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Risk Management with Multiple Value-at-Risk Constraints

Thai Nguyen, University of Ulm

About the speaker

- **Thai Nguyen** is a research associate (postdoct) at University of Ulm.
- Before joining the institute of Insurance Science (Ulm), he spent 4 years for a PhD in applied mathematics at University of Rouen (France).
- His research focuses on Financial and Insurance Mathematics.

Agenda

Introduction

One-VaR constrained problem

Two-VaR constrained problem

Numerical examples for power utility

Conclusion

Is VaR regulation good?

- a VaR measurement is based on the examination of the percentiles of the distribution $\longrightarrow$ easy to interpret,
- VaR can quantify statistically different risks and different components of a risk, and aggregate them into a single quantitative indicator.
- RM based on a singular terminal VaR constraint has been criticized by many empirical studies as it only controls the probability of a default.
- Because a VaR regulation ignores the magnitude of losses $\longrightarrow$ larger trading losses.
- VaR remains popular among practitioners and regulators.

Literature review

- Terminal VaR constraint: Basak und Shapiro (2001); Boyle und Tian (2007)

$$\max_{X_T} \mathbf{E}[U(X_T)] \quad \text{s.t.} \quad \mathbf{E}[\xi_T X_T] \le x_0 \quad \text{and} \quad \mathbf{P}(X_T < L) < \beta, \tag{1}$$

  where $\beta \in (0,1)$ is the maximum default probability.

- Dynamic VaR incremental constraint: Cuoco et al. (2008).
- In practice, VaR constraints are imposed over a yearly, monthly or weekly time horizon rather than continuously.

Related literature

- Schyns et al. (2010): multiple periodic VaR constraints in binomial models.
- Kraft und Steffensen (2013): possible intertemporal VaR or Expected Shortfall constraints (option-based approach+HJB)
- Cuoco und Liu (2006): capital requirement (a multiple of its reported VaR in each time period, VaR is determined *endogenously*)
- Shi und Werker (2012): two VaR constraints for long-term investors (numerically+stochastic interest rates).

What we do

- We provide an alternative (mathematically rigorously proven) reason to advocate the use of VaR.
- We extend the terminal VaR constrained problem by supposing that there are multiple VaR constraints $\Longrightarrow$ closed form solution.
- We compare the multiple-VaR solution with the terminal-VaR: analytically and numerically.
- Typical drawbacks of terminal-VaR RM can be overcome substantially by a realistic regulation with *multiple* VaR constraints.

One-VaR constrained solution

- Assume a complete Black-Scholes economy with one risky asset
- Utility function U (increasing, concave, differentiable) satisfies

$$\mathbf{E}[\xi_T I(y\xi_T)] < \infty \quad \text{and} \quad \mathbf{E}[U(I(y\xi_T))] < \infty, \forall y > 0, \quad I = (U')^{-1},$$

- For $x_0 \geq x_0^{min} := \mathbf{E}[L\xi_T \mathbf{1}_{\xi_T \leq \bar{\xi}}]$, the optimal terminal wealth is given by

$$\widehat{X}_T(\widehat{y}^{BS}, \xi_T) := I(\widehat{y}^{BS}\xi_T)\mathbf{1}_{\xi_T < \underline{\xi}} + L\mathbf{1}_{\underline{\xi} \leq \xi_T < \bar{\xi}} + I(\widehat{y}^{BS}\xi_T)\mathbf{1}_{\xi_T \geq \bar{\xi}}, \qquad (2)$$

where $\widehat{y}^{BS}$ is determined by $\mathbf{E}[\xi_T \widehat{X}_T] = x_0$, $\mathbf{P}(\xi_T > \bar{\xi}) = \beta$ and $\widehat{y}^{BS}\underline{\xi} = U'(L)$.

Comments

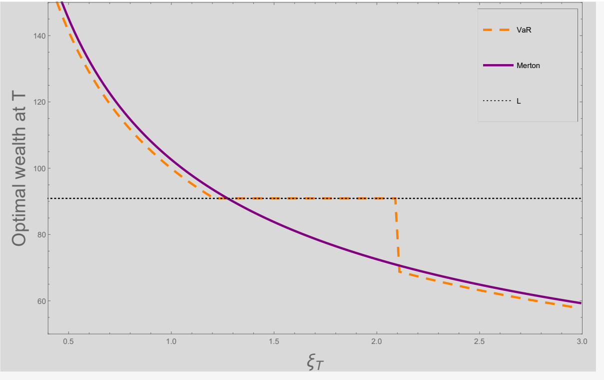

Figure: Optimal terminal wealth

Comments

- The uninsured loss states $\xi_T \geq \bar{\xi}$ is characterised by the quantity $\bar{\xi}$ (*independently* of preferences and the endowment).
- The risk manager will not insure losses in these states (occur with probability β) and hedge against the intermediate-loss states $\underline{\xi} \leq \xi_T < \bar{\xi}$.
- For good or bad market scenarios, the VaR terminal wealth performs worse than the unregulated terminal wealth for good and bad states.
- Losses could be more severe than those of the unconstrained problem is the most critical drawback of VaR-type regulations.

Two-VaR problem

- A terminal VaR constraint exclusively does not appropriately reflect a realistic regulatory framework like Solvency II.
- Adding multiple constraints partially mitigates the shifting of the risks into the tail:

$$\max_{(\pi_t)_{t\in[0,T]}} \mathbf{E}[U(X_T)], \tag{3}$$

$$\text{s.t.} \quad \mathbf{E}[\xi_T X_T] \leq x_0, \quad \text{and} \quad \mathbf{P}(X_{t_1} < L_{t_1}) < \beta_1, \quad \mathbf{P}(X_T < L_T | \mathcal{F}_{t_1}) < \beta_2.$$

- How to solve this problem?

Recursive solution

- Setting $X_{t_1} = x$, we first need to solve the problem

$$\max_{(\pi_t)_{t_1 \le t \le T}} \mathbf{E}[U(X_T)|X_{t_1} = x], \tag{4}$$

under the constraints

$$\mathbf{E}[\xi_T X_T | X_{t_1} = x] \le x\xi_{t_1} \quad \text{and} \quad \mathbf{P}(X_T < L_T | X_{t_1} = x) < \beta_2.$$

- The t_1-time minimum capital I_{t_1} defined by

$$I_{t_1} := \mathbf{E}[L_T \xi_T \xi_{t_1}^{-1} \mathbf{1}_{\xi_T \le \bar{\xi}_2} | \mathcal{F}_{t_1}], \tag{5}$$

Indirect utility at time t_1

- Given $x \geq l_{t_1}$, the indirect value function satisfies

$$\widehat{V}_{t_1}^{2VaR}(x) = \widehat{V}_{t_1}(x)\mathbf{1}_{x<x^{bind}} + \widehat{V}_{t_1}^{M}(x)\mathbf{1}_{x\geq x^{bind}}, \quad x \in (l_{t_1}, \infty), \tag{6}$$

  where $\widehat{V}_{t_1}(x) := \mathbf{E}[U(\widehat{X}_T)|X_{t_1} = x]$ and x^{bind} is explicitly defined.

- $\widehat{V}_{t_1}^{2VaR}$ is a globally strictly concave function satisfying Inada's condition on (l_{t_1}, ∞).

- Let $I_{t_1}^{2VaR,(1)} := ((\widehat{V}_{t_1}^{2VaR})')^{-1}$.

Theorem

Define $\bar{\xi}_1$ *such that* $\mathbf{P}(\xi_{t_1} > \bar{\xi}_1) = \beta_1$ *and assume that*

$$x_0 > \mathbf{E}[\max(L_{t_1}, l_{t_1})\xi_{t_1}\mathbf{1}_{\xi_{t_1} \le \bar{\xi}_1}] + \mathbf{E}[l_{t_1}\xi_{t_1}\mathbf{1}_{\xi_{t_1} > \bar{\xi}_1}] := x_0^{min,2VaR}. \tag{7}$$

Then, one of the following two possibilities holds:

1. *On* $L_{t_1} > l_{t_1}$

$$\widehat{X}_{t_1} = I_{t_1}^{2VaR,(1)}(\widehat{y}_1\xi_{t_1})\mathbf{1}_{\xi_{t_1} \le \underline{\xi}_1} + L_{t_1}\mathbf{1}_{\underline{\xi}_1 \le \xi_{t_1} < \bar{\xi}_1} + I_{t_1}^{2VaR,(1)}(\widehat{y}_1\xi_{t_1})\mathbf{1}_{\bar{\xi}_1 \le \xi_{t_1}}, \tag{8}$$

   where $I_{t_1}^{2VaR,(1)}\left(\widehat{y}_1\underline{\xi}_1\right) = L_{t_1}$. *The* t_1*-VaR constraint is binding iff* $\bar{\xi}_1 > \underline{\xi}_1$.

2. *On* $L_{t_1} \le l_{t_1}$, $\widehat{X}_{t_1} = I_{t_1}^{2VaR,(1)}(\widehat{y}_1\xi_{t_1})$.

The Lagrange multiplier $\widehat{y}_1$ *satisfies the budget constraint* $\mathbf{E}[\xi_{t_1}\widehat{X}_{t_1}] = x_0$.

Averaged terminal wealth

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Given the market state at maturity ξ_T,

$$\overline{X}(\xi_T) := \mathbf{E}_{\xi_{t_1}|\xi_T}[\widehat{X}_T(\widehat{y}_2, \xi_T)] = \int_0^\infty \widehat{X}_T(\widehat{y}_2(z), \xi_T)\mathrm{d}F_{m_{t_1}, v_{t_1}}(z), \tag{9}$$

where $F_{m_{t_1}, v_{t_1}}$ is the CDF of $LN(m_{t_1}, v_{t_1})$, defined by

$$m_{t_1} := -(r + \theta^2/2)t_1 + t_1[\ln \xi_T + (r + \theta^2/2)T]/T = \frac{t_1}{T} \ln \xi_T; \quad v_{t_1} := \theta^2 t_1(T - t_1)/T$$

Numerical examples for power utility

- Market parameters

$$x_0 = 100;\ \mu = 0.06;\ r = 0.01;\ \sigma = 0.2;\ \gamma = 2;\ t_1 = 1;\ T = 2, L_t = 90e^{0.005t}$$

- The t-year maximal default probability as $\beta(t) = 1 - (1 - 0.5\%)^t$,

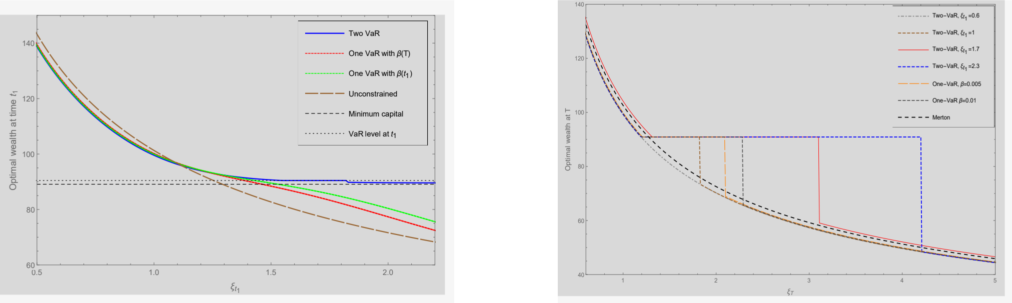
y^M	$y^{BS}_{\beta(2)}$	$y^{BS}_{\beta(1)}$	$\widehat{y}_1$
0.0000950041	0.000100174	0.000100892	0.000101424

Table: Lagrangian multipliers

$\underline{\xi}_1$	$\bar{\xi}_1$	$\underline{\xi}^{BS}_{\beta(1)}$	$\bar{\xi}^{BS}_{\beta(1)}$	$\underline{\xi}^{BS}_{\beta(2)}$	$\bar{\xi}^{BS}_{\beta(2)}$
1.54408	1.82706	1.19942	2.28918	1.20802	2.09661

Table: Market state limits

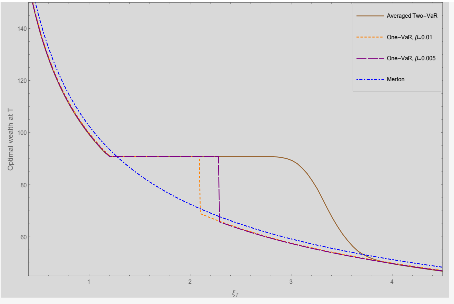
Numerical result (1):optimal wealth



(a) Optimal wealth at $t_1 = T/2 = 1$ (b) Optimal terminal wealth

Figure: Two-VaR optimal wealth

Numerical result(2):averaged wealth



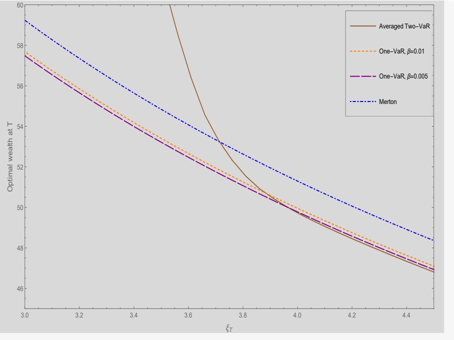
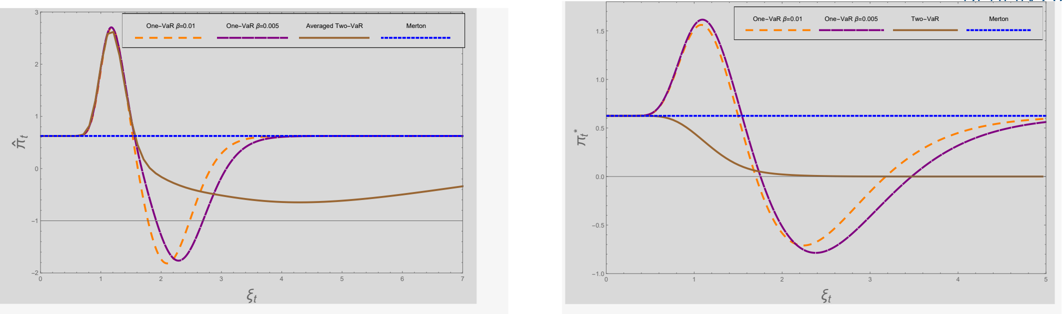


Figure: Averaged terminal wealth

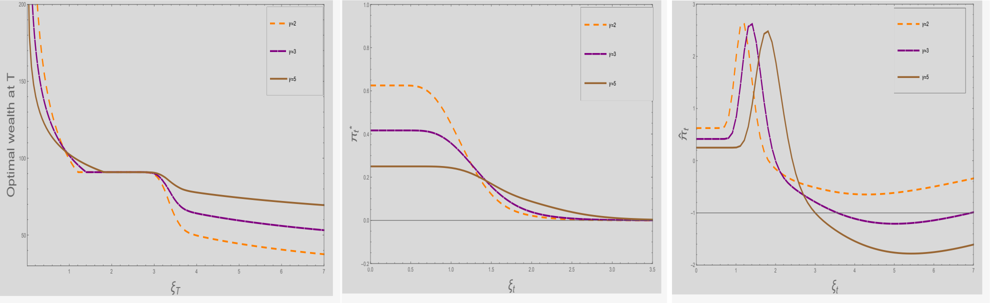
Numerical result(3):optimal strategy



(a) optimal strategies $t = 1.5$ in $[t_1, T]$. (b) optimal strategies at $t = 0.5$ in $[0, t_1]$.

Figure: Optimal strategy

Robustness



(a) average wealth at T. (b) strategy at $t = 0.5$. (c) strategy at $t = 1.5$.

Figure: Effect of γ

Conclusion

- Risk management with multiple VaR constraints is useful for loss prevention at intertemporal times.
- The multiple-VaR strategy on average may deliver a lower terminal wealth level than the one-VaR solution in good and very bad states.
- The multiple VaR terminal wealth can be on average lower than the one-VaR terminal wealth in a very rather limited range of very bad market scenarios.
- The result can be extended to the case with $n \geq 3$ VaR constraints.
- A multiple VaR framework induces a preference for less volatility.

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Thank you very much for your attention!

Contact details:

Thai Nguyen

address:	Helmholtzstrasse 20, R.1.48
	89069 Ulm, Germany
phone:	+49 731/50-31170
mail:	thai.nguyen@uni-ulm.de
web:	https://www.uni-ulm.de/mawi/ivw/institut/team

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