Robust evaluation of SCR for participating Life Insurance under Solvency II

D. Hainaut[1] (P. Devolder[1], A. Pelsser)

[1] ISBA, Université Catholique de Louvain

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Motivation

1) Mortality/longevity risks make the market incomplete then there is not an unique risk neutral measure for the valuation of the SCR.

2) Potential model misspecifications and uncertainty about economic/mortality parameters.

Contributions of this article:

a) Analytical formulas for the calculation of BE & NAV of participating life insurances with death benefits. Approximations of the SCR.

b) Model misspecifications and uncertainty managed by considering the worst case in a set of equivalent risk neutral measures, constrained by the entropy of the change of measure.

Financial Market

Financial market: d assets $S_t = (S_t^0, S_t^1, \dots, S_t^{d-1})^\top$. The asset S_t^0 is a bond (discount factor). If $B_t^S = (B_t^0, \dots, B_t^{d-1})$ are BM,

$$\frac{dS_t}{S_t} = \mu_S dt + \Sigma_S dB_t^S \,,$$

where $\mu_S = (\mu_0, ..., \mu_{d-1})$ and Σ_S is a $(d-1) \times (d-1)$ matrix.

Investment strategy: constant proportion: $\theta_S = (\theta_0, \dots, \theta_{d-1})^\top$. The total asset, denoted by A_t, is equal to $\theta'_S S_t$ and its dynamics is defined by:

$$\frac{dA_t}{A_t} = \theta_S^\top \mu_S dt + \theta_S^\top \Sigma_S dB_t^S .$$

Mortality rates

Individual of age x. The probability of survival till age $x+t$ is equal to the next expression:

$$ {}_t p_x := P(\tau > t) = \mathbb{E}\left(e^{-\int_0^t d\lambda_s} \,|\, \mathcal{F}_0 \right) . $$

If $\mu_d(s)$ is function of time where the hazard rate is a random process led by the following dynamics ($B_s = (B_t^0, \dots, B_t^d)$)

$$ d\lambda_s = \left(\mu_d(s) - \frac{\sigma_d^\top \sigma_d}{2} \right) ds + \sigma_d^\top dB_s \, , $$

We define a "mortality account", S_t^M, with a growth rate equal to the hazard rate:

$$ S_t^M := e^{\int_0^t d\lambda_s} $$

S_t^M is a geometric Brownian motion $\frac{dS_t^M}{S_t^M} = \mu_d(t)dt + \sigma_d^\top dB_t$ and is such that ${}_{T-t}p_{x+t} = \mathbb{E}\left(\frac{S_t^M}{S_T^M} | \mathcal{F}_t \right)$.

Liabilities

Saving contract purchased at age x

- Minimum guarantee rate g
- ρ participation rate to the appreciation of A_t, credited every periods of length Δ (times $t_1, \ldots, t_n = T$)

Survival: lump sum L_T paid at expiry T :

$$L_T = C \prod_{k=1}^{n} \left(e^{g\Delta} + \left[\rho \frac{A_{t_k}}{A_{t_{k-1}}} - e^{g\Delta} \right]_+ \right)$$

Death guarantee: if $t_{j-1} \leq \tau \leq t_j$, a multiple α of the saved capital is paid:

$$L_{t_j} = \alpha\, C \prod_{k=1}^{j} \left(e^{g\Delta} + \left[\rho \frac{A_{t_k}}{A_{t_{k-1}}} - e^{g\Delta} \right]_+ \right)$$

Best estimate

If the risk neutral measure is denoted Q, the Best Estimate at time t_i is given by

$$
\begin{aligned}
BE_i^Q = {} & C \prod_{k=1}^{i} \frac{S^0_{t_{k-1}}}{S^0_{t_k}} \left(e^{g\Delta} + \left[\rho \frac{A_{t_k}}{A_{t_{k-1}}} - e^{g\Delta} \right]_+ \right) \times \\
& \Bigg[\alpha \sum_{j=i+1}^{n} \prod_{k=i+1}^{j} \mathbb{E}^Q \Bigg(\Bigg(\frac{S^M_{t_{k-1}}}{S^M_{t_k \wedge t_{j-1}}} \frac{S^0_{t_{k-1}}}{S^0_{t_k}} - \frac{S^M_{t_{k-1}}}{S^M_{t_k}} \frac{S^0_{t_{k-1}}}{S^0_{t_k}} \Bigg) \\
& \left(e^{g\Delta} + \left[\rho \frac{A_{t_k}}{A_{t_{k-1}}} - e^{g\Delta} \right]_+ \right) |\mathcal{F}_0 \Bigg) + \sum_{j=i+1}^{n} \prod_{k=i+1}^{j} \\
& \mathbb{E}^Q \left(\frac{S^M_{t_{k-1}}}{S^M_{t_k}} \frac{S^0_{t_{k-1}}}{S^0_{t_k}} \left(e^{g\Delta} + \left[\rho \frac{A_{t_k}}{A_{t_{k-1}}} - e^{g\Delta} \right]_+ \right) |\mathcal{F}_{t_i} \right) \Bigg] .
\end{aligned}
\tag{1}
$$

Problem: the market is **incomplete** ($d+1$ BM , d assets, mortality) + **uncertainty** on parameters.

Risk neutral measures

We rewrite the joint dynamics of financial assets and the mortality account as follows:

$$\underbrace{\begin{pmatrix} \frac{dS_t^S}{S_t^S} \\ \frac{dS_t^M}{S_t^M} \end{pmatrix}}_{\frac{dS_t}{S_t}} = \underbrace{\begin{pmatrix} \mu_S \\ \mu_d(t) \end{pmatrix}}_{\mu} dt + \underbrace{\begin{pmatrix} \Sigma_S & 0_{d-1} \\ \sigma_{d\,(0:d-1)} & \sigma_{d\,(d)} \end{pmatrix}}_{\Sigma} \underbrace{\begin{pmatrix} dB_t^S \\ dB_t^d \end{pmatrix}}_{dB_t}$$

Any equivalent measure Q to the real one, is defined by the following Radon Nikodym derivative

$$\left.\frac{dQ}{dP}\right|_t = \exp\left(-\frac{1}{2}\int_0^t \Upsilon_s^T \Upsilon_s ds - \int_0^t \Upsilon_s^T dB_s\right),$$

if we choose $\Upsilon := \begin{pmatrix} \Sigma_S^{-1}\left(\mu_S - r\mathbf{1}_d\right) \\ \upsilon \end{pmatrix}$ then discounted assets prices $\frac{S_t^i}{S_t^0}$ are martingales under Q whatever the parameter $\upsilon \in \mathbb{R}$ and $r \in \mathbb{R}^+$! Uncertainty on both (v, r) and $r \neq \mu_0$...

Risk neutral measures

Under Q , we have

$$\frac{dS_t}{S_t} = \left(\begin{array}{c} r\mathbf{1} \\ \underbrace{\mu_d(t) + \sigma_d^\top \Upsilon}_{=\mu_M(t)} \end{array} \right) dt + \Sigma dW_t .$$

We consider as eligible pairs (v, r), those for which the entropy of $\frac{dQ}{dP}$ is bounded by

$$\mathbb{E}^Q \left(\ln \left. \frac{dQ}{dP} \right|_t \mathcal{F}_0 \right) \leq \frac{1}{2} U^2 t \qquad \forall t \in \mathbb{R}^+ .$$

where $U \in \mathbb{R}^+$ is a parameter chosen by the insurer and directly related to its level of risk aversion.

This constraint is equivalent to impose a bound on the integral

$$\frac{1}{2} \int_0^t \Upsilon_s^T \Upsilon_s ds \quad \leq \quad \frac{1}{2} U^2 t \qquad \forall t \in \mathbb{R}^+ .$$

Risk neutral measures

Set of equivalent measures

If the entropy is bounded, admissible pairs (r, v) are in the domain

$$\mathcal{A} = \{v, r) \,|\, v^{-} \leq v \leq v^{+} \,,\, r^{-}(v) \leq r \leq r^{+}(v)\}$$

where

$$v^{\pm} = \pm\sqrt{\frac{\left(\mu_S^{\top}\left(\Sigma_S\Sigma_S^{\top}\right)^{-1}\mathbf{1}\right)^2}{\mathbf{1}^{\top}\left(\Sigma_S\Sigma_S^{\top}\right)^{-1}\mathbf{1}} - \left(\mu_S^{\top}\left(\Sigma_S\Sigma_S^{\top}\right)^{-1}\mu_S - U^2\right)}$$

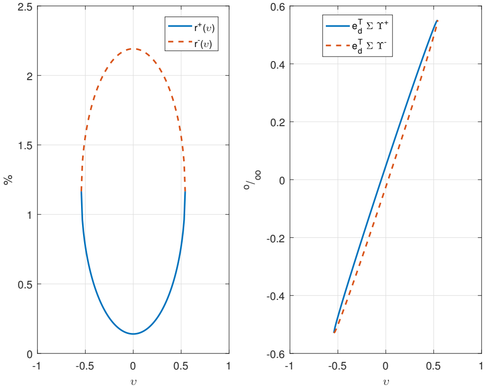
$$r^{\pm}(v) \quad = \quad \frac{2\mu_S^{\top}\left(\Sigma_S\Sigma_S^{\top}\right)^{-1}\mathbf{1} \pm \sqrt{D(v)}}{2\,\mathbf{1}^{\top}\left(\Sigma_S\Sigma_S^{\top}\right)^{-1}\mathbf{1}}\,,$$

$$D(v) = 4\left(\mu_S^{\top}\left(\Sigma_S\Sigma_S^{\top}\right)^{-1}\mathbf{1}\right)^2 - 4\left(\mathbf{1}^{\top}\left(\Sigma_S\Sigma_S^{\top}\right)^{-1}\mathbf{1}\right) \times\left(\mu_S^{\top}\left(\Sigma_S\Sigma_S^{\top}\right)^{-1}\mu_S - U^2 + v^2\right).$$

Risk neutral measures

The **robust best estimate** is defined by the maximum of non robust BE's over the set $\mathcal{A}$:

$$BE_i = \max_{\{\upsilon,r\}\in\mathcal{A}} BE_i^{Q(v,r)} \quad i = 0, 1, \dots T \tag{2}$$



Robust Best Estimate

The robust BE is easy to determine because $BE_i^{Q(v,r)}$ admits a closed from expression.

BE

Let us define

$$\Psi^1(r,v,i,k) = \mathbb{E}^Q\left(\mathbb{E}^Q\left(\frac{S^M_{t_{k-1}}S^0_{t_{k-1}}}{S^M_{t_k}S^0_{t_k}}\left[\rho\frac{A_{t_k}}{A_{t_{k-1}}} - e^{g\Delta}\right]_+ |\mathcal{F}_{t_{k-1}}\right)|\mathcal{F}_{t_i}\right)$$

$$= e^{-\left(r+\frac{1}{\Delta}\int_{t_{k-1}}^{t_k}\mu_M(s)ds\right)\Delta}\left(\rho e^{\left(r-\theta^\top\Sigma\Sigma^\top(e_0+e_d)\right)\Delta}\Phi\left(d_1(\theta)\right) - e^{g\Delta}\Phi\left(d_2(\theta)\right)\right)$$

$$\begin{aligned}\Psi^2(r,i,k) &:= \mathbb{E}^Q\left(S^0_{t_{k-1}}\mathbb{E}^Q\left(\frac{1}{S^0_{t_k}}\left[\rho\frac{A_{t_k}}{A_{t_{k-1}}} - e^{g\Delta}\right]_+ |\mathcal{F}_{t_{k-1}}\right)|\mathcal{F}_{t_i}\right)\\ &= e^{-r\Delta}\left(\rho e^{\left(r-\theta^\top\Sigma\Sigma^\top e_0\right)\Delta}\Phi\left(c_1(\theta)\right) - e^{g\Delta}\Phi\left(c_2(\theta)\right)\right).\end{aligned}$$

Where

$$d_1(\theta) = \frac{\ln\rho - \left(g-\mu_A-\sigma_A^2\right)\Delta}{\sigma_A\sqrt{\Delta}} \qquad c_1(\theta) = \frac{\ln\rho - \left(g-r-\frac{1}{2}\theta^\top\Sigma\Sigma^T\theta+\theta^\top\Sigma\Sigma^\top e_0\right)\Delta}{\sqrt{\theta^\top\Sigma\Sigma^\top\theta}\sqrt{\Delta}},$$

Robust Best Estimate

BE Cont'd

And $d_2(\theta) = d_1(\theta) - \sigma_A\sqrt{\Delta}$, $c_2(\theta) = c_1(\theta) - \sqrt{\theta^\top\Sigma\Sigma^\top\theta}\sqrt{\Delta}$. The best estimate is then

$$BE_i \quad = \quad C\prod_{k=1}^{i}\frac{S^0_{t_{k-1}}}{S^0_{t_k}}\left(e^{g\Delta}+\left[\rho\frac{A_{t_k}}{A_{t_{k-1}}}-e^{g\Delta}\right]_+\right)\times V(i)$$

where $V(i)$ is defined as follows:

$$\begin{aligned}
V(i) \quad := \quad & \max_{\{\upsilon,r\}\in\mathcal{A}}\Bigg[\alpha\sum_{j=i+1}^{n}\prod_{k=i+1}^{j}\Bigg(e^{(g-r)\Delta-\int_{t_{k-1}}^{t_k\wedge t_{j-1}}\mu_M(s)ds} \\
& +\left(\Psi^1(r,\upsilon,i,k)1_{\{k<j\}}+\Psi^2(r,i,j)1_{\{k=j\}}\right)\Bigg) \\
& -\alpha\sum_{j=i+1}^{n}\prod_{k=i+1}^{j}\left(e^{(g-r)\Delta-\int_{t_{k-1}}^{t_k}\mu_M(s)ds}+\Psi^1(r,\upsilon,i,k)\right) \\
& +\prod_{k=i+1}^{n}\left(e^{(g-r)\Delta-\int_{t_{k-1}}^{t_k}\mu_M(s)ds}+\Psi^1(r,\upsilon,0,k)\right)\Bigg].
\end{aligned}$$

Robust NAV & SCR

The NAV is then a measure of profitability defined by:

$$NAV_0 \quad := \quad A_0 - BE_0 \tag{3}$$

The SCR for a level of confidence $\beta = 0.5\%$ is

$$P\left(NAV_0 - NAV_{t_1} \geq SCR_0^{reg}\right) \quad = \quad \beta\,.$$

This is simply an approached value of the 0.5% Value at Risk (VaR) of the NAV:

$$P\left(\mathbb{E}\left(NAV_{t_1}\,|\,\mathcal{F}_0\right) - NAV_{t_1} \geq SCR_0\right) \quad = \quad \beta\,.$$

under the assumption that $NAV_0 = \mathbb{E}\left(NAV_{t_1}\,|\,\mathcal{F}_0\right)$. A natural definition for the SCR & prospective SCR is then

$$P\left(\mathbb{E}\left(NAV_{t_{j+1}}\,|\,\mathcal{F}_{t_0}\right) - NAV_{t_{j+1}} \geq SCR_{t_j}\right) \quad = \quad 1-(1-\beta)^{j+1}\,,$$

Robust NAV & SCR

Distribution of the NAV is unknown but we can approximate it by matching a random to its moments, that admits closed form expressions

NAV Moments

The NAV moment of order u at time t_j:

$$\mathbb{E}\left(NAV_{t_j}^{u}|\mathcal{F}_0\right)=\sum_{m=0}^{u}\left[\binom{u}{m}\left(-C\times V(j)\right)^{m}A_0^{u-m}\prod_{k=1}^{j}\left[\sum_{l=0}^{m}\right.\right.$$

$$\left.\left.\left[\binom{m}{l}\left(e^{g\Delta}\right)^{m-l}\sum_{p=0}^{l}\left[\binom{l}{p}\left(-e^{g\Delta}\right)^{l-p}(\rho)^{p}\,h(k,u,l,m,p)\right]\right]\right]\right]$$

where $h(k,u,l,m,p)$ is the following expectation under the real measure:

$$h(.)\quad=\quad\mathbb{E}\left(\frac{S^M_{t_{k-1}}}{S^M_{t_k}}\left(\frac{A_{t_k}}{A_{t_{k-1}}}\right)^{u-m+p}\left(\frac{S^0_{t_{k-1}}}{S^0_{t_k}}\right)^{m}\left(\mathbb{I}_{\left\{\rho\frac{A_{t_k}}{A_{t_{k-1}}}>e^{g\Delta}\right\}}\right)^{\mathbb{I}_{\{l\neq 0\}}}|\mathcal{F}_0\right)$$

Robust NAV & SCR

The function $h(k,u,l,m,p)$ admits a closed form expression. E.g. If $l\neq 0$, $h(k,u,l,m,p)$ is equal to

$$h(.) = \exp\left((u-m+p)\left(\theta_S^\top\mu_S - \frac{1}{2}\theta^\top\Sigma\Sigma^\top\theta\right)\Delta - m\left(\mu_0 - \frac{1}{2}e_0^\top\Sigma\Sigma^\top e_0\right)\Delta\right)$$

$$\times\exp\left(-\int_{t_{k-1}}^{t_k}\mu_d(s)ds + \frac{1}{2}e_d^\top\Sigma\Sigma^\top e_d\Delta\right)\times\exp\left(\frac{1}{2}\gamma_Y^2\right)\left(1-\Phi(x_{min}-\gamma_Y\rho_{XY})\right)$$

where $\Phi(.)$ is the cdf of a $N(0,1)$ and γ_Y, ρ_{XY} , x_{min} are constant and equal to

$$\gamma_Y \quad := \quad \sqrt{((u-m+p)\theta - m\,e_0 - e_d)^\top\,\Sigma\Sigma^\top\,((u-m+p)\theta - m\,e_0 - e_d)}\sqrt{\Delta}\,.$$

$$\rho_{XY} \quad := \quad \frac{\left[\theta^\top\Sigma\Sigma^\top\,((u-m+p)\theta - m\,e_0 - e_d)\right]\sqrt{\Delta}}{\gamma_Y\sqrt{\theta^\top\Sigma\Sigma^\top\theta}}\,.$$

$$x_{min} := \frac{g\Delta - \ln\rho - \left(\theta_S^\top\mu_S - \frac{1}{2}\theta^\top\Sigma\Sigma^\top\theta\right)\Delta}{\sqrt{\theta^\top\Sigma\Sigma^\top\theta}\sqrt{\Delta}}$$

Robust NAV & SCR

The distribution of the NAV for the SCR calculation is approached:

1. **By a Gaussian law**

$$\tilde{NAV}_{t_j} \sim N\left(\mu_j^{gaus}, \sigma_j^{gaus}\right)$$

(matching of the 2 first NAV moments for determining $\mu_j^{gaus}, \sigma_j^{gaus}$).

2. **By a NIG law**

$$\tilde{NAV}_{t_j} \sim NIG\left(\mu_j^{gaus}, \alpha_j^{nig}, \beta_j^{nig}, \delta_j^{nig}\right)$$

(matching of the 4 first NAV moments for determining $\mu_j^{gaus}, \alpha_j^{nig}, \beta_j^{nig}, \delta_j^{nig}$)

Numerical illustration

2 assets: S^0 cash and S^1.

	Value		Value
μ_S	$\begin{pmatrix} 1\% \\ 5\% \end{pmatrix}$	Std. Dev.	$\begin{pmatrix} 2\% \\ 15\% \end{pmatrix}$
$\mu_d(t)$	Gompertz-Makeham	σ_d	0.10%
	Correlation	$\begin{pmatrix} S^0 \\ S^1 \\ S^M \end{pmatrix} = \begin{pmatrix} 1 & -0.20 & 0.05 \\ -0.20 & 1 & 0.05 \\ 0.05 & 0.05 & 1 \end{pmatrix}$	

Numerical illustration

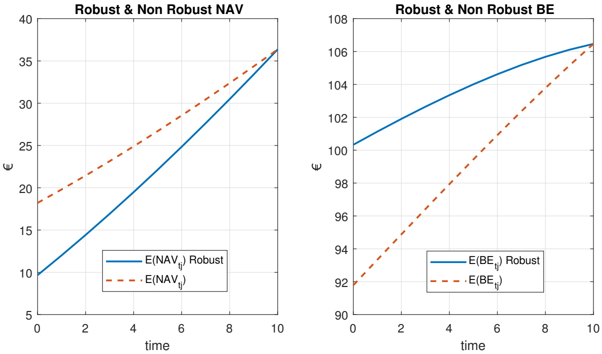
Parameters of the participating policy:

Parameters	Value	Parameters	Value
g	1%	α	1
ρ	90%	C	100
x	50	T	10
θ_1	60%	θ_2	40%
A_0	110	U	0.75

The non robust BE and NAV are evaluated with the assumptions that $r =$ risk free rate and $\upsilon = 0$

Numerical illustration

Robust NAV and BE ($NAV_0 \approx 10$, $BE_0 \approx 100$) are more conservative than their non robust equivalents ($NAV_0 \approx 18$, $BE_0 \approx 92$)



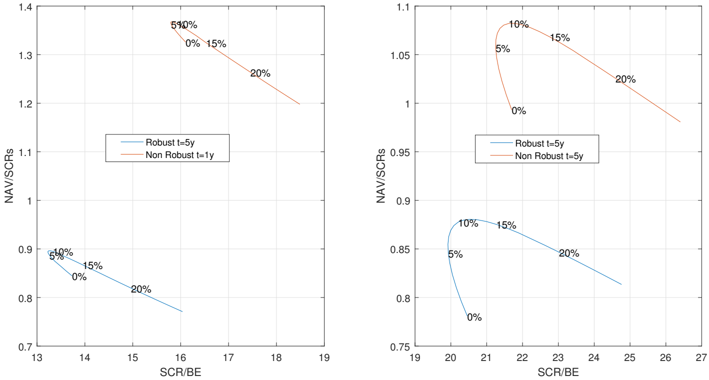
Numerical illustration

The impact of Robustness on SCR is limited. The Gaussian approximation underestimates the SCR.

	NIG approximation		Non Robust	Gaussian approximation	
t	$\frac{\text{R. SCR}}{\text{BE}}$(%)	R. SCR(€)	SCR(€)	$\frac{\text{R. SCR}}{\text{BE}}$(%)	R. SCR(€)
1	20.89	21.12	21.65	16.71	16.90
2	25.21	25.69	26.10	21.90	22.32
3	28.02	28.75	29.15	25.70	26.37
4	30.20	31.20	31.60	28.87	29.84
5	32.08	35.38	33.75	31.69	32.96

Numerical illustration

ALM strategy: the performance, $\frac{\mathbb{E}(NAV_t)}{SCR_t}$, and risk, $\frac{SCR_t}{\mathbb{E}(BE_t)}$, can be adjusted by the asset allocation. Here measured at year 1 and 5 (% of S^1).



Conclusions

- Misspecifications and uncertainty are managed by considering a set of equivalent measures, with bounded entropy.
- This bound can be calibrated to match BE and SCR estimates from a more complex internal model.
- The relative simplicity of the model allows us to obtain closed form expressions for most of quantities of interest as BE and NAV moments.
- Potential shortcomings induced by the Brownian dynamics are partly compensated by the robustness of the procedure.
- Using A robust model leads to a prudent estimate of the NAV.
- However, this does not systematically increase the solvency capital requirement.
- Efficient tool to optimize the asset allocation strategy.

Thanks for your attention!

References: Robust SCR valuation of participating life insurance policies in the Solvency II Framework. Insurance: Mathematics & Economics, Volume 79, March 2018, Pages 107-123