

## Pricing Pension Buy-outs under Stochastic Interest and Mortality Rates

Joint work with Yolcu-Okur, Y., Şahin, Ş. and Uğur, Ö.

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## About the speaker



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# Agenda

Aim of the Study

The Model Framework

Liability Model

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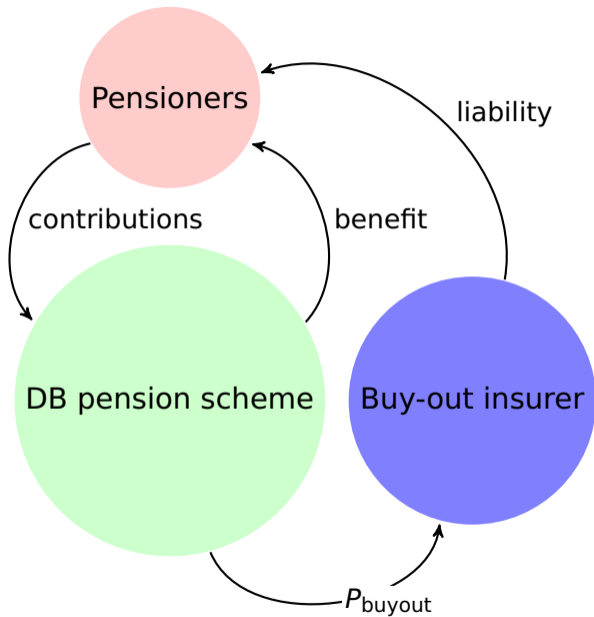
# Aim of the Study



## The main purpose and contribution of the study

Introduce a general pricing structure for pension buy-outs

1. under the **dependence assumption** between interest and mortality rates risks.
2. in a continuous time framework.



# The Model Framework

$$\begin{aligned} P_{\text{buyout}}(t) &= \frac{\sum_{t_i > t}^{t_M} E^{\mathbb{Q}} \left[ e^{-\int_t^{t_i} r(s) ds} \max\{PA(t_i^+) - (PA(t_i) - N(t_i) \cdot C), 0\} | \mathcal{I}_t \right]}{L(t)} \\ &= \frac{\sum_{t_i > t}^{t_M} B(t, t_i, r(t)) E^{t_i} [\max\{PA(t_i^+) - (PA(t_i) - N(t_i) \cdot C), 0\} | \mathcal{I}_t]}{L(t)} \end{aligned}$$

- $r(\cdot)$  : the stochastic short rate for  $t \geq 0$ .
- $L(\cdot)$  is the liability of the pension scheme.
- $a(\cdot)$  is introduced in Equation (1).
- $B(\cdot)$  is the fair price of a zero coupon bond.
- $PA(t_i^+)$  is the value of the pension portfolio at the beginning of time  $t_{i+1}$ .

## Definition

**(Liability Process)** The pension liability process  $L(t)$  at time  $t$ ,  $t \in [0, T]$ , is determined as

$$L(t) = N(t) \times a(t, x),$$

where

- $N(t)$  : the number of survivors in the model at time  $t$  (determined according to the force of mortality rate dynamics).
- $a(t, x)$  is given in Equation (1).

## Liability Model (Continued)

$$a(t, x) = \sum_{t_i=t_1}^{t_M} B_S(t, t_i, x, C) \quad (1)$$

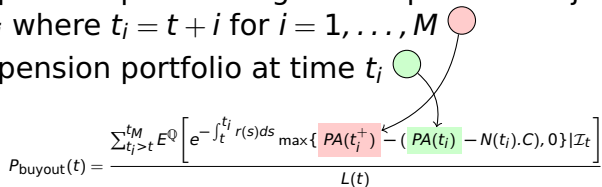
- $B_S(t, t_i, x, C)$ : the fair price of a pure endowment contract where  $t_i = t + i$  for  $i = 1, \dots, M$  and  $t_M = \inf\{t : N(t) = 0\}$ .

$$B_S(t, t_i, x, C) = \mathbb{1}_{\tau(x) > t} \times C \times B(t, t_i, r(t)) \times \tilde{p}(t, t_i, x)$$

- $\tau(x)$ : the future lifetime of an individual aged  $x$ .
- $B(t, t_i, r(t)) = E^{\mathbb{Q}} \left[ \exp \left( - \int_t^{t_i} r(s) ds \right) \middle| \mathcal{I}_t \right]$  and  $\tilde{p}(t, t_i, x) = E^{t_i} \left[ \exp \left( - \int_t^{t_i} \mu(s, x + s) ds \right) \middle| \mathcal{I}_t \right]$

## The Model Framework (Continued)

- The value of pension portfolio right after possible adjustments at the end of time  $t_i$  where  $t_i = t + i$  for  $i = 1, \dots, M$
- The value of pension portfolio at time  $t_i$

$$P_{\text{buyout}}(t) = \frac{\sum_{t_i > t}^{t_M} E^Q \left[ e^{-\int_t^{t_i} r(s) ds} \max\{ PA(t_i^+) - (PA(t_i) - N(t_i) \cdot C), 0 \} | \mathcal{I}_t \right]}{L(t)}$$


- Here,

$$PA(t_i^+) = \max\{ PA(t_i) - N(t_i) \times C, L(t_i) \}.$$

- The value of pension assets under  $\mathbb{P}$

$$dA_k(t) = A_k(t)[\alpha_k dt + \sigma_k dW_k(t)] \quad (2)$$

- $\text{Cov}(A_k(t), A_l(t)) = \rho_{kl}\sigma_k\sigma_l$
- $\text{Cov}(W_k(t), W_l(t)) = \rho_{kl}t$ ,  $k = 1, 2, 3$ ;  $l = 1, 2, 3$ ;  $k \neq l$ .
  1.  $\rho_{kl}$ : the correlation coefficient between assets  $k$  and  $l$ .
  2.  $\alpha_k$ : the drift term for asset  $k$  where  $k = 1, 2, 3$ .
  3.  $\sigma_k$ : the instantaneous volatility

## Financial Market Model (Continued)

The value of the pension portfolio under  $\mathbb{Q}$

$$d \log PA(t) = \left( r - \frac{1}{2} \sigma_W^2 \right) dt + \sum_{k=1}^3 \pi_k(t) \sigma_k dW_k^{\mathbb{Q}}(t)$$

1.  $r$  is the risk free rate (stochastic).
2.  $\pi(t) = [\pi_1(t), \pi_2(t), \pi_3(t)]$  are the weights of the assets in the portfolio.
3.  $\sigma_W^2 = \sum_{k,l=1}^3 \pi_k(t) \pi_l(t) \rho_{kl} \sigma_k \sigma_l$
4.  $dW_k^{\mathbb{P}}(t) = dW_k^{\mathbb{Q}}(t) - \left( \frac{\sum_1^3 \pi_k(t) \alpha_k - r}{\sum_1^3 \pi_k(t) \sigma_k} \right) dt.$

## Assumptions

1. No annual contributions. No pension gap at inception.
2. No inflation risk.
3.  $x = 65$ ,  $N(0) = 10000$ ,  $C = 60000$ ,  $dt = 1/252$
4. UK male population for ages 65 to 110 between 1928 and 2013
5.  $\pi_{UK} = [0.10, 0.85, 0.05]$

## Application (Continued)

Table 1: Model assumptions for the numerical illustrations

	<b>Mortality Model</b>	<b>Short Rate Model</b>
<b>Scenario I</b>	Ornstein-Uhlenbeck	Vasicek
<b>Scenario II</b>	Ornstein-Uhlenbeck	Cox-Ingersoll-Ross
<b>Scenario III</b>	Lee-Carter	Vasicek
<b>Scenario IV</b>	Lee-Carter	Cox-Ingersoll-Ross

- S&P UK stock total return index  $A_1(t)$ , the Merrill Lynch UK Sterling corporate bond index  $A_2(t)$  and the 3-month UK cash total return index  $A_3(t)$ .

Table 2: The estimated parameter values for Investment Strategy

Parameter	Estimate	Parameter	Estimate
$\alpha_1$	0.0448	$\sigma_1$	0.1600
$\alpha_2$	0.0215	$\sigma_2$	0.0716
$\alpha_3$	0.0001	$\sigma_3$	0.0077

- The correlation matrix

$$\rho^{(1)} = \begin{bmatrix} 1 & \rho_{12} & \rho_{13} \\ \rho_{21} & 1 & \rho_{23} \\ \rho_{31} & \rho_{32} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0.3483 & -0.1002 \\ 0.3483 & 1 & -0.1772 \\ -0.1002 & -0.1772 & 1 \end{bmatrix}$$

Table 3: Estimated parameter values in all scenarios

<b>Parameter set for the application</b>	
<b>Contract details</b>	$C = 60000, N(0) = 10000, PA(0) = L(0), x = 65$
<b>Vasicek model</b>	$a^r = 0.045398, b^r = 0.090070, c^r = 0.003789$
<b>CIR model</b>	$a^r = 0.2, b^r = 0.04, \sigma^r = 0.1, r(0) = 0.04$
<b>LC model 1</b>	$\delta = -0.422705, \sigma = 1.636589, \eta = 0.0943$
<b>LC model 2</b>	$\delta = -0.422705, \sigma = 1.636589, \eta = 1.197179$

# Scenario I



## Definition

**(Short rate and mortality rate dynamics under measure  $\mathbb{Q}$ )** Let us introduce interest and mortality rates dynamics as

$$dr(t) = a^r(b^r - r(t))dt + c^r dW^r(t) \quad (3)$$

$$d\mu(t) = a^\mu \mu(t)dt + c^\mu dW^\mu(t), \quad (4)$$

where  $a^r$ ,  $b^r$ ,  $c^r$ ,  $a^\mu$  and  $c^\mu$  are positive constants. Here,  $W^r(t)$  and  $W(t)$  are independent Wiener processes under measure  $\mathbb{Q}$  and

$$dW^\mu(t) = \rho dW^r(t) + \sqrt{1 - \rho^2} dW(t), \quad (5)$$

where  $\rho$  is a correlation coefficient (Liu et al., 2014).

## Scenario II

### Definition

**(Short rate and mortality rate dynamics under measure  $\mathbb{Q}$ )** Let us define interest and mortality rate dynamics as

$$dr(t) = a^r (b^r - r(t))dt + \sigma^r \sqrt{r(t)}dW^r(t) \quad (6)$$

$$d\mu(t) = a^\mu \mu(t)dt + c^\mu dW^\mu(t),$$

where  $a^r$ ,  $b^r$ ,  $\sigma^r$ ,  $a^\mu$  and  $c^\mu$  are positive constants. Here,

$$dW^\mu(t) = \rho dW^r(t) + \sqrt{1 - \rho^2} dW(t).$$

## Scenario III

### Definition

**(Short rate and mortality rate dynamics under measure  $\mathbb{Q}$ )** The respective dynamics under  $\mathbb{Q}$  of  $r(t)$  and  $\mu(t)$  are as follows:

$$dr(t) = a^r(b^r - r(t))dt + c^r dW^r(t)$$

$$d\mu^j(t) = \mu^j(t)(\tilde{\delta}^\mu(t)dt + \sigma^\mu(t)dW^\mu(t)), \quad (7)$$

where  $\tilde{\delta}^\mu(t) = \delta^\mu(t) - \eta\sigma^\mu(t)$  and  $dW^\mu(t) = \rho dW^r(t) + \sqrt{1 - \rho^2}dW(t)$ . Here,  $j$  represents the number of insureds and  $j \in \{1, \dots, m\}$ .

## Scenario IV

### Definition

**(Short rate and mortality rate dynamics under measure  $\mathbb{Q}$ )** The respective dynamics under  $\mathbb{Q}$  of  $r(t)$  and  $\mu(t)$  are as follows:

$$dr(t) = a^r(b^r - r(t))dt + \sigma^r \sqrt{r(t)}dW^r(t)$$

$$d\mu^j(t) = \mu^j(t)(\tilde{\delta}^\mu(t)dt + \sigma^\mu(t)dW^\mu(t)),$$

where  $\tilde{\delta}^\mu(t) = \delta^\mu(t) - \eta\sigma^\mu(t)$  and  $dW^\mu(t) = \rho dW^r(t) + \sqrt{1 - \rho^2}dW(t)$ . Here,  $j$  represents the number of insureds with  $j \in \{1, \dots, m\}$ .

# Buy-out Premiums

**Table 4:** Actuarial fair prices of the buy-out deal under 50000 MC samples according to different correlation coefficients for each scenario

		$\rho = -0.9$	$\rho = 0$	$\rho = 0.9$
<b>Scenario I</b>		0.276702	0.275262	0.273804
<b>Scenario II</b>		0.280131	0.276051	0.272238
<b>Scenario III</b>	$\eta = 0.0943$	0.404026	0.403342	0.402736
	$\eta = 1.197179$	0.443413	0.442993	0.442623
<b>Scenario IV</b>	$\eta = 0.0943$	0.411471	0.409774	0.408163
	$\eta = 1.197179$	0.455282	0.454347	0.453450

## 95% Confidence Interval for $P_{\text{buyout}}(0)$

The confidence interval for  $P_{\text{buyout}}(0)$  is calculated as follows:

$$P_{\text{buyout}}(0)_{-} = \frac{\sum_{t_i=1}^{t_M} PV_{\text{payoff}}(t_i)_{-}}{L(0)}$$

$$P_{\text{buyout}}(0)_{-} = \frac{\sum_{t_i=1}^{t_M} PV_{\text{payoff}}(t_i)^{-}}{L(0)}, \quad (8)$$

where  $P_{\text{buyout}}(0)_{-}$  and  $P_{\text{buyout}}(0)^{-}$  show the lower and upper bounds of the confidence interval respectively. Here,  $PV_{\text{payoff}}(t_i) = E^{\mathbb{Q}} \left[ e^{-\int_0^{t_i} r(s) ds} H(t_i) \right]$  and

$$PV_{\text{payoff}}(t_i)_{-} = \mu_{\text{payoff}}(t_i) - 1.96[\sigma_{\text{payoff}}(t_i)/\sqrt{N}]$$

$$PV_{\text{payoff}}(t_i)^{-} = \mu_{\text{payoff}}(t_i) + 1.96[\sigma_{\text{payoff}}(t_i)/\sqrt{N}].$$

# Scenario I

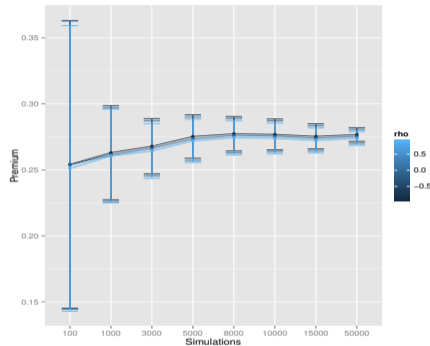


Figure 1: Buy-out premiums with the confidence intervals for all  $\rho$  based on Scenario I

## Scenario II

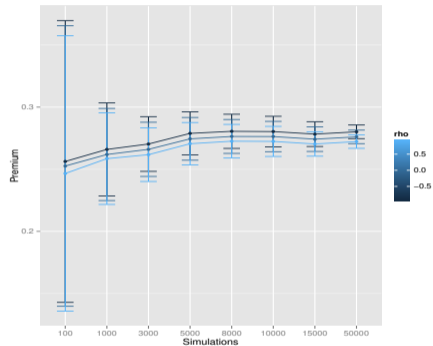


Figure 2: Buy-out premiums with the confidence intervals for all  $\rho$  based on Scenario II

## Scenario III

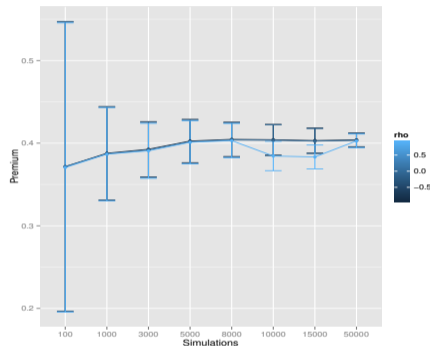


Figure 3: Buy-out premiums with the confidence intervals for all  $\rho$  based on Scenario III according to the calibration of the UK data when  $\eta = 0.0943$

## Scenario III (Continued)

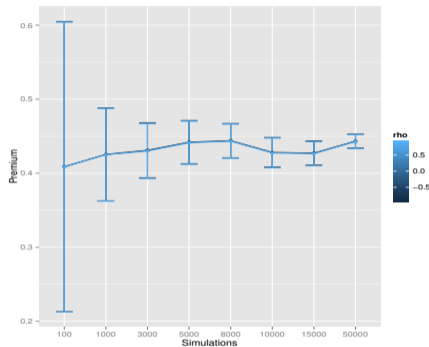


Figure 4: Buy-out premiums with the confidence intervals for all  $\rho$  based on Scenario III according to the calibration of the UK data when  $\eta = 1.197179$

## Scenario IV

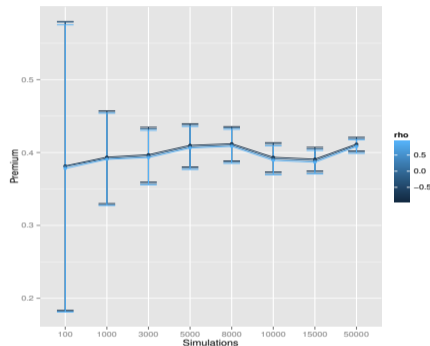
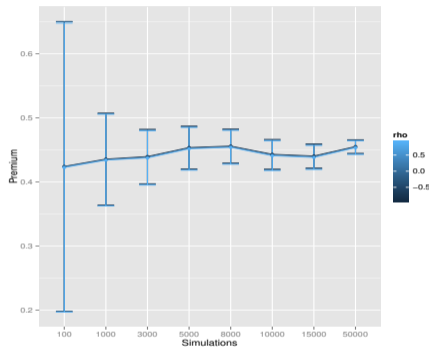


Figure 5: Buy-out premiums with the confidence intervals for all  $\rho$  based on Scenario IV according to the calibration of the UK data when  $\eta = 0.0943$

## Scenario IV (Continued)



**Figure 6:** Buy-out premiums with the confidence intervals for all  $\rho$  based on Scenario IV according to the calibration of the UK data when  $\eta = 1.197179$

## Further Research

- Different approaches for modeling mortality rates like Cairns-Blake-Dowd model
- Enhance the single stopping time assumption in LC model
- Different dependency structures
  1. the dependency between pension assets and mortality rates
  2. Markov-type shocks which affect both asset returns and mortality rates
  3. the dependency between interest rate and pension assets

**Thank you very much for your attention!**

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