(s,t)$ is consistent w.p.1 if the support of Z is included in that of C and admits a weak convergence result.

Possible applications to goodness-of-fit testing for models based on cross-odds ratios specification (see Scheike and Sun, 2012).

LTC insurance data

- Database from a large French LTC insurer (see also Guibert and Planchet, 2014)
- $209,939$ contracts observed on period 1998-2010 after cleaning the database and almost 70% are censored

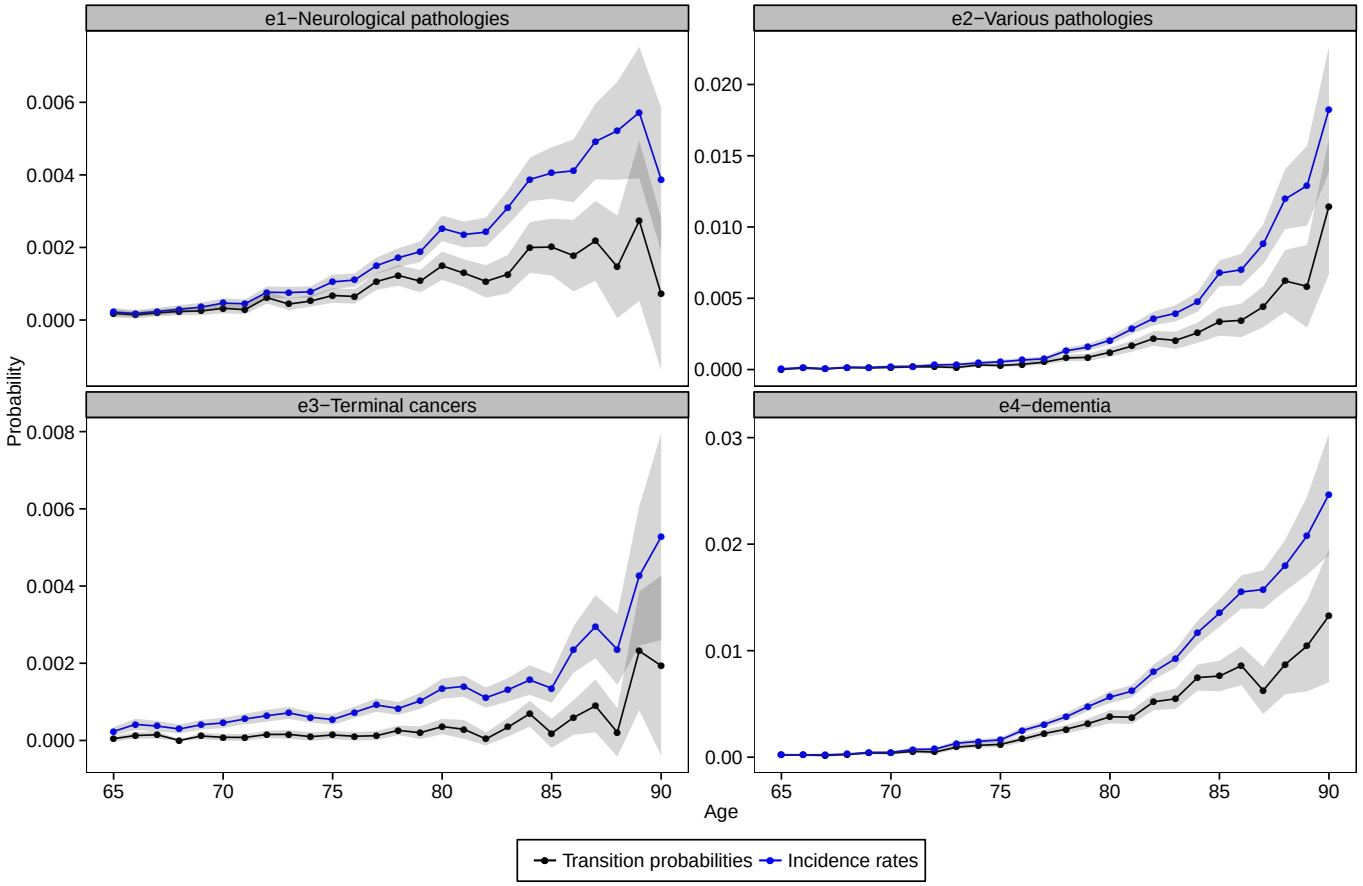


4 types of pathology and 2 direct exit causes.

	Exit causes	$\%$
e_1	Neurologic pathologies	2.5%
e_2	Various pathologies	2.7%
e_3	Terminal cancers	2.4%
e_4	Dementia	5.4%
d_1	Death	52.2%
d_2	Cancel	34.8%

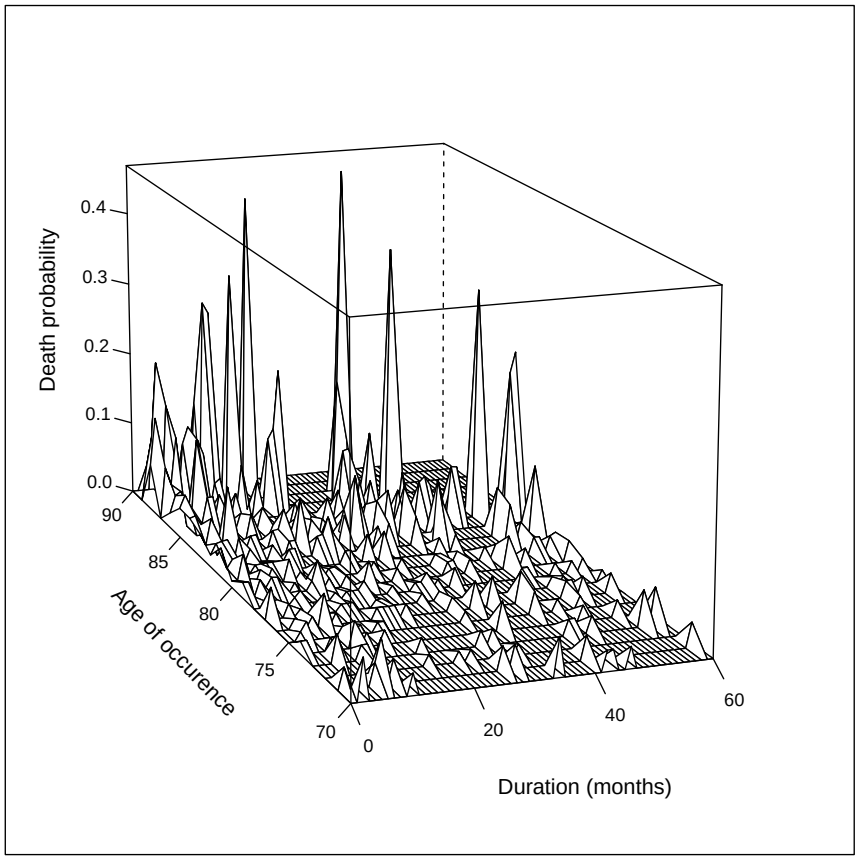
Transition probabilities

- Estimate annual transition probabilities to become dependent and stay at least one month in a disability state
- Compute pointwise 95% confidence interval from 500 bootstrap resamples

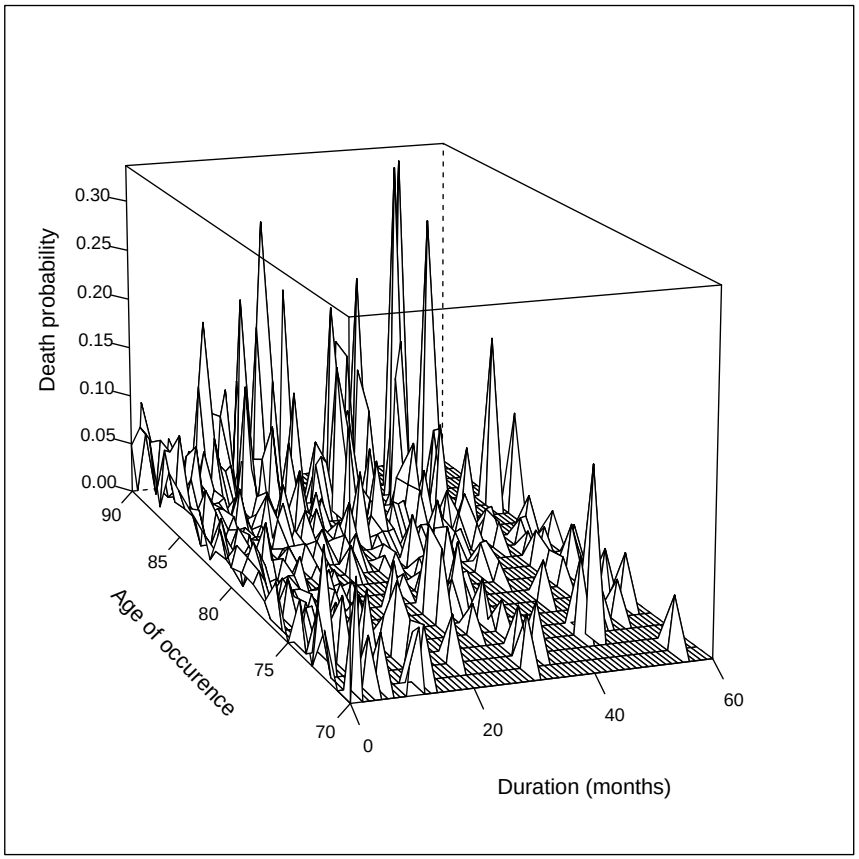


Transition probabilities

- Estimated surface of monthly death rates from each dependent state but quality is low due to missing data



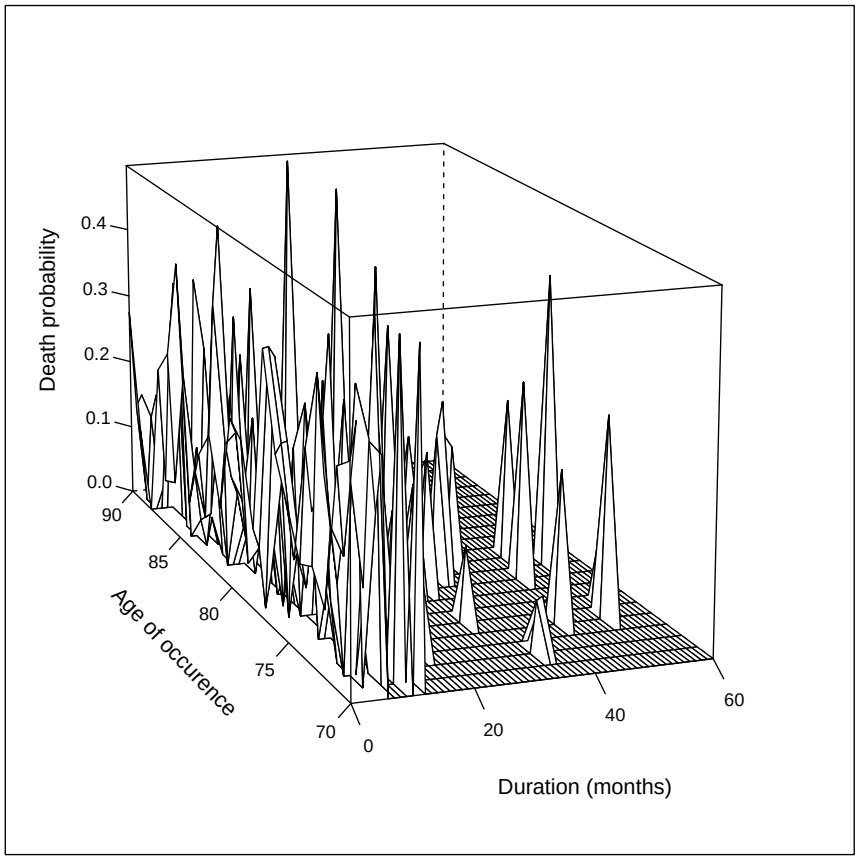
e_1-Neurologic pathologies.



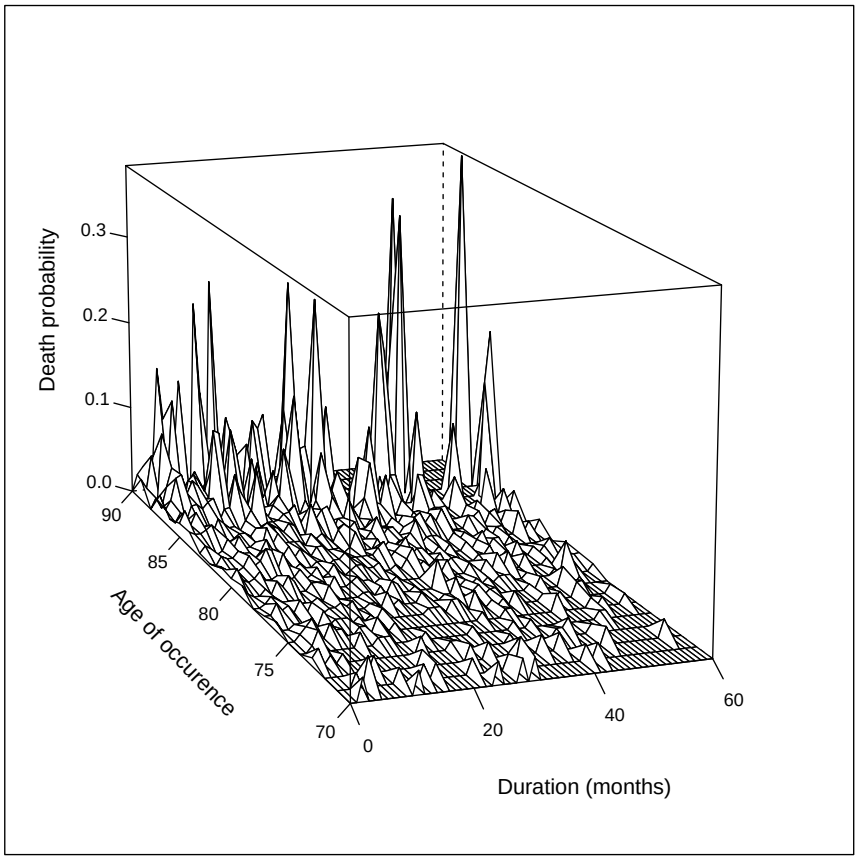
e_2-Various pathologies.

Transition probabilities

- Estimated surface of monthly death rates from each dependent state but quality is low due to missing data



e_3-Terminal cancers.



e_4-Dementia.

Summary

- Non-parametric estimation for AJ-integrals that we apply to estimate this type of acyclic multi-state model under right-censoring
- These estimators and their properties stay valid if we consider covariates
- We provide new non-parametric estimators for transition probabilities
- We exhibit a non-parametric estimator for local association measures
- We application them to LTC insurance data to estimate key probabilities

- Many outlooks
  - Consider framework for regression models
  - Regard more relevant bootstrap approach for AJ-integrals estimation
  - Develop semi-parametric approaches based on our local association measure

Thank you for your kind attention.

Some References I

Andersen, P. K., Borgan, r., Gill, R. D., and Keiding, N. (1993). *Statistical Models Based on Counting Processes*. Springer Series in Statistics. Springer-Verlag New York Inc.

Cheng, Y., Fine, J. P., and Kosorok, M. R. (2007). Nonparametric Association Analysis of Bivariate Competing-Risks Data. *Journal of the American Statistical Association*, **102**(480), 1407–1415.

Christiansen, M. C. (2012). Multistate models in health insurance. *Advances in Statistical Analysis*, **96**(2), 155–186.

Datta, S. and Satten, G. A. (2002). Estimation of Integrated Transition Hazards and Stage Occupation Probabilities for Non-Markov Systems Under Dependent Censoring. *Biometrics*, **58**(4), 792–802.

Guibert, Q. and Planchet, F. (2014). Construction de lois d'expérience en présence d'évènements concurrents – Application à l'estimation des lois d'incidence d'un contrat dépendance. *Bulletin Français d'Actuariat*, **13**(27), 5–28.

Haberman, S. and Pitacco, E. (1998). *Actuarial Models for Disability Insurance*. Chapman and Hall/CRC, 1 edition.

Lakhal, L., Rivest, L.-P., and Abdous, B. (2008). Estimating survival and association in a semicompeting risks model. *Biometrics*, **64**(1), 180–188.

Meira-Machado, L., de Uña-Álvarez, J., and Cadarso-Suárez, C. (2006). Nonparametric estimation of transition probabilities in a non-Markov illness–death model. *Lifetime Data Analysis*, **12**(3), 325–344.

Some References II

Meira-Machado, L., de Uña-Álvarez, J., and Datta, S. (2014). Nonparametric estimation of conditional transition probabilities in a non-Markov illness-death model. *Computational Statistics*, pages 1–21.

Peng, L. and Fine, J. P. (2006). Nonparametric estimation with left-truncated semicompeting risks data. *Biometrika*, **93**(2), 367–383.

Scheike, T. H. and Sun, Y. (2012). On cross-odds ratio for multivariate competing risks data. *Biostatistics*, **13**(4), 680–694.

Stute, W. (1993). Consistent Estimation Under Random Censorship When Covariables Are Present. *Journal of Multivariate Analysis*, **45**(1), 89–103.

Stute, W. (1995). The Central Limit Theorem Under Random Censorship. *The Annals of Statistics*, **23**(2), 422–439.

Suzukawa, A. (2002). Asymptotic properties of Aalen-Johansen integrals for competing risks data. *Journal of the Japan Statistical Society*, **32**, 77–93.