Robust evaluation of SCR for participating life insurances under Solvency II

Donatien Hainaut*, Pierre Devolder*
ISBA, Université Catholique de Louvain, Belgium
Antoon Pelsser†
Dept. of Finance, Maastricht University , Netherlands

March 26, 2018

Abstract

This article proposes a robust framework to evaluate the solvency capital requirement (SCR) of a participating life insurance with death benefits. The preference for robustness arises from the ambiguity caused by the market incompleteness, model shortcomings and parameters misspecifications. To incorporate the uncertainty in the procedure of evaluation, we consider a set of potential equivalent pricing measures in the neighborhood of the real one. In this framework, closed form expressions for the net asset value (NAV) and for its moments are found. The SCR is next approximated by the Value at Risk of Gaussian or normal inverse Gaussian (NIG) random variables, approaching the NAV distribution and fitted by moments matching.

KEYWORDS : Solvency II, robustness, ORSA, life insurance.

1 Introduction

In the Solvency II regulation (first pillar), the Solvency Capital Requirement (SCR) is meant to cover one year of deterioration of the Net Asset Value (NAV). The NAV, that is a market consistent evaluation of future profits yield by the company. It is evaluated by the difference between the market value of assets and the Best Estimate (BE) provisions. The BE is appraised by a market to model approach, as the expected sum of future discounted benefits. Whereas, the total market value of liabilities is the sum of this BE and of the Risk Margin (RM).

However, the Solvency II framework presents some operational drawbacks. Firstly, due to the complexity of guidelines, SCR and NAV are evaluated exclusively by Monte-Carlo simulations in most of insurance companies. As underlined in Floryszczak et al. (2016), programs computing the SCR are black boxes, extremely demanding in terms of resources and not adapted for decision making. For this reason, the regulator introduced the second Pillar, called "Own Risk Solvency Assessment" (ORSA) which ensures that the management has a holistic view of risks. In the ORSA, the capital assessment may substantially differ from the first pillar, and the insurer has the freedom to develop alternative approaches to manage risks. Furthermore, the undertaking should ensure that its assessment of the overall solvency needs is forward-looking, including a medium term or long term perspective as appropriate. Within this context, Bonnin et al. (2014) and Combes et al. (2016) propose analytical models to pilot the asset-liability management (ALM) policy of participating policies. This article proposes a new alternative to these approaches.

The Solvency II regulation also raises interesting theoretical questions. Firstly, a single insurance claim cannot be hedged on an individual basis. Instead, insurers rely on the law of large numbers and pool risks to reduce their exposure to claims. Due to this incompleteness, multiple equivalent pricing measures exist and each one reflects the insurer's risk aversion for unhedgeable risks. Solvency II recommends to

*Postal address: Voie du Roman Pays 20, 1348 Louvain-la-Neuve (Belgium) . E-mail to: donatien.hainaut(at)uclouvain.be, E-mail to: pierre.devolder(at)uclouvain.be

†Postal address: Tongersestraat 53, P.O. Box 616 6200 MD Maastricht (Netherlands). E-mail to: a.pelsser(at)maastrichtuniversity.nl

evaluate BE provisions under a risk neutral measure with realistic assumptions about unhedgeable risks. But there are no guidelines to determine admissible measures. A second point concerns the model risk. The SCR is evaluated by complex programs and today the potential impact of model misspecifications on the SCR is not addressed by Solvency II. This article is a first attempt to incorporate this uncertainty in the SCR valuation.

However, the uncertainty about the risk neutral measure, parameters and the model gives rise to a current of research in the literature that focuses on robustness. A model is qualified as robust if it takes into account the potential misspecifications. The theory of Robustness was pioneered in economics by Hansen and Sargent (1995), (2001) or (2007). This theory is an alternative to Bayesian approaches that are typically limited to parametric versions of model uncertainty. In a robust approach, there is no need to make any assumptions about the a priori distribution of parameters and the uncertainty concerns the entire drift function. To formulate model misspecifications, Hansen and Sargent employ a relative entropy factor. This relative entropy captures the perturbation between the estimated model and the unobservable true model. Anderson et al. (2003) extend the robust control theory with the theory of semi-groups. Balter and Pelsser (2015) use a similar approach and propose a robust pricing method in an incomplete market. Maenhout (2004) propose a robust solution to the consumption and portfolio problem of Merton (1969). Instead of explicitly bounding the entropy, a penalty term is introduced in the infinitesimal generator of the value function. This additional term penalizes alternative models that are too far away from the reference model. This approach was extended in Maenhout (2006) to mean reverting risk premiums. The literature on robustness is vast and we refer to Guidolin and Rinaldi (2010) for a detailed review.

This paper contributes to the literature in two main directions. Firstly, it proposes a robust and simple model to estimate the SCR of participating life insurances with death benefits. The evaluation scheme is mainly based on analytical expressions of NAV and BE, without any recourse to simulations. Our approach is also compliant with ORSA guidelines and provide a simple method to estimate prospective SCR. Secondly, this article addresses the uncertainty surrounding the model specifications. The distance between the estimated model and the unobservable true specification is delimited by an entropic constraint on eligible real and pricing measures. The bound on the entropy is directly related to the level of confidence in results produced by the model. The ambiguity around model assumptions may then be counterbalanced by adjusting this entropic bound. If we position our work in the scope of the ORSA, the constraint on entropy may be calibrated so as to match BE and SCR estimates computed by our model with those obtained with a complex internal model. Our approach may also be reconciled with the rationale of Cochrane and Saa-Requejo (2000)'s Good-Deal-Bound. Their idea is to bound the Sharpe ratios of all possible assets in the market and thus exclude Sharpe ratios which are considered to be too large. The idea was streamlined and extended to models with jumps in Bjork and Slinko (2006) or models with switching regimes as in Donnelly (2011).

This article is structured in the following way. Section 2 defines the multivariate Brownian motion driving the financial market in which the insurance company invested collected premiums. The model for the human mortality is presented in section 3. The following section introduces the specifications of the participating insurance contract with death benefits. In section 5, we discuss the choice of the pricing measure and of the entropic constraint. Closed form expressions for robust and non-robust best estimate provisions are next provided. In section 7, we infer closed form expressions for the net asset value (NAV) and for its moments. In section 8, the SCR is calculated by a Value at Risk of Gaussian or normal inverse Gaussian (NIG) random variables, approaching the NAV distribution and fitted by moments matching. Section 9 discusses the problem of ambiguity under the real measure.

2 The financial market

We consider an insurance company that proposes participating life contracts with a minimal guarantee, and death benefits. Before detailing the specifications of policies, we firstly introduce the financial market. We consider d assets driven by a multivariate geometric Brownian motion of dimension d. This process is defined on a probability space Ω, endowed with a filtration $(\mathcal{F}_t)_t$ under a real probability measure denoted by P. There is considerable piece of evidence suggesting that the Brownian motion with constant drift and standard deviation are not appropriate to model stocks returns, due to extreme comovements. However, it is analytically tractable and its shortcomings are compensated in section 5 by integrating preferences for robustness. The assets prices are denoted by S_t^i for $i = 0$ to $d-1$ and obey to following stochastic differential equations:

$$\underbrace{\begin{pmatrix} \frac{dS_t^0}{S_t^0} \\ \frac{dS_t^1}{S_t^1} \\ \vdots \\ \frac{dS_t^{d-1}}{S_t^{d-1}} \end{pmatrix}}_{\frac{dS_t^S}{S_t^S}} = \underbrace{\begin{pmatrix} \mu_0 \\ \mu_1 \\ \vdots \\ \mu_{d-1} \end{pmatrix}}_{\mu_S} dt + \underbrace{\begin{pmatrix} \sigma_{00} & 0 & \dots & 0 \\ \sigma_{10} & \sigma_{11} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \sigma_{d-1,0} & \sigma_{d-1,1} & \dots & \sigma_{d-1,d-1} \end{pmatrix}}_{\Sigma_S} \underbrace{\begin{pmatrix} dB_t^0 \\ dB_t^1 \\ \vdots \\ dB_t^{d-1} \end{pmatrix}}_{dB_t^S}. \tag{1}$$

Where $(B_t^0, \dots, B_t^{d-1})$ are independent Brownian motions under P. The matrix of diffusion Σ_S is constant, positive definite, and invertible. Using the Itô's lemma, we can show that the vector of prices, $S_t = (S_t^0, S_t^1, \dots, S_t^{d-1})^\top$, satisfies the following relation :

$$d \ln S_t = \left(\mu_S - \frac{diag(\Sigma_S \Sigma_S^\top)}{2} \right) dt + \Sigma_S dB_t^S .$$

The asset S_t^0 is the numeraire: the present value at time t of a cash-flow F paid at time T is equal to $\mathbb{E}^Q\left(\frac{S_t^0}{S_T^0} F | \mathcal{F}_t\right)$, where Q is a risk neutral measure. The numeraire is e.g. a short term rolling bond. In this case, μ_0 is the instantaneous interest rate (under the real measure P). Investments of the insurance company are continuously rebalanced and proportions invested in each assets are summarized by the vector $\theta_S = (\theta_0, \dots, \theta_{d-1})^\top$. The total asset, denoted by A_t, is equal to $\theta_S^{'} S_t$ and its dynamics is defined by:

$$\frac{dA_t}{A_t} = \theta_S^\top \mu_S dt + \theta_S^\top \Sigma_S dB_t^S.$$

From the Itô's lemma, we infer that the total asset value is a lognormal random variable:

$$A_t = A_0 \exp\left(\left(\theta_S^\top \mu_S - \frac{1}{2} \theta_S^\top \Sigma_S \Sigma_S^\top \theta_S \right) t + \int_0^T \theta_S^\top \Sigma_S dB_t^S \right).$$

Notice that the financial market is incomplete. To underline this point, let us consider a change of measure

$$\left. \frac{dQ^{\mu^Q}}{dP} \right|_t = \exp\left(-\frac{1}{2} \int_0^t \chi^\top \chi ds - \int_0^t \chi^\top dB_s \right),$$

where $\chi = \Sigma_S^{-1} \left(\mu_S - \mu^Q \mathbf{1}_{d-1} \right)$ and $\mu^Q \in \mathbb{R}^+$ is an arbitrary constant. Under Q^{μ^Q}, the drift of all assets, including the numeraire, is equal to μ^Q and discounted assets prices are martingales. This entails that Q^{μ^Q} is a risk neutral measure, whatsoever the value chosen for μ^Q[1]. In practice, actuaries set μ^Q to the current risk free rate μ_0 but nothing prevent us (in our framework) to choose a different value which corresponds to the expected return on cash, under Q. This modified rate is adjusted to take into account the uncertainty about the future evolution of interests. The consequences of this incompleteness are addressed in section 5.

3 The mortality risk

The insurance contract plans the payment of a multiple of the technical provision in case of premature death. The presence of this risk of mortality is an important source of incompleteness. Before detailing this point, we introduce a model for mortality. We assimilate the decease of an individual of age x to the first jump of a point process $(N_t)_t$. This process is defined on Ω and its natural filtration is denoted by

[1] The incompleteness of the financial market results from the absence of pure discount bond. If a discount bond of maturity T is defined as $B(t,T) = \mathbb{E}^Q\left(\frac{S_t^0}{S_T^0} | \mathcal{F}_t\right) = \exp\left(-\int_t^T \mu^Q ds\right)$, then we can infer from the observation of its price the value of μ^Q.

$(\mathcal{H}_t)_{t\geq 0}$. The time of death is a stopping time noted τ, with respect to $\mathcal{H}_t$. The probability of survival till age $x+t$ is equal to the next expression:

$$
\begin{aligned}
{}_tp_x &:= P(\tau > t) = \mathbb{E}\left(\boldsymbol{I}_{\{\tau>t\}} | \mathcal{F}_0\right) \\
&= P(N_t = 0) = \mathbb{E}\left(\boldsymbol{I}_{\{N_t=0\}} | \mathcal{F}_0\right) ,
\end{aligned}
$$

where I is the indicator variable. The hazard rate of N_t is noted λ_t. λ_t is a hidden stochastic process defined on a filtration $(\mathcal{O}_t)_t$. This filtration carries information that is not observable about λ_t and differs then from $\mathcal{H}_t$ that contains the visible information about the individual's survival. Conditionally to the sample path followed by λ_t, N_t is a Poisson process with an intensity λ_t. The survival probability (given $\mathcal{O}_t$) is in this case equal to

$$
\begin{aligned}
P(N_t = 0 \,|\, \mathcal{O}_t \wedge \mathcal{H}_0) &= \mathbb{E}\left(e^{-\int_0^t \lambda_s ds} \,|\, \mathcal{O}_t \wedge \mathcal{H}_0\right) \\
&= \boldsymbol{I}_{\{\tau>0\}} e^{-\int_0^t \lambda_s ds} .
\end{aligned}
$$

As $\mathcal{O}_t$ is not visible, using nested expectations allows us to infer that the probability of survival is given by the next expectation:

$$
\begin{aligned}
{}_tp_x &= \mathbb{E}\left(N_t = 0 \,|\, \mathcal{H}_0\right) \\
&= \mathbb{E}\left(\mathbb{E}\left(N_t = 0 \,|\, \mathcal{O}_t \vee \mathcal{H}_0\right) \,|\, \mathcal{H}_0\right) \\
&= \boldsymbol{I}_{\{\tau>0\}} \mathbb{E}\left(e^{-\int_0^t \lambda_s ds} \,|\, \mathcal{H}_0\right) .
\end{aligned}
$$

We assume that the hazard rate is a random process led by the following dynamics

$$
\lambda_s ds = \left(\mu_d(s) - \frac{\sigma_d^\top \sigma_d}{2}\right) ds + \sigma_d^\top dB_s ,
$$

where $\sigma_d = (\sigma_{d,0}, ..., \sigma_{d,d})^\top$ is a vector of positive constant. $B_t := \left(B_t^0, ..., B_t^d\right)^\top$ is the vector of Brownian motions ruling financial markets to which we add B_t^d a Brownian motion driving the evolution of mortality rates. The drift, $\mu_d(s)$, is a positive function of time. In numerical illustrations, $\mu_d(s)$ is a Makeham function, detailed in appendix A.

Given that the hazard rate is Brownian, the intensity λ_t may become negative. From a theoretical point of view, this assumption is then inappropriate. However, calibrating the model to real demographic data reveals that this scenario occurs with a negligible probability because the standard deviation of mortality rates is small compared to their mean. To illustrate this point, we report in table 1 the means, standard deviations of mortality rates, for the male French population, over the period 1954-2014. The last column of this table presents the probability of observing negative mortality rates under the assumption of normality when the drift and deviation are set to their historical values. Above 30 years old, the probability of such an event never exceeds 1 basis point (0.0001). For a 20 years old man, this probability climbs to 0.0004, which is still negligible. These figures confirm that working with Gaussian mortality rate is an acceptable assumption. Luciano et al. (2017) or Luciano and Vigna (2008) draw the same conclusion whereas Bauer et al. (2010) adopt a similar approach to evaluate longevity linked securities.

	Log-forces of mortality		
Age,	mean in $^o/_{oo}$	std in $^o/_{oo}$	$P(. \leq 0)$ in %
20	1,3223	0,3977	0,044
30	1,5506	0,3991	0,005
40	2,9821	0,403	0,004
50	7,4489	0,4242	0,005
60	16,8335	0,4498	0,005
70	37,5161	0,4939	0,007
80	95,5457	0,524	0,006

Table 1: This table reports the mean and standard deviation (std) of log-forces of mortality, measured over the period 1954-2014, for the French male population. The last column shows the probability of negative mortality rates.

Remark that our model allows for correlation between financial markets and morbidity if $\sigma_{d,(0:d-1)}$ are not null. Such a dependence is considered in the article of Deelstra et al. (2016). We infer that the survival probability is given by

$$\begin{aligned} {}_tp_x &= \boldsymbol{I}_{\{\tau>0\}}\mathbb{E}\left(\exp\left(-\int_0^t\left(\mu_d(s)-\frac{\sigma_d^\top\sigma_d}{2}\right)ds-\int_0^t\sigma_d^\top dB_s\right)\mid\mathcal{H}_0\right)\\ &= \boldsymbol{I}_{\{\tau>0\}}\exp\left(\int_0^t\left(-\mu_d(s)\right)ds\right). \end{aligned}$$

For later developments, we introduce what we call a "mortality account", S_t^M, with a growth rate equal to the hazard rate:

$$\begin{aligned} S_t^M &:= e^{\int_0^t\lambda_s ds}\\ &= \exp\left(\int_0^t\left(\mu_d(s)-\frac{\sigma_d^\top\sigma_d}{2}\right)ds+\int_0^t\sigma_d^\top dB_s\right), \end{aligned}$$

S_t^M is a geometric Brownian motion with the next dynamics

$$\frac{dS_t^M}{S_t^M} = \mu_d(t)dt+\sigma_d^\top dB_t\,. \tag{2}$$

The survival probability can then be rewritten as the expectation of a ratio of mortality accounts:

$${}_{T-t}p_{x+t}=\mathbb{E}\left(\frac{S_t^M}{S_T^M}|\mathcal{H}_t\right).$$

This reformulation of the survival probability is used in later developments. To end this section, we mention that the combination of the financial and insurance market is incomplete for three reasons. The first one is the absence of a risk free asset like a discount bond. The second one is that the individual's mortality cannot be hedged. Finally, the mortality hazard rate is stochastic and also unhedgeable. The consequence on the incompleteness on pricing is further discussed in section 5.

4 Liabilities

Participating life insurances are saving contracts that provide a minimum guarantee combined with a system of participation to the appreciation of the total asset. In addition, the capital is reimbursed if the individual deceases before the maturity of the contract. The minimum return that is guaranteed is noted g. The participation is credited at the end of periods of length Δ, till the expiry of the contract. The participating contract is purchased by an individual of age x. It pays a lump sum payment L_T at expiry T , if the insured is still alive:

$$L_T = C\prod_{k=1}^{n}\left(e^{g\Delta}+\left[\rho\frac{A_{t_k}}{A_{t_{k-1}}}-e^{g\Delta}\right]_+\right)$$

where ρ is the participation rate and C is the initial deposit. The participation to assets appreciation is calculated at times $t_1,\dots,t_n=T$. If the individuals passes away at time $t_{j-1}\leq\tau\leq t_j$ then the insurance company pays a multiple of the saved capital

$$L_{t_j} = \alpha\, C\prod_{k=1}^{j}\left(e^{g\Delta}+\left[\rho\frac{A_{t_k}}{A_{t_{k-1}}}-e^{g\Delta}\right]_+\right)$$

at time t_j where the multiplier is $\alpha\in\mathbb{N}$. Notice that the participation in our model is purely financial: it is totally independent from contingent gains coming from a deviation between the real and forecast mortality. The Best Estimate (BE) provision of this contract is the expected sum of future discounted benefits, forecast in a risk neutral world. For the moment, BE's are evaluated under a risk neutral measure Q chosen by the insurer. If we denote by $\mathcal{F}_t=\mathcal{H}_t\vee\mathcal{G}_t$, the augmented filtration that carries the visible information about the morbidity and financial markets, the BE provision at time $t=0$ is defined by the following expectation:

$$\begin{aligned} BE_0^Q &:= \alpha C \sum_{j=1}^{n} \mathbb{E}^Q \left(\boldsymbol{I}_{\{t_{j-1} \leq \tau \leq t_j\}} \frac{S_0^0}{S_{t_j}^0} \prod_{k=1}^{j} \left(e^{g\Delta} + \left[\rho \frac{A_{t_k}}{A_{t_{k-1}}} - e^{g\Delta} \right]_+ \right) |\mathcal{F}_0 \right) \\ &\quad + C \mathbb{E}^Q \left(\boldsymbol{I}_{\{t_n \leq \tau\}} \frac{S_0^0}{S_{t_n}^0} \prod_{k=1}^{n} \left(e^{g\Delta} + \left[\rho \frac{A_{t_k}}{A_{t_{k-1}}} - e^{g\Delta} \right]_+ \right) |\mathcal{F}_0 \right) \end{aligned}$$

We specify in the next section how the risk neutral measure is determined. For the moment, we consider that Q is perfectly identified. If we nest this last expectation by the enlarged filtration $\mathcal{O}_{t_j} \vee \mathcal{F}_0$, the BE may be rewritten as a function of mortality accounts, S_t^M:

$$\begin{aligned} BE_0^Q &= \alpha C \sum_{j=1}^{n} \mathbb{E}^Q \left(\left(1 - \frac{S_{t_{j-1}}^M}{S_{t_j}^M} \right) \frac{S_0^M}{S_{t_{j-1}}^M} \frac{S_0^0}{S_{t_j}^0} \prod_{k=1}^{j} \left(e^{g\Delta} + \left[\rho \frac{A_{t_k}}{A_{t_{k-1}}} - e^{g\Delta} \right]_+ \right) |\mathcal{F}_0 \right) \\ &\quad + C \, \mathbb{E}^Q \left(\frac{S_0^M}{S_{t_n}^M} \frac{S_0^0}{S_{t_n}^0} \prod_{k=1}^{n} \left(e^{g\Delta} + \left[\rho \frac{A_{t_k}}{A_{t_{k-1}}} - e^{g\Delta} \right]_+ \right) |\mathcal{F}_0 \right) \end{aligned}$$

As we don't assume the independence between the mortality and financial conditions, we cannot isolate factors related to the morbidity, as usually done in the actuarial literature. However, the independence between increments of all processes allows us to rewrite the BE provision as a product of expectations:

$$\begin{aligned} BE_0^Q &= \alpha C \sum_{j=1}^{n} \prod_{k=1}^{j} \mathbb{E}^Q \left(\left(\frac{S_{t_{k-1}}^M}{S_{t_k \wedge t_{j-1}}^M} \frac{S_{t_{k-1}}^0}{S_{t_k}^0} - \frac{S_{t_{k-1}}^M}{S_{t_k}^M} \frac{S_{t_{k-1}}^0}{S_{t_k}^0} \right) \left(e^{g\Delta} + \left[\rho \frac{A_{t_k}}{A_{t_{k-1}}} - e^{g\Delta} \right]_+ \right) |\mathcal{F}_0 \right) \quad (3) \\ &\quad + C \prod_{k=1}^{n} \mathbb{E}^Q \left(\frac{S_{t_{k-1}}^M}{S_{t_k}^M} \frac{S_{t_{k-1}}^0}{S_{t_k}^0} \left(e^{g\Delta} + \left[\rho \frac{A_{t_k}}{A_{t_{k-1}}} - e^{g\Delta} \right]_+ \right) |\mathcal{F}_0 \right) \end{aligned}$$

On the other hand, the best estimate provision for a given risk neutral measure Q at time t_i has the following expression

$$\begin{aligned} BE_i^Q &= C \prod_{k=1}^{i} \frac{S_{t_{k-1}}^0}{S_{t_k}^0} \left(e^{g\Delta} + \left[\rho \frac{A_{t_k}}{A_{t_{k-1}}} - e^{g\Delta} \right]_+ \right) \times \quad (4) \\ &\left[\alpha \sum_{j=i+1}^{n} \prod_{k=i+1}^{j} \mathbb{E}^Q \left(\left(\frac{S_{t_{k-1}}^M}{S_{t_k \wedge t_{j-1}}^M} \frac{S_{t_{k-1}}^0}{S_{t_k}^0} - \frac{S_{t_{k-1}}^M}{S_{t_k}^M} \frac{S_{t_{k-1}}^0}{S_{t_k}^0} \right) \left(e^{g\Delta} + \left[\rho \frac{A_{t_k}}{A_{t_{k-1}}} - e^{g\Delta} \right]_+ \right) |\mathcal{F}_0 \right) \right. \\ &\left. + \sum_{j=i+1}^{n} \prod_{k=i+1}^{j} \mathbb{E}^Q \left(\frac{S_{t_{k-1}}^M}{S_{t_k}^M} \frac{S_{t_{k-1}}^0}{S_{t_k}^0} \left(e^{g\Delta} + \left[\rho \frac{A_{t_k}}{A_{t_{k-1}}} - e^{g\Delta} \right]_+ \right) |\mathcal{F}_{t_i} \right) \right] . \end{aligned}$$

Equations (3) and (4) clearly emphasize that BE provisions may be represented as a product of several call options. But before evaluating them, we have to clarify the choice of the risk neutral measure.

5 Risk neutral measures and pricing of uncertainty

As evoked in previous sections, the market is incomplete due to the absence of risk free asset, to mortality and to uncertainty mortality rates. Consequence: there exists then an infinity of equivalent risk measures that may be considered as risk neutral. On the other hand, the multivariate Brownian motion driving assets and mortality offers a compromise between analytical tractability and realism. But important features of assets returns, like skewness, leptokurticity or non-identically distributed increments, are not replicated by such a dynamic. To circumvent these drawbacks, we integrate a preference for robustness in the valuation procedure. For this purpose, we rewrite the joint dynamics of financial assets and the mortality account as follows:

$$\underbrace{\begin{pmatrix} \frac{dS_t^S}{S_t^S} \\ \frac{dS_t^M}{S_t^M} \end{pmatrix}}_{\frac{dS_t}{S_t}} = \underbrace{\begin{pmatrix} \mu_S \\ \mu_d(t) \end{pmatrix}}_{\mu} dt + \underbrace{\begin{pmatrix} \Sigma_S & 0_{d-1} \\ \sigma_{d\,(0:d-1)} & \sigma_{d\,(d)} \end{pmatrix}}_{\Sigma} \underbrace{\begin{pmatrix} dB_t^S \\ dB_t^d \end{pmatrix}}_{dB_t} . \quad (5)$$

Any equivalent measure Q to the real one, is defined by the following Radon Nikodym derivative

$$\left.\frac{dQ}{dP}\right|_t = \exp\left(-\frac{1}{2}\int_0^t \Upsilon_s^T \Upsilon_s ds - \int_0^t \Upsilon_s^T dB_s\right),$$

where Υ_s is a d-vector of $\mathcal{F}_t-$adapted processes. Here Υ_s is assumed constant: $\Upsilon = (w_0, \ldots, w_d)$ where $w_{i=0,\ldots,d}$ are risk premiums related to each Brownian motions. Under the Q measure, the vector $W_t = (W_t^0, ..., W_t^d)^\top$ defined by

$$\begin{pmatrix} dW_t^0 \\ \vdots \\ dW_t^d \end{pmatrix} = \begin{pmatrix} dB_t^0 \\ \vdots \\ dB_t^d \end{pmatrix} + \begin{pmatrix} w_0 \\ \vdots \\ w_d \end{pmatrix} dt$$

is a vector of independent Brownian motions. For any arbitrary $r \in \mathbb{R}$ and $v \in \mathbb{R}$, if we choose $\Upsilon := \begin{pmatrix} \Sigma_S^{-1}(\mu_S - r\mathbf{1}_d) \\ v \end{pmatrix}$ then discounted assets prices $\frac{S_t^i}{S_t^0}$ are martingales under Q. The equivalent measure defined by this manner is then eligible as risk neutral one.

We insist on the fact that $r \in \mathbb{R}$ is a parameter defining Υ and is chosen by the insurer. r is not necessarily set to the current risk free rate[2]. Due to incompleteness, r is an expected rate, adjusted to take into account the insurer's risk aversion to an adverse evolution of interest rates. Under the equivalent measure, the mortality account S_t^M has a drift equal to $\mu_M(s) = \mu_d(s) + \sigma_d^\top \Upsilon$ whereas all assets grow at an average rate, r. To summarize, the joint dynamics of financial assets and the mortality account under Q is defined by:

$$\frac{dS_t}{S_t} = \begin{pmatrix} r\mathbf{1} \\ \mu_M(s) \end{pmatrix} dt + \Sigma dW_t.$$

where $\mathbf{1}$ is a d vector of ones. On the other hand, the total asset satisfies the following relation under Q:

$$A_t = A_0 \exp\left(\left(r - \frac{1}{2}\theta_S^\top \Sigma_S \Sigma_S^\top \theta_S\right) t + \int_0^t \theta_S^\top \Sigma_S dW_s\right),$$

where the vector $\theta_S = (\theta_0, \ldots, \theta_{d-1})^\top$ contains portfolio weights. In theory, any arbitrary value for r is allowed and leads to different estimates of BE provisions. In practice, r may be assimilated to a prudent estimate of the average risk free rate. Among all available measures, a natural choice consists to set r equal to μ_0, plus a risk premium. However, the degree of uncertainty over r may be high if we price long term contracts. To take into account the risk aversion to ambiguity, the best estimate provisions is appraised by its maximum value reached over a set of equivalent measures Q. This set is delimited by an entropic constraint of the change of measure from P to Q. The entropy is a distance that quantifies the distortion between the real and risk neutral measures. With other words, to limit the exposure to model and parameters misspecifications, BE provisions are evaluated in the worst case scenario selected in a subset of risk neutral worlds. The entropic distance is the expectation under Q of the logarithm of the Radon-Nikodym derivative $\left.\frac{dQ}{dP}\right|_t$ and is constrained as follows:

$$\mathbb{E}^Q\left(\ln \left.\frac{dQ}{dP}\right|_t \mathcal{F}_0\right) \leq \frac{1}{2}U^2 t \qquad \forall t \in \mathbb{R}^+. \tag{6}$$

where $U \in \mathbb{R}^+$ is a parameter chosen by the insurer and directly related to its level of risk aversion. The entropic distance and its bound both depend on the considered time horizon. This constraint is equivalent to impose a bound on the integral

$$\frac{1}{2}\int_0^t \Upsilon_s^T \Upsilon_s ds \leq \frac{1}{2}U^2 t \qquad \forall t \in \mathbb{R}^+. \tag{7}$$

Remark that the parameter U is an additional degree of freedom. Then, it is eventually possible to calibrate it so as to match the output of our model (e.g. NAV and SCR) with results from a complex internal model. As we consider that Υ is constant: $\Upsilon = \begin{pmatrix} \Sigma_S^{-1}(\mu_S - r\mathbf{1}) \\ v \end{pmatrix}$, the entropic constraint delimits an elliptic domain for (μ_S, v) in $\mathbb{R}^2$ as stated in the next proposition:

[2]Remember that in our framework, the numeraire is S_0 and the current risk free rate may be assimilated to μ_0.

Proposition 5.1. *The entropy is bounded by a constraint (6), if and only if*

$$U^2 \geq \mu_S^\top \left(\Sigma_S \Sigma_S^\top\right)^{-1} \mu_S - \frac{\left(\mu_S^\top \left(\Sigma_S \Sigma_S^\top\right)^{-1} \mathbf{1}\right)^2}{\mathbf{1}^\top \left(\Sigma_S \Sigma_S^\top\right)^{-1} \mathbf{1}} \tag{8}$$

In this case, $\upsilon^- \leq \upsilon \leq \upsilon^+$ *where*

$$\upsilon^{\pm} = \pm \sqrt{\frac{\left(\mu_S^\top \left(\Sigma_S \Sigma_S^\top\right)^{-1} \mathbf{1}\right)^2}{\mathbf{1}^\top \left(\Sigma_S \Sigma_S^\top\right)^{-1} \mathbf{1}} - \left(\mu_S^\top \left(\Sigma_S \Sigma_S^\top\right)^{-1} \mu_S - U^2\right)} \tag{9}$$

and if $r^-(\upsilon) \leq r \leq r^+(\upsilon)$ *where*

$$r^{\pm}(\upsilon) \quad = \quad \frac{2\mu_S^\top \left(\Sigma_S \Sigma_S^\top\right)^{-1} \mathbf{1} \pm \sqrt{D(\upsilon)}}{2\,\mathbf{1}^\top \left(\Sigma_S \Sigma_S^\top\right)^{-1} \mathbf{1}}\,, \tag{10}$$

and

$$D(\upsilon) = 4\left(\mu_S^\top \left(\Sigma_S \Sigma_S^\top\right)^{-1} \mathbf{1}\right)^2 - 4\left(\mathbf{1}^\top \left(\Sigma_S \Sigma_S^\top\right)^{-1} \mathbf{1}\right)\left(\mu_S^\top \left(\Sigma_S \Sigma_S^\top\right)^{-1} \mu_S - U^2 + \upsilon^2\right).$$

Proof. The constraint on entropy (7) is equivalent to

$$r^2\,\mathbf{1}^\top \left(\Sigma_S \Sigma_S^\top\right)^{-1} \mathbf{1} - 2r\mu_S^\top \left(\Sigma_S \Sigma_S^\top\right)^{-1} \mathbf{1} + \left(\mu_S^\top \left(\Sigma_S \Sigma_S^\top\right)^{-1} \mu_S - U^2 + \upsilon^2\right) \quad \leq \quad 0 \tag{11}$$

The left hand term in this last equation is a second order polynomial for which the discriminant is

$$D(\upsilon) = 4\left(\mu_S^\top \left(\Sigma_S \Sigma_S^\top\right)^{-1} \mathbf{1}\right)^2 - 4\left(\mathbf{1}^\top \left(\Sigma_S \Sigma_S^\top\right)^{-1} \mathbf{1}\right)\left(\mu_S^\top \left(\Sigma_S \Sigma_S^\top\right)^{-1} \mu_S - U^2 + \upsilon^2\right).$$

The discriminant is positive if and only if

$$\begin{aligned} 4\left(\mu_S^\top \left(\Sigma_S \Sigma_S^\top\right)^{-1} \mathbf{1}\right)^2 - 4\left(\mathbf{1}^\top \left(\Sigma_S \Sigma_S^\top\right)^{-1} \mathbf{1}\right)\left(\mu_S^\top \left(\Sigma_S \Sigma_S^\top\right)^{-1} \mu_S - U^2\right) \\ \geq 4\left(\mathbf{1}^\top \left(\Sigma_S \Sigma_S^\top\right)^{-1} \mathbf{1}\right)\upsilon^2 \end{aligned}$$

as $\Sigma_S \Sigma_S^\top$ is semi positive definite, this last expression simplifies as follows

$$\frac{\left(\mu_S^\top \left(\Sigma_S \Sigma_S^\top\right)^{-1} \mathbf{1}\right)^2}{\mathbf{1}^\top \left(\Sigma_S \Sigma_S^\top\right)^{-1} \mathbf{1}} - \left(\mu_S^\top \left(\Sigma_S \Sigma_S^\top\right)^{-1} \mu_S - U^2\right) \geq \upsilon^2$$

and we infer the bound (9) on υ. On the other hand, the two roots of polynomial in the left hand term of (11) are

$$r^{\pm}(\upsilon) \quad = \quad \frac{2\mu_S^\top \left(\Sigma_S \Sigma_S^\top\right)^{-1} \mathbf{1} \pm \sqrt{D(\upsilon)}}{2\,\mathbf{1}^\top \left(\Sigma_S \Sigma_S^\top\right)^{-1} \mathbf{1}}$$

□

For any admissible parameters (υ, r), the drift of mortality rates under the equivalent measure is provided by the following expression:

$$\begin{aligned} \mu_M(s) &= \mu_d(s) + \sigma_d^\top \Upsilon(\upsilon, r) \\ &= \mu_d(s) + \sigma_{d\,(0:d-1)}^\top \Sigma_S^{-1} \left(\mu_S - r\mathbf{1}_d\right) + \sigma_{d\,(d)}\upsilon\,. \end{aligned}$$

In absence of dependence between the mortality and financial markets ($\sigma_{d\,(0:d-1)}^\top = 0$), the set of admissible equivalent mortality rates is an ellipse centered around $\mu_d(s)$. Whereas the set of admissible discount rate under Q (defined by relation (10)), is an ellipse that is centered around $\frac{2\mu_S^\top \left(\Sigma_S \Sigma_S^\top\right)^{-1} \mathbf{1}}{2\,\mathbf{1}^\top \left(\Sigma_S \Sigma_S^\top\right)^{-1} \mathbf{1}}$.

This central return may be interpreted as the expected return of a portfolio with weights equal to the marginal contribution of each asset to the total variance of the market.

We expect that the real value of a contingent claim instrument that pays a $\mathcal{F}_t$-adapted cash-flow H_T at time T is in an interval:

$$Value \in \left[\min_{\{\upsilon,r\}\in\mathcal{A}} \mathbb{E}^Q\left(\frac{S_t^0}{S_T^0} H_T \,|\, \mathcal{F}_t\right), \max_{\{\upsilon,r\}\in\mathcal{A}} \mathbb{E}^Q\left(\frac{S_t^0}{S_T^0} H_T \,|\, \mathcal{F}_t\right)\right] \tag{12}$$

where $\mathcal{A}$ is the set of parameters that defines equivalent risk neutral measures with an entropy bounded by equation (6):

$$\mathcal{A} = \{\upsilon, r) \,|\, \upsilon^- \leq \upsilon \leq \upsilon^+ , r^-(\upsilon) \leq r \leq r^+(\upsilon)\}$$

The size of the interval in equation (12) measures the model risk and the uncertainty over parameters. What we call "robust price" of a contingent claim is precisely the maximum value attained over the set of eligible equivalent measures:

$$\text{Robust Price} = \max_{\{\upsilon,r\}\in\mathcal{A}} \mathbb{E}^Q\left(\frac{S_t^0}{S_T^0} H_T \,|\, \mathcal{F}_t\right) .$$

If we apply this principle of valuation to provisions, the robust best estimate is defined by the maximum of non robust BE's over the set $\mathcal{A}$:

$$BE_i = \max_{\{\upsilon,r\}\in\mathcal{A}} BE_i^Q \quad i = 0, 1, ...T \tag{13}$$

Remark that our approach is compatible with the rationale of Cochrane and Saa-Requejo (2000)'s Good-Deal-Bound. Here, the expected assets return r is located in an interval $[r^-(\upsilon), r^+(\upsilon)]$ which is close to Cochrane and Saa-Requejo's idea that consists to bound sharpe ratios of assets.

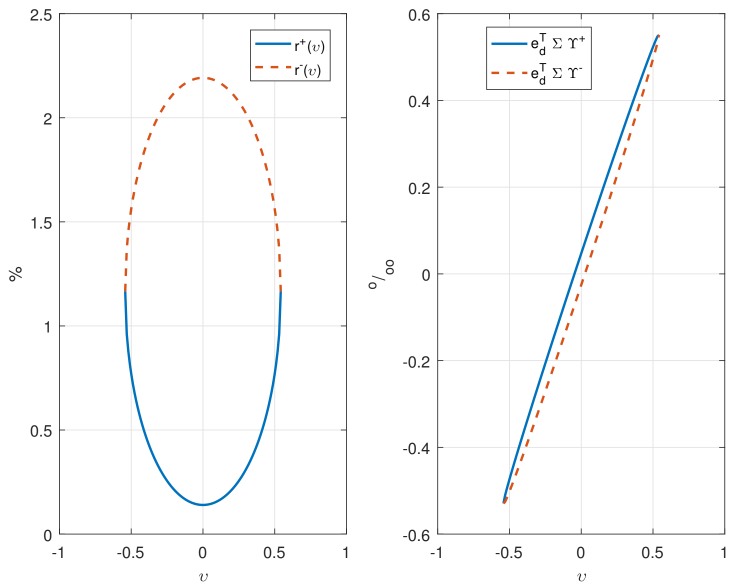

Figure 1: The left plot shows the domain of admissible risk neutral returns r, for the evaluation of contingent claims, when the entropy is bounded by $U = 0.75$. The right plot presents the domain of mortality risk premiums,$\sigma_d^\top \Upsilon(\upsilon, r) = e_d^\top \Sigma \Upsilon(\upsilon, r)$, under Q in function of υ. These graphs are obtained with parameters of table 2.

Parameters	Value	Parameters	Value
μ_S	$\begin{pmatrix} 1\% \\ 5\% \end{pmatrix}$	Std. Dev.	$\begin{pmatrix} 2\% \\ 15\% \\ 0.10\% \end{pmatrix}$
Σ	$\begin{pmatrix} 0.02 & 0 & 0 \\ -0.03 & 0.1470 & 0 \\ 0 & 0.0001 & 0.0010 \end{pmatrix}$	Correlation	$\begin{pmatrix} 1 & -0.20 & 0.05 \\ -0.20 & 1 & 0.05 \\ 0.05 & 0.05 & 1 \end{pmatrix}$

Table 2: The first column reports the parameters μ_S and Σ defining the assets and mortality, used in all numerical illustrations of this article. The second column presents statistics related to Σ. We report the standard deviations of $S^1_{t=1}$, $S^2_{t=1}$, $S^M_{t=1}$ and their correlations.

To conclude this section, we present some numerical results to illustrate the proposition 5.1. The figure (1) shows the domains of $r(\upsilon)$ and of the risk premium $\sigma_d^\top \Upsilon(\upsilon)$ added to the mortality rate under Q, as a function of υ. Parameters used for this exercise are reported in table 2. Given that the constraint on entropy is reformulated as a quadratic constraint on parameters, domains for $r(\upsilon)$ and the mortality risk premiums are elliptical. Any pair of parameters $\left(r, \sigma_d^\top \Upsilon(\upsilon, r)\right)$ inside these ellipsoids corresponds to an eligible risk neutral measure, with an entropy lower than $\exp\left(\frac{1}{2}U^2 t\right)$.

6 Evaluation of best estimate provisions

The first part of this section focuses on the evaluation of non-robust best estimate provisions such as defined by equation (3) and (4). For a given risk neutral measure Q, the evaluation of provisions requires to price multiple European call options of the type:

$$\begin{aligned} &\mathbb{E}^Q\left(\frac{S^M_{t_{k-1}}}{S^M_{t_k}}\frac{S^0_{t_{k-1}}}{S^0_{t_k}}\left[\rho\frac{A_{t_k}}{A_{t_{k-1}}} - e^{g\Delta}\right]_+ |\mathcal{F}_{t_i}\right) \\ &= \mathbb{E}^Q\left(S^M_{t_{k-1}}S^0_{t_{k-1}}\mathbb{E}^Q\left(\frac{1}{S^M_{t_k}S^0_{t_k}}\left[\rho\frac{A_{t_k}}{A_{t_{k-1}}} - e^{g\Delta}\right]_+ |\mathcal{F}_{t_{k-1}}\right)|\mathcal{F}_{t_i}\right). \end{aligned} \tag{14}$$

where $t_i \leq t_k$.Whereas for the last period before expiry of the contract, we have to appraise the following call option:

$$\begin{aligned} &\mathbb{E}^Q\left(\frac{S^0_{t_{k-1}}}{S^0_{t_k}}\left[\rho\frac{A_{t_k}}{A_{t_{k-1}}} - e^{g\Delta}\right]_+ |\mathcal{F}_{t_i}\right) \\ &= \mathbb{E}^Q\left(S^0_{t_{k-1}}\mathbb{E}^Q\left(\frac{1}{S^0_{t_k}}\left[\rho\frac{A_{t_k}}{A_{t_{k-1}}} - e^{g\Delta}\right]_+ |\mathcal{F}_{t_{k-1}}\right)|\mathcal{F}_{t_i}\right). \end{aligned} \tag{15}$$

In order to price these contingent claims, we define an equivalent forward measure denoted by $F(k)$. The numeraire that defines this forward measure is a bond with a payoff linked to the mortality. This kind of mortality bond pays one monetary unit at time t_k if the individual of age $x+t$ survives till age $x+t_k$ and nothing if the person passes away before. The value of such a bond is the expectation of the discounted payoff:

$$\begin{aligned} S_t^{F(k)} &= \mathbb{E}^Q\left(\frac{S^M_t}{S^M_{t_k}}\frac{S^0_t}{S^0_{t_k}} \mid \mathcal{F}_t\right) \\ &= e^{-\left(r+\frac{1}{t_k-t}\int_t^{t_k}\mu_M(s)ds\right)(t_k-t)} \end{aligned} \tag{16}$$

where $t < t_k$ and r is the discount rate under Q. At expiry, the value of this mortality bond is $S^{F(k)}_{t_k} = 1$. The change of measure toward $F(k)$ is defined by a Radon-Nykodym derivative:

$$\frac{dF(k)}{dQ}|_{\mathcal{F}_t} = \frac{S^{F(k)}_{t_k}}{S^M_{t_k}S^0_{t_k}}\frac{S^M_t S^0_t}{S^{F(k)}_t}. \tag{17}$$

We may check that this ratio is a martingale and fulfills all the conditions to be used as a change of measure. On the other hand, under the equivalent forward measure $F(k)$, we have that

$$\mathbb{E}^{F(k)}\left(\left[\rho\frac{A_{t_k}}{A_{t_{k-1}}}-e^{g\Delta}\right]_+ \mid \mathcal{F}_{t_{k-1}}\right) = \frac{\mathbb{E}^{Q}\left(\frac{S_{t_k}^{F(k)}}{S_{t_k}^{M}S_{t_k}^{0}}\frac{S_{t_{k-1}}^{M}S_{t_{k-1}}^{0}}{S_{t_{k-1}}^{F(k)}}\left[\rho\frac{A_{t_k}}{A_{t_{k-1}}}-e^{g\Delta}\right]_+ \mid \mathcal{F}_{t_{k-1}}\right)}{\mathbb{E}^{Q}\left(\frac{S_{t_k}^{F(k)}}{S_{t_k}^{M}S_{t_k}^{0}}\frac{S_{t_{k-1}}^{M}S_{t_{k-1}}^{0}}{S_{t_{k-1}}^{F(k)}} \mid \mathcal{F}_{t_{k-1}}\right)}$$

$$= \frac{\mathbb{E}^{Q}\left(\frac{1}{S_{t_k}^{M}S_{t_k}^{0}}\left[\rho\frac{A_{t_k}}{A_{t_{k-1}}}-e^{g\Delta}\right]_+ \mid \mathcal{F}_{t_{k-1}}\right)}{\mathbb{E}^{Q}\left(\frac{1}{S_{t_k}^{M}S_{t_k}^{0}} \mid \mathcal{F}_{t_{k-1}}\right)}.$$

This last result allows us to rewrite the expected payoff of the option defined by equation (14) as the product of a discount factor times the expected cash-flow under $F(k)$:

$$\mathbb{E}^{Q}\left(\frac{1}{S_{t_k}^{M}S_{t_k}^{0}}\left[\rho\frac{A_{t_k}}{A_{t_{k-1}}}-e^{g\Delta}\right]_+ |\mathcal{F}_{t_{k-1}}\right) = \frac{e^{-\left(r+\frac{1}{\Delta}\int_{t_{k-1}}^{t_k}\mu_M(s)ds\right)\Delta}}{S_{t_{k-1}}^{M}S_{t_{k-1}}^{0}}\times \mathbb{E}^{F(k)}\left(\left[\rho\frac{A_{t_k}}{A_{t_{k-1}}}-e^{g\Delta}\right]_+ \mid \mathcal{F}_{t_{k-1}}\right). \tag{18}$$

So as to calculate the expected payoff in this equation, we need to determine the dynamics of assets returns under $F(k)$. For this purpose, additional notations are required. Firstly, we introduce the following notation:

$$\theta := (\theta_S\,,0) = (\theta_1,\,...,\theta_{d-1},\,0)$$

which is the vector of portfolio weights, complemented by zero. Secondly, we denote by $X_t := \ln\frac{A_t}{A_0}$, the log-return of the total asset. The dynamics of X_t under Q is then provided by a stochastic differential equation (SDE):

$$dX_t = \underbrace{\left(r-\frac{1}{2}\theta^\top\Sigma\Sigma^\top\theta\right)}_{\mu_X}dt+\underbrace{\theta^\top\Sigma}_{\Sigma_X}dW_t\,,$$

where $\mu_X\in\mathbb{R}$ and $\Sigma_X\in\mathbb{R}^{d+1}$ are respectively the constant drift and diffusion coefficients of X_t. We next define a new process, $Y_t =: \ln\frac{S_t^M S_t^0}{S_0^M S_0^0}$ that is led by the following dynamics under Q:

$$dY_t = \underbrace{\left(r+\mu_M(t)-\frac{\left(e_0^\top+e_d^\top\right)\Sigma\Sigma^\top\left(e_0+e_d\right)}{2}\right)}_{\mu_Y(t)}dt+\underbrace{\left(e_0^\top+e_d^\top\right)\Sigma}_{\Sigma_Y}dW_t\,.$$

where e_0 and e_d are $d+1$ vectors, $(1,0,\dots,0)$ and $(0,0,\dots,1)$. the time-varying drift and diffusion coefficients of Y_t are respectively denoted by $\mu_Y(t)$ and Σ_Y. The next proposition is a key result for evaluating the option (14).

Proposition 6.1. *For $t\le s\le t_k$, the moment generating function (mgf) of the total asset log-return, X_s, under $F(k)$ is given by the next expression*

$$\mathbb{E}^{F(k)}\left(e^{\omega X_s}\mid\mathcal{F}_t\right) = \exp\left(\omega X_t+\left(\omega\left(\mu_X-\Sigma_X\Sigma_Y^\top\right)+\frac{1}{2}\Sigma_X\Sigma_X^\top\omega^2\right)(s-t)\right) \tag{19}$$

and its dynamics under $F(k)$ is provided by the following SDE:

$$\begin{aligned} dX_t &= \left(\mu_X-\Sigma_X\Sigma_Y^T\right)dt+\Sigma_X dW_t \\ &= \left(r-\frac{1}{2}\theta^\top\Sigma\Sigma^T\theta-\theta^\top\Sigma\Sigma^\top\left(e_0+e_d\right)\right)dt+\theta^\top\Sigma dW_t \end{aligned} \tag{20}$$

Proof. By construction, the moment generating function of X_s under $F(k)$ is equal to the next ratio under Q

$$\mathbb{E}^{F(k)}\left(e^{\omega X_s}\,|\,\mathcal{F}_t\right) \;=\; \frac{\mathbb{E}^{Q}\left(e^{\omega X_s - Y_{t_k}}\,|\,\mathcal{F}_t\right)}{\mathbb{E}^{Q}\left(e^{-Y_{t_k}}\,|\,\mathcal{F}_t\right)}\,. \tag{21}$$

Given that $\omega X_s - Y_{t_k}$ conditionally to the filtration $\mathcal{F}_t$, is the difference between two Gaussian processes:

$$\begin{aligned}\omega X_s - Y_{t_k}\,|\,\mathcal{F}_t \;&=\; \omega X_t - Y_t + \left(\omega\mu_X(s-t) - \int_t^{t_k}\mu_Y(s)\,ds\right)\\ &\quad+\left(\left(\omega\Sigma_X - \Sigma_Y\right)\left(W_s - W_t\right) - \Sigma_Y\left(W_{t_k} - W_s\right)\right)\,.\end{aligned}$$

The last term of this equation is the sum of two independent normal variables and is then also normal with the following specifications:

$$\begin{aligned}&\left(\omega\Sigma_X - \Sigma_Y\right)\left(W_s - W_t\right) - \Sigma_Y\left(W_{t_k} - W_s\right)\\ &\qquad\sim N\left(0, \sqrt{\left(\omega\Sigma_X - \Sigma_Y\right)\left(\omega\Sigma_X - \Sigma_Y\right)^{\top}(s-t) + \Sigma_Y\Sigma_Y^{\top}\left(t_k - s\right)}\right)\,.\end{aligned}$$

We infer then that the numerator of (21) is the expectation of a lognormal random variable:

$$\begin{aligned}\mathbb{E}^{Q}\left(e^{\omega X_s - Y_{t_k}}\,|\,\mathcal{F}_t\right) &= \exp\left(\omega X_t - Y_t + \omega\mu_X(s-t) - \int_t^{t_k}\mu_Y(s)\,ds\right)\\ &\quad\times\exp\left(\frac{1}{2}\left(\omega\Sigma_X - \Sigma_Y\right)\left(\omega\Sigma_X - \Sigma_Y\right)^{\top}(s-t) + \frac{1}{2}\Sigma_Y\Sigma_Y^{\top}\left(t_k - s\right)\right)\,.\end{aligned}$$

As the denominator of (21) is equal to

$$\mathbb{E}^{Q}\left(e^{-Y_{t_k}}\,|\,\mathcal{F}_t\right) \;=\; \exp\left(-Y_t - \int_t^{t_k}\mu_Y(s)\,ds + \frac{1}{2}\Sigma_Y\Sigma_Y^{\top}\left(t_k - t\right)\right)$$

And after simplification, we finally deduce that the moment generating function of X_t under $F(k)$ is equal to

$$\begin{aligned}\frac{\mathbb{E}^{Q}\left(e^{\omega X_s - Y_{t_k}}\,|\,\mathcal{F}_t\right)}{\mathbb{E}^{Q}\left(e^{-Y_{t_k}}\,|\,\mathcal{F}_t\right)} &= \exp\left(\omega X_t + \left(\omega\mu_X - \omega\Sigma_X\Sigma_Y^{\top}\right)(s-t)\right)\\ &\quad\times\exp\left(\frac{1}{2}\omega\Sigma_X\Sigma_X^{\top}\omega\,(s-t)\right)\,.\end{aligned}$$

This is also the mgf of a process driven by the dynamics proposed in equation (20).

□

From this last proposition, we infer that under the forward measure $F(k)$ and for any $t \le t_k$, the dynamics of the total asset log-return is given by

$$d\ln\frac{A_t}{A_0} \;=\; \left(r - \frac{1}{2}\theta^{\top}\Sigma\Sigma^{T}\theta - \theta^{\top}\Sigma\Sigma^{\top}\left(e_0 + e_d\right)\right)dt + \theta^{\top}\Sigma dW_t\,.$$

Applying the Itô's lemma, allows us to establish the following expression for the total asset under $F(k)$:

$$A_t \;=\; A_0\exp\left(\left(r - \frac{1}{2}\theta^{\top}\Sigma\Sigma^{T}\theta - \theta^{\top}\Sigma\Sigma^{\top}\left(e_0 + e_d\right)\right)t + \theta^{\top}\Sigma W_t\right) \tag{22}$$

and that A_t is a geometric Brownian motion:

$$\frac{dA_t}{A_t} \;=\; \left(r - \theta^{\top}\Sigma\Sigma^{\top}\left(e_0 + e_d\right)\right)dt + \theta^{\top}\Sigma dW_t\,.$$

These features are used in the next proposition to find a closed form expression of options (14) and (15), that are used later as building blocks for evaluating the best estimate provisions.

Proposition 6.2. *The option (14) is equal to*

$$\mathbb{E}^Q\left(\frac{1}{S^M_{t_k}S^0_{t_k}}\left[\rho\frac{A_{t_k}}{A_{t_{k-1}}}-e^{g\Delta}\right]_+|\mathcal{F}_{t_{k-1}}\right) = \frac{e^{-\left(r+\frac{1}{\Delta}\int_{t_{k-1}}^{t_k}\mu_M(s)ds\right)\Delta}}{S^M_{t_{k-1}}S^0_{t_{k-1}}}\left(\rho e^{\left(r-\theta^\top\Sigma\Sigma^\top(e_0+e_d)\right)\Delta}\Phi\left(d_1(\theta)\right)-e^{g\Delta}\Phi\left(d_2(\theta)\right)\right) \tag{23}$$

where $\Phi(.)$ is the cumulative distribution function of a standard normal variable and with:

$$\begin{aligned} d_1(\theta) &= \frac{\ln\rho-\left(g-r-\frac{1}{2}\theta^\top\Sigma\Sigma^\top\theta+\theta^\top\Sigma\Sigma^\top\left(e_0+e_d\right)\right)\Delta}{\sqrt{\theta^\top\Sigma\Sigma^\top\theta}\sqrt{\Delta}}, \\ d_2(\theta) &= d_1(\theta)-\sqrt{\theta^\top\Sigma\Sigma^\top\theta}\sqrt{\Delta}. \end{aligned}$$

Proof. If we remember equation (18), the option (14) is the product of a discount factor and of the expected payoff under the forward measure:

$$\mathbb{E}^Q\left(\frac{1}{S^M_{t_k}S^0_{t_k}}\left[\rho\frac{A_{t_k}}{A_{t_{k-1}}}-e^{g\Delta}\right]_+|\mathcal{F}_{t_{k-1}}\right) = \frac{e^{-\left(r+\int_{t_{k-1}}^{t_k}\mu_M(s)ds\right)\Delta}}{S^M_{t_{k-1}}S^0_{t_{k-1}}}\mathbb{E}^{F(k)}\left(\left[\rho\frac{A_{t_k}}{A_{t_{k-1}}}-e^{g\Delta}\right]_+|\mathcal{F}_{t_{k-1}}\right).$$

The payoff can be reformulated as follows:

$$\mathbb{E}^{F(k)}\left(\left[\rho\frac{A_{t_k}}{A_{t_{k-1}}}-e^{g\Delta}\right]_+|\mathcal{F}_{t_{k-1}}\right) = \rho\mathbb{E}^{F(k)}\left(\left[\frac{A_{t_k}}{A_{t_{k-1}}}-e^{g\Delta-\ln\rho}\right]_+|\mathcal{F}_{t_{k-1}}\right). \tag{24}$$

On the other hand, we know from equation (22) that the total asset is a log-normal random variable under the forward measure $F(k)$:

$$\ln\frac{A_{t_k}}{A_{t_{k-1}}} \sim N\left(\underbrace{\left(r-\frac{1}{2}\theta^\top\Sigma\Sigma^\top\theta-\theta^\top\Sigma\Sigma^\top\left(e_0+e_d\right)\right)}_{\mu_A}\Delta,\underbrace{\sqrt{\theta^\top\Sigma\Sigma^\top\theta}}_{\sigma_A}\sqrt{\Delta}\right)$$

where μ_A and σ_A are respectively the drift and the standard deviation of the total asset log-return. The expectation in the right hand term of equation (24) may then be rewritten by:

$$\begin{aligned} \mathbb{E}^{F(k)}\left(\left[\frac{A_{t_k}}{A_{t_{k-1}}}-e^{g\Delta-\ln\rho}\right]_+|\mathcal{F}_{t_{k-1}}\right) &= \int_{z_{inf}}^{+\infty}\left(e^{\mu_A\Delta+\sigma_A\sqrt{\Delta}z}-e^{g\Delta-\ln\rho}\right)\phi(z)dz \\ &= e^{\mu_A\Delta}\int_{z_{inf}}^{+\infty}\left(e^{\sigma_A\sqrt{\Delta}z}\right)\phi(z)dz-\left(e^{g\Delta-\ln\rho}\right)\left(1-\Phi\left(z_{inf}\right)\right) \end{aligned}$$

where $\phi(z)=\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}$ is the density of a standard normal variable and z_{inf} satisfies $\mu_A\Delta+\sigma_A\sqrt{\Delta}z_{inf}=g\Delta-\ln\rho$ or

$$z_{inf} = \frac{\left(g-\mu_A\right)\Delta-\ln\rho}{\sigma_A\sqrt{\Delta}}.$$

Given that

$$\left(e^{\sigma_A\sqrt{\Delta}z}\right)\phi(z) = \exp\left(\frac{1}{2}\sigma_A^2\Delta\right)\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}\left(z-\sigma_A\sqrt{\Delta}\right)^2\right)$$

the right hand term of equation (24) becomes:

$$\begin{aligned} \mathbb{E}^{F(k)}\left(\left[\frac{A_{t_k}}{A_{t_{k-1}}}-e^{g\Delta-\ln\rho}\right]_+|\mathcal{F}_{t_{k-1}}\right) &= e^{\mu_A\Delta+\frac{1}{2}\sigma_A^2\Delta}\left(1-\Phi\left(z_{inf}-\sigma_A\sqrt{\Delta}\right)\right) \\ &\quad -\left(e^{g\Delta-\ln\rho}\right)\left(1-\Phi\left(z_{inf}\right)\right). \end{aligned}$$

The standard Gaussian random variable being symmetric, $1-\Phi(z_{inf})=\Phi(-z_{inf})$ and the previous expression is then equal to

$$\begin{aligned}\mathbb{E}^{F(k)}\left(\left[\frac{A_{t_k}}{A_{t_{k-1}}}-e^{g\Delta-\ln\rho}\right]_+|\mathcal{F}_{t_{k-1}}\right) &= e^{\mu_A\Delta+\frac{1}{2}\sigma_A^2\Delta}\Phi\left(-z_{inf}+\sigma_A\sqrt{\Delta}\right)-\left(e^{g\Delta-\ln\rho}\right)\Phi\left(-z_{inf}\right)\\ &= e^{\left(\mu_A+\frac{1}{2}\sigma_A^2\right)\Delta}\Phi\left(d_1(\theta)\right)-\left(e^{g\Delta-\ln\rho}\right)\Phi\left(d_2(\theta)\right)\end{aligned}$$

where

$$\begin{aligned}d_1(\theta) &= -\frac{(g-\mu_A)\,\Delta-\ln\rho}{\sigma_A\sqrt{\Delta}}+\sigma_A\sqrt{\Delta}\\ &= \frac{\ln\rho-\left(g-\mu_A-\sigma_A^2\right)\Delta}{\sigma_A\sqrt{\Delta}}\\ d_2(\theta) &= d_1(\theta)-\sigma_A\sqrt{\Delta}\,.\end{aligned}$$

we can conclude that the option (14) admits the closed form expression (23). □

It is interesting to notice that the participation to profits, calculated with the formula (23), depends on mortality through the correlation between financial markets and mortality. If we remember the expression (3) and (4) of best estimate provisions, the option for the last period is independent from the mortality. The next corollary reports the analytical expression of this option.

Corollary 6.3. *The option for the last period of capitalization, embedded in BE provisions (3) and (4) has a value equal to*

$$\begin{aligned}&\mathbb{E}^Q\left(\frac{1}{S^0_{t_k}}\left[\rho\frac{A_{t_k}}{A_{t_{k-1}}}-e^{g\Delta}\right]_+|\mathcal{F}_{t_{k-1}}\right)=\\ &e^{-r\Delta}\left(\rho e^{\left(r-\theta^\top\Sigma\Sigma^\top e_0\right)\Delta}\Phi\left(c_1(\theta)\right)-e^{g\Delta}\Phi\left(c_2(\theta)\right)\right)\end{aligned}$$

where $\Phi(.)$ is the cumulative distribution function of a standard normal variable and where:

$$\begin{aligned}c_1(\theta) &= \frac{\ln\rho-\left(g-r-\frac{1}{2}\theta^\top\Sigma\Sigma^T\theta+\theta^\top\Sigma\Sigma^\top e_0\right)\Delta}{\sqrt{\theta^\top\Sigma\Sigma^\top\theta}\sqrt{\Delta}}\,,\\ c_2(\theta) &= c_1(\theta)-\sqrt{\theta^\top\Sigma\Sigma^\top\theta}\sqrt{\Delta}\,.\end{aligned}$$

We have now all the elements to evaluate the robust best estimate provisions such as defined by equation (13). However, so as to rewrite provisions in a concise way, we introduce the following notations for the options defined by equations (14) and (15):

$$\begin{aligned}\Psi^1(r,\upsilon,i,k) &:= \mathbb{E}^Q\left(S^M_{t_{k-1}}S^0_{t_{k-1}}\mathbb{E}^Q\left(\frac{1}{S^M_{t_k}S^0_{t_k}}\left[\rho\frac{A_{t_k}}{A_{t_{k-1}}}-e^{g\Delta}\right]_+|\mathcal{F}_{t_{k-1}}\right)|\mathcal{F}_{t_i}\right) \qquad (25)\\ &= e^{-\left(r+\frac{1}{\Delta}\int_{t_{k-1}}^{t_k}\mu_M(s)ds\right)\Delta}\left(\rho e^{\left(r-\theta^\top\Sigma\Sigma^\top(e_0+e_d)\right)\Delta}\Phi\left(d_1(\theta)\right)-e^{g\Delta}\Phi\left(d_2(\theta)\right)\right)\,,\end{aligned}$$

and

$$\begin{aligned}\Psi^2(r,i,k) &:= \mathbb{E}^Q\left(S^0_{t_{k-1}}\mathbb{E}^Q\left(\frac{1}{S^0_{t_k}}\left[\rho\frac{A_{t_k}}{A_{t_{k-1}}}-e^{g\Delta}\right]_+|\mathcal{F}_{t_{k-1}}\right)|\mathcal{F}_{t_i}\right) \qquad (26)\\ &= e^{-r\Delta}\left(\rho e^{\left(r-\theta^\top\Sigma\Sigma^\top e_0\right)\Delta}\Phi\left(c_1(\theta)\right)-e^{g\Delta}\Phi\left(c_2(\theta)\right)\right)\,.\end{aligned}$$

In these equations, r and υ determine the risk neutral measure, i is the time index of the reference filtration and k is the index of the accumulation period. If we remember the expression (3) for the BE^Q_0

at time 0, the robust BE_0 is the maximum of provisions over the set $\mathcal{A}$ of admissible parameters:

$$
\begin{aligned}
BE_0 \quad = \quad & C \max_{\{\upsilon,r\}\in\mathcal{A}} \Bigg[\alpha \sum_{j=1}^{n} \prod_{k=1}^{j} \Big(e^{(g-r)\Delta - \int_{t_{k-1}}^{t_k \wedge t_{j-1}} \mu_M(s)ds} \\
& + \big(\Psi^1(r,\upsilon,0,k) 1_{\{k<j\}} + \Psi^2(r,0,j) 1_{\{k=j\}} \big) \Big) \\
& -\alpha \sum_{j=1}^{n} \prod_{k=1}^{j} \Big(e^{(g-r)\Delta - \int_{t_{k-1}}^{t_k} \mu_M(s)ds} + \Psi^1(r,\upsilon,0,k) \Big) \\
& + \prod_{k=1}^{n} \Big(e^{(g-r)\Delta - \int_{t_{k-1}}^{t_k} \mu_M(s)ds} + \Psi^1(r,\upsilon,0,k) \Big) \Bigg] .
\end{aligned}
$$

Similarly the robust best estimate provision at time t_i, just after crediting the participation, is given by:

$$
BE_i \quad = \quad C \prod_{k=1}^{i} \frac{S^0_{t_{k-1}}}{S^0_{t_k}} \left(e^{g\Delta} + \left[\rho \frac{A_{t_k}}{A_{t_{k-1}}} - e^{g\Delta} \right]_+ \right) \times V(i) \tag{27}
$$

where $V(i)$ is defined as follows:

$$
\begin{aligned}
V(i) \quad := \quad & \max_{\{\upsilon,r\}\in\mathcal{A}} \Bigg[\alpha \sum_{j=i+1}^{n} \prod_{k=i+1}^{j} \Big(e^{(g-r)\Delta - \int_{t_{k-1}}^{t_k \wedge t_{j-1}} \mu_M(s)ds} \\
& + \big(\Psi^1(r,\upsilon,i,k) 1_{\{k<j\}} + \Psi^2(r,i,j) 1_{\{k=j\}} \big) \Big) \\
& -\alpha \sum_{j=i+1}^{n} \prod_{k=i+1}^{j} \Big(e^{(g-r)\Delta - \int_{t_{k-1}}^{t_k} \mu_M(s)ds} + \Psi^1(r,\upsilon,i,k) \Big) \\
& + \prod_{k=i+1}^{n} \Big(e^{(g-r)\Delta - \int_{t_{k-1}}^{t_k} \mu_M(s)ds} + \Psi^1(r,\upsilon,0,k) \Big) \Bigg] .
\end{aligned} \tag{28}
$$

We can draw a parallel between the robust approach and the Solvency II regulation. In Solvency II, the total value of liabilities is the sum of BE and the risk margin. This risk margin is the cost of capital needed to cover intrinsic risks of the insurance contract. In the robust evaluation scheme, the value of BE already includes a risk premium for adverse deviation of these intrinsic risks.

Before concluding this section, we present in figure 2 the relation between the average return r of assets under Q and the non-robust best estimate provisions of three contracts with different participation rates and guarantees. The subscriber is a 50 years old man and the average mortality $\mu_d(t)$ is a Makeham function detailed in appendix A. The duration of policies is T =10 years and $\alpha = 1$. The invested capital in the contract is C=100 whereas the insurer's investment strategy is $(\theta_1, \theta_2) = (60\%, 40\%)$. The other characteristics of assets are these presented in table 2.

The three plots of figure 2 reveal an important feature of participating contracts: the sensitivity of BE^Q to r differs widely between contracts. For the first contract ($g = 1\%$ and $\rho = 90\%$), the BE is inversely proportional to r. For the second and third contracts, the non-robust BE is a convex function of r that admits a local minimum. Under certain circumstances, the robust best estimate provision (which is the maximum of BE_i^Q over the set $\mathcal{A}$) is not necessarily obtained with the lowest admissible r over $\mathcal{A}$. In our example, this mainly occurs for contracts with a high participation rate and a negative guarantee. However this type of contract is currently not proposed by insurance companies.

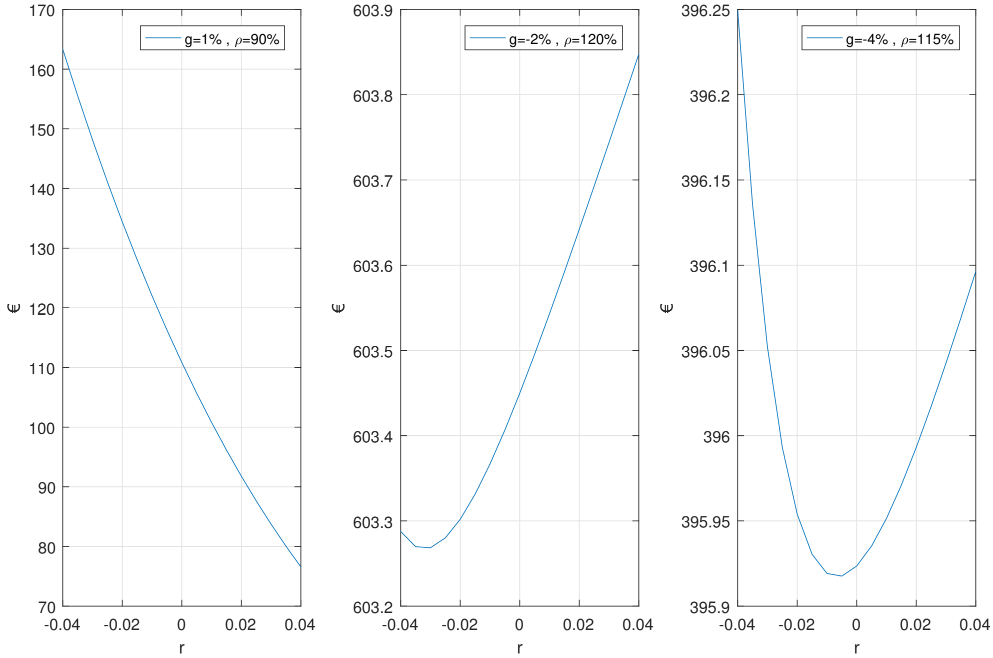

Figure 2: Example of non-robust best-estimate provisions for different level of r.

The figure 3 exhibits the surface of best estimate provisions for all admissible pairs $(\upsilon, r) \in \mathcal{A}$. We consider two contracts, subscribed by a 50 years old man, with the same guarantee and participation rate ($g = 1\%$, $\rho = 0.90$). The first contract foresees the payment of the provision in case of death ($\alpha = 1$). The death benefit for the second policy is equal to ten times the provision[3] ($\alpha = 10$). As in the previous example, the duration is ten years, the capital is $C = 100$ and $(\theta_1, \theta_2) = (60\%, 40\%)$. The entropic parameter U is equal to 0.75. The crosses point the robust best estimate provisions out. These values and the corresponding (υ, r) are reported in table 3. For the first contract, the robust provision is computed with the lowest return available in $\mathcal{A}$, that is $r = r^-(\upsilon)$ with $\upsilon = 0$. This is not the case for the second contract. The robust provision is obtained with $r(\upsilon) \geq r^-(0)$ for $\upsilon = 0.1734$ and with a risk premium added to the reference mortality table. The robust BE provision is then not necessarily computed with the lowest r, particularly if the death benefit is high compared to the provision.

Parameters	υ_{robust}	r_{robust}	BE_{robust}
$g = 1\%\, \rho = 0.90$ and $\alpha = 1$	0	0.14%	109.37
$g = 1\%\, \rho = 0.90$ and $\alpha = 10$	0.1734	0.19%	164.96

Table 3: Robust best estimate provisions and corresponding pair of parameters (υ, r).

7 The robust Net Asset Value (NAV) and Best Estimate (BE)

In the remainder of this work, we assume that the participation is calculated on a yearly basis ($\Delta = 1$). The net asset value (NAV) at time 0 is the difference between the total asset and the best estimate provision. The NAV may be interpreted as the market value of future incomes earned by the insurance company, which is also the market capitalization of the firm. The NAV is then a measure of profitability defined by:

$$NAV_0 \quad := \quad A_0 - BE_0 \tag{29}$$

[3] In practice, death allowances are often lower but our purpose is here to clearly emphasize the role of mortality on the determination of robust BE.

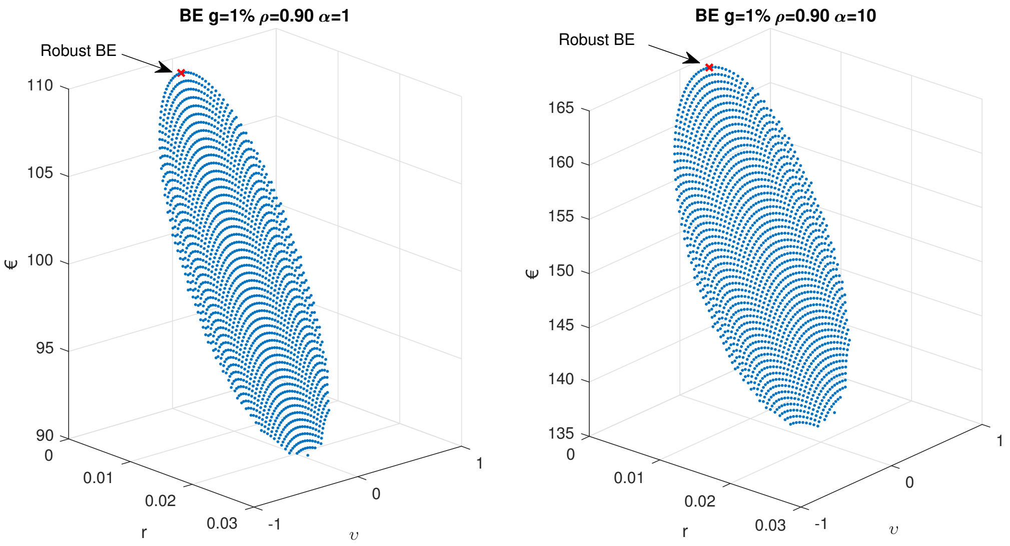

Figure 3: The left and right plots present the surface of BE_0^Q over $\mathcal{A}$ for two contracts. The entropy parameter is set to $U = 0.75$. The red dots point the robust best estimates out.

where BE_0 is the robust best estimate provision, such as defined by equation (27). According to the solvency II regulation, the solvency capital requirement (SCR) corresponds to the economic capital a (re)insurance undertaking needs to hold in order to limit the probability of ruin to 0.5%, i.e. ruin would occur once every 200 years. According to this recommendation, the solvency capital is a percentile of the NAV distribution in one year, under the real measure P. However, the regulator recommends a slightly different mathematical definition which is:

$$P\left(NAV_0 - NAV_{t_1} \geq SCR_0^{reg}\right) \quad = \quad \beta\,. \tag{30}$$

where $\beta = 0.5\%$ is the confidence level. In fact, the SCR defined by this way is simply an approached value of the 0.5% Value at Risk (VaR) of the NAV:

$$P\left(\mathbb{E}\left(NAV_{t_1}\,|\,\mathcal{F}_0\right) - NAV_{t_1} \geq SCR_0\right) \quad = \quad \beta\,. \tag{31}$$

for a confidence level of β=0.5% where the expectation is here evaluated under the real measure P. The solvency capital SCR_0 calculated by this last formula would be higher than SCR_0^{reg} for any profitable insurance company as on average $\mathbb{E}\left(NAV_{t_1}\,|\,\mathcal{F}_0\right) > NAV_0$ in this case. As the solvency capital defined by equation (31) is more conservative then the one obtained with the regulator's formula, we adopt it as definition in the remainder of this article.

The ORSA (Own Risk and Solvency Assessment) is an internal process for evaluating risks and is a tool of control for the Supervisory authorities. The undertaking should develop for the ORSA its own processes with appropriate and adequate techniques, tailored to fit into its risk-management system and taking into consideration the nature, scale and complexity of the risks inherent to the business. Recognition and valuation bases for the ORSA may be different from the Solvency II bases. Furthermore, the undertaking should ensure that its assessment of the overall solvency needs is forward-looking, including a medium term or long-term perspective as appropriate. However, the regulation does not provide detailed guidelines to evaluate the prospective SCR. In our framework, applying the same principle as the one used to evaluate SCR_0, leads to the following definition for prospective SCR_{t_j} for $j \geq 1$:

$$P\left(\mathbb{E}\left(NAV_{t_{j+1}}\,|\,\mathcal{F}_{t_j}\right) - NAV_{t_{j+1}} \geq SCR_{t_j}\,|\,\mathcal{F}_{t_j}\right) \quad = \quad \beta\,. \tag{32}$$

However, in this case SCR_{t_j} is $\mathcal{F}_{t_j}$ adapted. This means that at any times before t_j, the SCR_{t_j} is a random variable. The exact value of SCR_{t_j} is then unknown before t_j. At our knowledge, the distribution of SCR_{t_j} defined in this way is only calculable with Monte Carlo simulations. We also wonder what is the relevance of this measure of risk for asset-liability management or to communicate with shareholders. As authorized by the ORSA, this motivates us to adopt a more natural definition for the SCR_{t_j}:

$$P\left(\mathbb{E}\left(NAV_{t_{j+1}}\,|\,\mathcal{F}_{t_0}\right) - NAV_{t_{j+1}} \geq SCR_{t_j}\right) \quad = \quad 1-(1-\beta)^{j+1}\,, \tag{33}$$

where the confidence level is adjusted year on year. In this approach, the SCR is the Value at risk for a time horizon t_{j+1}, computed with a yearly confidence level of β. Using this method presents two advantages. Firstly, the SCR defined by this way is no more a random variable but a scalar. Secondly, the SCR may be computed without recourse to simulations, if the probability density function (pdf) of $NAV_{t_{j+1}}$ is known. However, this is not the case for the insurance contract that we study. For this reason, we approach the NAV by another random variable, fitted by moments matching. The first step to deploy this method consists to evaluate the moments of $NAV_{t_{j+1}}$. If the individual is still alive at time t_j, this $NAV_{t_{j+1}}$ is equal to

$$NAV_{t_j} = \mathbf{I}_{\{\tau > t_j\}} \left(A_{t_j} - C \prod_{k=1}^{i} \frac{S^0_{t_{k-1}}}{S^0_{t_k}} \left(e^{g\Delta} + \left[\rho \frac{A_{t_k}}{A_{t_{k-1}}} - e^{g\Delta} \right]_+ \right) V(j) \right) \tag{34}$$

where $V(j)$ is defined by equation (28). Due to the independence of increments, the expected $NAV_{t_{j+1}}$ is rewritten as the difference between the expected total asset and a product of an option payoff:

$$\mathbb{E}\left(NAV_{t_j}|\mathcal{F}_0\right) = \mathbb{E}\left(\frac{S^M_0}{S^M_{t_j}} A_{t_j} |\mathcal{F}_0 \right) - C \prod_{k=1}^{j} \mathbb{E}\left[\frac{S^M_{t_{k-1}}}{S^M_{t_k}} \frac{S^0_{t_{k-1}}}{S^0_{t_k}} \left(e^{g\Delta} + \left[\rho \frac{A_{t_k}}{A_{t_{k-1}}} - e^{g\Delta} \right]_+ \right) |\mathcal{F}_0 \right] V(j) \, .$$

This expectation and the other moments of the NAV are provided in the next proposition.

Proposition 7.1. *If $\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$ the expected moment of order u for the NAV at time t_j is provided by the following equation*

$$\mathbb{E}\left(NAV^u_{t_j}|\mathcal{F}_0\right) = \sum_{m=0}^{u} \left[\binom{u}{m} \left(-C \times V(j)\right)^m A_0^{u-m} \prod_{k=1}^{j} \left[\sum_{l=0}^{m} \left[\binom{m}{l} \left(e^{g\Delta}\right)^{m-l} \sum_{p=0}^{l} \left[\binom{l}{p} \left(-e^{g\Delta}\right)^{l-p} (\rho)^p \, h(k,u,l,m,p) \right] \right] \right] \right] \tag{35}$$

where the function $h(k,u,l,m,p)$ is the following expectation under the real measure:

$$h(k,u,l,m,p) = \mathbb{E}\left(\frac{S^M_{t_{k-1}}}{S^M_{t_k}} \left(\frac{A_{t_k}}{A_{t_{k-1}}} \right)^{u-m+p} \left(\frac{S^0_{t_{k-1}}}{S^0_{t_k}} \right)^m \left(\mathbf{I}_{\left\{ \rho \frac{A_{t_k}}{A_{t_{k-1}}} > e^{g\Delta} \right\}} \right)^{\mathbf{I}_{\{l \neq 0\}}} |\mathcal{F}_0 \right) \tag{36}$$

Proof. Given that $\left(\mathbf{I}_{\{\tau > t_j\}}\right)^u = \mathbf{I}_{\{\tau > t_j\}}$ for all $u > 0$, we have that

$$NAV^u_{t_j} = \mathbf{I}_{\{\tau > t_j\}} \left(A_{t_j} - C \prod_{k=1}^{i} \frac{S^0_{t_{k-1}}}{S^0_{t_k}} \left(e^{g\Delta} + \left[\rho \frac{A_{t_k}}{A_{t_{k-1}}} - e^{g\Delta} \right]_+ \right) V(j) \right)^u .$$

Applying the Newton's binomial theorem to this expression and taking its expectation under P leads to the following equality

$$\begin{aligned} &\mathbb{E}\left(NAV^u_{t_j}|\mathcal{F}_0\right) = \\ &\mathbb{E}\left(\sum_{m=0}^{u} \binom{u}{m} \frac{S^M_0}{S^M_{t_j}} \left(A_{t_j}\right)^{u-m} \left(\prod_{k=1}^{j} \frac{S^0_{t_{k-1}}}{S^0_{t_k}} \left(e^{g\Delta} + \left[\rho \frac{A_{t_k}}{A_{t_{k-1}}} - e^{g\Delta} \right]_+ \right) \left(-C \times V(j,\theta)\right) \right)^m |\mathcal{F}_0 \right) = \\ &\sum_{m=0}^{u} \binom{u}{m} \left(-C \times V(j)\right)^m A_0^{u-m} \, \mathbb{E}\left(\frac{S^M_0}{S^M_{t_j}} \left(\frac{A_{t_j}}{A_0} \right)^{u-m} \left(\prod_{k=1}^{j} \frac{S^0_{t_{k-1}}}{S^0_{t_k}} \left(e^{g\Delta} + \left[\rho \frac{A_{t_k}}{A_{t_{k-1}}} - e^{g\Delta} \right]_+ \right) \right)^m |\mathcal{F}_0 \right) \end{aligned} \tag{37}$$

On another hand, the decomposition

$$\frac{S^M_0}{S^M_{t_j}} \left(\frac{A_{t_j}}{A_0} \right)^{u-m} = \frac{S^M_0}{S^M_{t_1}} \left(\frac{A_{t_1}}{A_0} \right)^{u-m} \cdots \frac{S^M_{t_{j-2}}}{S^M_{t_{j-1}}} \left(\frac{A_{t_{j-1}}}{A_{t_{j-2}}} \right)^{u-m} \frac{S^M_{t_{j-1}}}{S^M_{t_j}} \left(\frac{A_{t_j}}{A_{t_{j-1}}} \right)^{u-m}$$

and the independence of increments, allows us to rewrite the expected product of option payoffs in equation (37) as a product of their expectations:

$$\mathbb{E}\left(NAV_{t_j}^u|\mathcal{F}_0\right) = \sum_{m=0}^{u}\binom{u}{m}\left(-C\times V(j)\right)^m A_0^{u-m}$$
$$\times\prod_{k=1}^{j}\mathbb{E}\left(\frac{S_{t_{k-1}}^M}{S_{t_k}^M}\left(\frac{A_{t_k}}{A_{t_{k-1}}}\right)^{u-m}\left(\frac{S_{t_{k-1}}^0}{S_{t_k}^0}\right)^m\left(e^{g\Delta}+\left[\rho\frac{A_{t_k}}{A_{t_{k-1}}}-e^{g\Delta}\right]_+\right)^m|\mathcal{F}_0\right).$$

If we apply the Newton's binomial theorem to the last term in these expectations, we infer that

$$\left(e^{g\Delta}+\left[\rho\frac{A_{t_k}}{A_{t_{k-1}}}-e^{g\Delta}\right]_+\right)^m = \sum_{l=0}^{m}\binom{m}{l}\left(e^{g\Delta}\right)^{m-l}\left(\left[\rho\frac{A_{t_k}}{A_{t_{k-1}}}-e^{g\Delta}\right]_+\right)^l$$

and

$$\mathbb{E}\left(NAV_{t_j}^u|\mathcal{F}_0\right) = \sum_{m=0}^{u}\binom{u}{m}\left(-C\times V(j)\right)^m A_0^{u-m}$$
$$\times\prod_{k=1}^{j}\sum_{l=0}^{m}\binom{m}{l}\left(e^{g\Delta}\right)^{m-l}\mathbb{E}\left(\frac{S_{t_{k-1}}^M}{S_{t_k}^M}\left(\frac{A_{t_k}}{A_{t_{k-1}}}\right)^{u-m}\left(\frac{S_{t_{k-1}}^0}{S_{t_k}^0}\right)^m\left(\left[\rho\frac{A_{t_k}}{A_{t_{k-1}}}-e^{g\Delta}\right]_+\right)^l|\mathcal{F}_0\right).$$

Given that $\left[\rho\frac{A_{t_k}}{A_{t_{k-1}}}-e^{g\Delta}\right]_+ = \left(\rho\frac{A_{t_k}}{A_{t_{k-1}}}\mathbf{I}_{\left\{\rho\frac{A_{t_k}}{A_{t_{k-1}}}>e^{g\Delta}\right\}}-e^{g\Delta}\,\mathbf{I}_{\left\{\rho\frac{A_{t_k}}{A_{t_{k-1}}}>e^{g\Delta}\right\}}\right)$, we deduce the next equality:

$$\left(\left[\rho\frac{A_{t_k}}{A_{t_{k-1}}}-e^{g\Delta}\right]_+\right)^l = \left(\rho\frac{A_{t_k}}{A_{t_{k-1}}}-e^{g\Delta}\right)^l\left(\mathbf{I}_{\left\{\rho\frac{A_{t_k}}{A_{t_{k-1}}}>e^{g\Delta}\right\}}\right)^{\mathbf{I}_{l\neq 0}}$$
$$= \sum_{p=0}^{l}\binom{l}{p}\left(-e^{g\Delta}\right)^{l-p}\left(\rho\frac{A_{t_k}}{A_{t_{k-1}}}\right)^p\left(\mathbf{I}_{\left\{\rho\frac{A_{t_k}}{A_{t_{k-1}}}>e^{g\Delta}\right\}}\right)^{\mathbf{I}_{l\neq 0}}$$

And finally, we conclude that

$$\mathbb{E}\left(\frac{S_{t_{k-1}}^M}{S_{t_k}^M}\left(\frac{A_{t_k}}{A_{t_{k-1}}}\right)^{u-m}\left(\frac{S_{t_{k-1}}^0}{S_{t_k}^0}\right)^m\left(\left[\rho\frac{A_{t_k}}{A_{t_{k-1}}}-e^{g\Delta}\right]_+\right)^l|\mathcal{F}_0\right)$$
$$=\sum_{p=0}^{l}\binom{l}{p}\left(-e^{g\Delta}\right)^{l-p}(\rho)^p\,\mathbb{E}\left(\frac{S_{t_{k-1}}^M}{S_{t_k}^M}\left(\frac{A_{t_k}}{A_{t_{k-1}}}\right)^{u-m+p}\left(\frac{S_{t_{k-1}}^0}{S_{t_k}^0}\right)^m\left(\mathbf{I}_{\left\{\rho\frac{A_{t_k}}{A_{t_{k-1}}}>e^{g\Delta}\right\}}\right)^{\mathbf{I}_{l\neq 0}}|\mathcal{F}_0\right).$$

□

In a similar manner, we establish the formula for the moments of the robust best estimate provisions:

Proposition 7.2. *The expected best estimate provision of order u for the NAV at time t_j is provided by the next equation*

$$\mathbb{E}\left(BE_{t_j}^u|\mathcal{F}_0\right) = \left(C\,V(j)\right)^u\prod_{k=1}^{j}\sum_{l=0}^{u}\binom{u}{l}\left(e^{g\Delta}\right)^{u-l}\sum_{p=0}^{l}\binom{l}{p}\left(-e^{g\Delta}\right)^{l-p}(\rho)^p\,h(k,u,l,u,p) \quad (38)$$

where the function $h(k,u,l,u,p)$ is defined by equation (36).

Proof. Given that $\left(\mathbf{I}_{\{\tau>t_j\}}\right)^u = \mathbf{I}_{\{\tau>t_j\}}$ for all $u>0$, we have that

$$BE_{t_j}^u = \mathbf{I}_{\{\tau>t_j\}}C^uV(j)^u\left(\prod_{k=1}^{i}\frac{S_{t_{k-1}}^0}{S_{t_k}^0}\left(e^{g\Delta}+\left[\rho\frac{A_{t_k}}{A_{t_{k-1}}}-e^{g\Delta}\right]_+\right)\right)^u.$$

Applying the Newton's binomial theorem to the NAV expression (34) leads to the following equality

$$\mathbb{E}\left(BE^u_{t_j}|\mathcal{F}_0\right) =$$
$$(CV(j))^u\,\mathbb{E}\left(\frac{S_0^M}{S_{t_j}^M}\prod_{k=1}^{j}\left(\frac{S^0_{t_{k-1}}}{S^0_{t_k}}\right)^u\left(e^{g\Delta}+\left[\rho\frac{A_{t_k}}{A_{t_{k-1}}}-e^{g\Delta}\right]_+\right)^u|\,\mathcal{F}_0\right)$$

On another hand, the decomposition

$$\frac{S_0^M}{S_{t_j}^M} = \frac{S_0^M}{S_{t_1}^M}\cdots\frac{S_{t_{j-2}}^M}{S_{t_{j-1}}^M}\frac{S_{t_{j-1}}^M}{S_{t_j}^M}$$

and the independence of increments, allows us to rewrite the expectation of the product of option payoffs as the product of their expectations:

$$\mathbb{E}\left(BE^u_{t_j}|\mathcal{F}_0\right) = (CV(j))^u\prod_{k=1}^{j}\mathbb{E}\left(\frac{S_{t_{k-1}}^M}{S_{t_k}^M}\left(\frac{S^0_{t_{k-1}}}{S^0_{t_k}}\right)^u\left(e^{g\Delta}+\left[\rho\frac{A_{t_k}}{A_{t_{k-1}}}-e^{g\Delta}\right]_+\right)^u|\,\mathcal{F}_0\right)$$

If we apply the Newton's binomial theorem to the last term in these expectations, we infer that

$$\left(e^{g\Delta}+\left[\rho\frac{A_{t_k}}{A_{t_{k-1}}}-e^{g\Delta}\right]_+\right)^u = \sum_{l=0}^{u}\binom{u}{l}\left(e^{g\Delta}\right)^{u-l}\left(\left[\rho\frac{A_{t_k}}{A_{t_{k-1}}}-e^{g\Delta}\right]_+\right)^l .$$

Given that:

$$\begin{aligned}\left(\left[\rho\frac{A_{t_k}}{A_{t_{k-1}}}-e^{g\Delta}\right]_+\right)^l &= \left(\rho\frac{A_{t_k}}{A_{t_{k-1}}}-e^{g\Delta}\right)^l\left(\mathbf{I}_{\left\{\rho\frac{A_{t_k}}{A_{t_{k-1}}}>e^{g\Delta}\right\}}\right)^{\mathbf{I}_{\{l\neq 0\}}}\\ &= \sum_{p=0}^{l}\binom{l}{p}\left(-e^{g\Delta}\right)^{l-p}\left(\rho\frac{A_{t_k}}{A_{t_{k-1}}}\right)^p\left(\mathbf{I}_{\left\{\rho\frac{A_{t_k}}{A_{t_{k-1}}}>e^{g\Delta}\right\}}\right)^{\mathbf{I}_{\{l\neq 0\}}}\end{aligned}$$

we can conclude that the BE provisions are given by the expression (38) □

The moments of the net asset value and of the best estimate provisions both depends upon a function $h(.)$ that admits an analytical representation:

Proposition 7.3. *The expectations denoted by $h(k,u,l,m,p)$ have the following closed-form expressions:*

If $l=0$,

$$\begin{aligned}h(k,u,0,m,p) &= \exp\left((u-m+p)\left(\theta_S^\top\mu_S-\frac{1}{2}\theta^\top\Sigma\Sigma^\top\theta\right)\Delta-m\left(\mu_0-\frac{1}{2}e_0^\top\Sigma\Sigma^\top e_0\right)\Delta\right) \qquad (39)\\ &\times\exp\left(\frac{1}{2}\left((u-m+p)\theta-m\,e_0-e_d\right)^\top\Sigma\Sigma^\top\left((u-m+p)\theta-m\,e_0-e_d\right)\right)\\ &\times\exp\left(-\int_{t_{k-1}}^{t_k}\mu_d(s)ds+\frac{1}{2}e_d^\top\Sigma\Sigma^\top e_d\Delta\right)\end{aligned}$$

If $l\neq 0$,

$$\begin{aligned}h(k,u,l,m,p) = \exp&\left((u-m+p)\left(\theta_S^\top\mu_S-\frac{1}{2}\theta^\top\Sigma\Sigma^\top\theta\right)\Delta-m\left(\mu_0-\frac{1}{2}e_0^\top\Sigma\Sigma^\top e_0\right)\Delta\right) \qquad (40)\\ &\times\exp\left(-\int_{t_{k-1}}^{t_k}\mu_d(s)ds+\frac{1}{2}e_d^\top\Sigma\Sigma^\top e_d\Delta\right)\times\exp\left(\frac{1}{2}\gamma_Y^2\right)\left(1-\Phi(x_{min}-\gamma_Y\rho_{XY})\right)\end{aligned}$$

where $\Phi(.)$ is the cdf of a $N(0,1)$ and γ_Y, ρ_{XY} , x_{min} are constant and equal to

$$\gamma_Y := \sqrt{\left((u-m+p)\theta-m\,e_0-e_d\right)^\top\Sigma\Sigma^\top\left((u-m+p)\theta-m\,e_0-e_d\right)}\sqrt{\Delta}\,.$$

$$\rho_{XY} \quad := \quad \frac{\left[\theta^\top \Sigma\Sigma^\top \left((u-m+p)\theta - m\,e_0 - e_d\right)\right]\sqrt{\Delta}}{\gamma_Y \sqrt{\theta^\top \Sigma\Sigma^\top \theta}}\,.$$

$$x_{min} := \frac{g\Delta - \ln\rho - \left(\theta_S^\top \mu_S - \frac{1}{2}\theta^\top \Sigma\Sigma^\top\theta\right)\Delta}{\sqrt{\theta^\top \Sigma\Sigma^\top\theta}\sqrt{\Delta}}$$

Proof. When $l = 0$, $h(k,u,0,m,p)$ is given by

$$h(k,u,0,m,p) \quad = \quad \mathbb{E}\left(\frac{S^M_{t_{k-1}}}{S^M_{t_k}}\left(\frac{A_{t_k}}{A_{t_{k-1}}}\right)^{u-m+p}\left(\frac{S^0_{t_{k-1}}}{S^0_{t_k}}\right)^m |\mathcal{F}_0\right) \tag{41}$$

and as

$$\begin{aligned}
&\frac{S^M_{t_{k-1}}}{S^M_{t_k}}\left(\frac{A_{t_k}}{A_{t_{k-1}}}\right)^{u-m+p}\left(\frac{S^0_{t_{k-1}}}{S^0_{t_k}}\right)^m \\
&= \exp\left((u-m+p)\left(\theta_S^\top\mu_S - \frac{1}{2}\theta^\top\Sigma\Sigma^\top\theta\right)\Delta - m\left(\mu_0 - \frac{1}{2}e_0^\top\Sigma\Sigma^\top e_0\right)\Delta\right) \\
&\times \exp\left(-\int_{t_{k-1}}^{t_k}\mu_d(s)ds + \frac{1}{2}e_d^\top\Sigma\Sigma^\top e_d\Delta\right) \\
&\times \exp\left(\left((u-m+p)\theta - m\,e_0 - e_d\right)^\top\Sigma\left[B_{t_k} - B_{t_{k-1}}\right]\right)
\end{aligned}$$

we find the relation (39). When $l \neq 0$

$$h(k,u,l,m,p) \quad = \quad \mathbb{E}\left(\frac{S^M_{t_{k-1}}}{S^M_{t_k}}\left(\frac{A_{t_k}}{A_{t_{k-1}}}\right)^{u-m+p}\left(\frac{S^0_{t_{k-1}}}{S^0_{t_k}}\right)^m \mathbf{I}_{\left\{\rho\frac{A_{t_k}}{A_{t_{k-1}}} > e^{g\Delta}\right\}}|\mathcal{F}_0\right) \tag{42}$$

and the function $h(.)$ becomes:

$$\begin{aligned}
h(k,u,m,p) = \exp&\left((u-m+p)\left(\theta_S^\top\mu_S - \frac{1}{2}\theta^\top\Sigma\Sigma^\top\theta\right)\Delta - m\left(\mu_0 - \frac{1}{2}e_0^\top\Sigma\Sigma^\top e_0\right)\Delta\right) \\
&\times \exp\left(-\int_{t_{k-1}}^{t_k}\mu_d(s)ds + \frac{1}{2}e_d^\top\Sigma\Sigma^\top e_d\Delta\right) \\
&\times\mathbb{E}\left(\exp\left(\left((u-m+p)\theta - m\,e_0 - e_d\right)^\top\Sigma\left[B_{t_k} - B_{t_{k-1}}\right]\right)\mathbf{I}_{\left\{\rho\frac{A_{t_k}}{A_{t_{k-1}}} > e^{g\Delta}\right\}}|\mathcal{F}_0\right)
\end{aligned} \tag{43}$$

Under the real measure, the ratio $\frac{A_{t_k}}{A_{t_{k-1}}}$ is equal to the following exponential:

$$\frac{A_{t_k}}{A_{t_{k-1}}} \quad = \quad \exp\left(\left(\theta_S^\top\mu_S - \frac{1}{2}\theta^\top\Sigma\Sigma^\top\theta\right)\Delta + \theta^\top\Sigma\left[B_{t_k} - B_{t_{k-1}}\right]\right)$$

and the condition $\rho\frac{A_{t_k}}{A_{t_{k-1}}} > e^{g\Delta}$ is equivalent to

$$\frac{\theta^\top\Sigma\left[B_{t_k} - B_{t_{k-1}}\right]}{\sqrt{\theta^\top\Sigma\Sigma^\top\theta}\sqrt{\Delta}} \quad > \quad \underbrace{\frac{g\Delta - \ln\rho - \left(\theta_S^\top\mu_S - \frac{1}{2}\theta^\top\Sigma\Sigma^\top\theta\right)\Delta}{\sqrt{\theta^\top\Sigma\Sigma^\top\theta}\sqrt{\Delta}}}_{x_{min}}$$

The left hand term in this last inequality is a standard normal random variable, that we denote by X in the rest of the proof:

$$X \quad := \quad \frac{\theta^\top\Sigma\left[B_{t_k} - B_{t_{k-1}}\right]}{\sqrt{\theta^\top\Sigma\Sigma^\top\theta}\sqrt{\Delta}}\,.$$

If we define another standard normal random variable, Y, as follows

$$Y \quad := \quad \frac{\left((u-m+p)\theta - m\,e_0 - e_d\right)^\top\Sigma\left[B_{t_k} - B_{t_{k-1}}\right]}{\sqrt{\left((u-m+p)\theta - m\,e_0 - e_d\right)^\top\Sigma\Sigma^\top\left((u-m+p)\theta - m\,e_0 - e_d\right)}\sqrt{\Delta}}$$

the random vector is a standard bivariate Gaussian variable

$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho_{XY} \\ \rho_{XY} & 1 \end{pmatrix}\right)$$

To lighten developments we introduce the following notation:

$$\gamma_Y := \sqrt{((u-m+p)\theta - m\, e_0 - e_d)^\top \Sigma\Sigma^\top ((u-m+p)\theta - m\, e_0 - e_d)}\sqrt{\Delta}\,.$$

that allows us to define ρ_{XY}, the correlation between X and Y :

$$\rho_{XY} := \frac{\left[\theta^\top \Sigma\Sigma^\top ((u-m+p)\theta - m\, e_0 - e_d)\right]\sqrt{\Delta}}{\gamma_Y \sqrt{\theta^\top \Sigma\Sigma^\top \theta}}\,.$$

The expectation in the intermediate expression (43) of $h(.)$ is then equal to $\mathbb{E}\left(\exp(\gamma_Y\, Y)\,\mathbf{I}_{\{X>x_{min}\}}|\mathcal{F}_0\right)$. To evaluate this expectation, we reformulate the random variables X and Y as a linear combination of two independent standard normal variables X_1 and X_2:

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \rho_{XY} & \sqrt{1-\rho_{XY}^2} \end{pmatrix}\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$$

The independence between X_1 and X_2 allows us to decompose the expectation $\mathbb{E}\left(\exp(\gamma_Y\, Y)\,\mathbf{I}_{\{X>x_{inf}\}}|\mathcal{F}_0\right)$ as follows

$$\begin{aligned} \mathbb{E}\left(\exp(\gamma_Y\, Y)\,\mathbf{I}_{\{X>x_{inf}\}}|\mathcal{F}_0\right) &= \mathbb{E}\left(\exp\left(\gamma_Y \rho_{XY} X_1 + \gamma_Y\sqrt{1-\rho_{XY}^2}X_2\right)\mathbf{I}_{\{X_1>x_{inf}\}}|\mathcal{F}_0\right) \\ &= \mathbb{E}\left(\exp(\gamma_Y \rho_{XY} X_1)\,\mathbf{I}_{\{X_1>x_{inf}\}}|\mathcal{F}_0\right)\times \mathbb{E}\left(\exp\left(\gamma_Y\sqrt{1-\rho_{XY}^2}X_2\right)|\mathcal{F}_0\right) \end{aligned} \tag{44}$$

The second expectation is equal to

$$\mathbb{E}\left(\exp\left(\gamma_Y\sqrt{1-\rho_{XY}^2}X_2\right)|\mathcal{F}_0\right) = \exp\left(\frac{1}{2}\gamma_Y^2\left(1-\rho_{XY}^2\right)\right)$$

The first expectation in the equation (44) is given by

$$\mathbb{E}\left(\exp(\gamma_Y\, \rho_{XY}\, X_1)\,\mathbf{I}_{\{X_1>x_{min}\}}|\mathcal{F}_0\right) = \int_{x_{min}}^{+\infty} e^{\gamma_Y\, \rho_{XY}\, x}\phi(x)dx$$

where $\phi(x)$ is the pdf of standard $N(0,1)$. After simplifications, we obtain that

$$\begin{aligned} \mathbb{E}\left(\exp(\gamma_Y \rho_{XY} X_1)\,\mathbf{I}_{\{X_1>x_{min}\}}|\mathcal{F}_0\right) &= e^{\frac{1}{2}(\gamma_Y\rho_{XY})^2}\int_{x_{min}}^{+\infty}\phi(x-\gamma_Y\rho_{XY})dx \\ &= e^{\frac{1}{2}(\gamma_Y\rho_{XY})^2}\left(1-\Phi(x_{min}-\gamma_Y\rho_{XY})\right) \end{aligned}$$

and finally,

$$\begin{aligned} \mathbb{E}\left(\exp(\gamma_Y\, Y)\,\mathbf{I}_{X>x_{inf}}|\mathcal{F}_0\right) &= e^{\frac{1}{2}\gamma_Y^2\left(1-\rho_{XY}^2\right)}e^{\frac{1}{2}(\gamma_Y\rho_{XY})^2}\left(1-\Phi(x_{min}-\gamma_Y\rho_{XY})\right) \\ &= e^{\frac{1}{2}\gamma_Y^2}\left(1-\Phi(x_{min}-\gamma_Y\rho_{XY})\right)\,. \end{aligned}$$

□

The figure 4 presents the expected future NAV and BE calculated with propositions (7.1) and (7.2), for the participating contract having the specifications reported in table 4. The upper graphs show the robust estimates obtained with an entropy parameter $U = 0.75$. The lower plots exhibit the non robust BE and NAV, evaluated with the assumptions that $r = \mu_0$ and $\upsilon = 0$ (which are the natural assumptions done in practice by actuaries). As we could forecast, the robust NAV and BE are much more conservative that their non robust equivalents. However, we will see in the next section that working with a prudent estimate of the NAV, does not necessarily raise the solvency capital requirement.

Parameters	Value	Parameters	Value
g	1%	α	1
ρ	90%	C	100
x	50	T	10
θ_1	60%	θ_2	40%
A_0	110	U	0.75

Table 4: Parameters of the participating policy used to construct the figure 4.

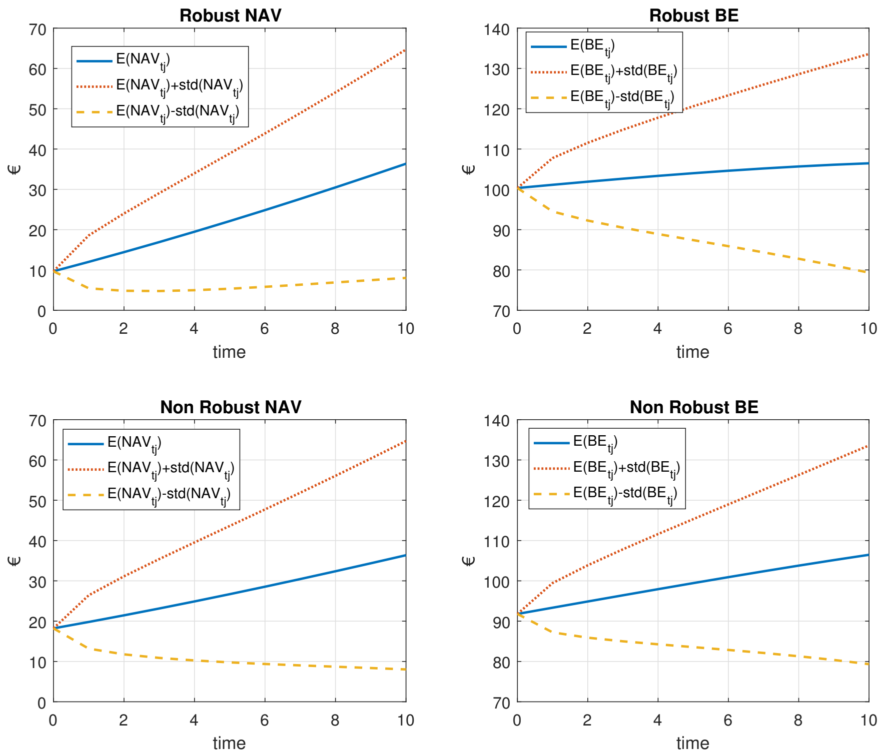

Figure 4: Upper plots show the average robust expected NAV and BE for the contract with the specifications of table 4. The lower graph presents the non robust equivalents, calculated with $r = \mu_0$ and $\upsilon = 0$.

8 Evaluation of the SCR

To calculate the initial and prospective solvency capital requirements as defined by relations (30) and (33), we approach the pdf of the NAV by another random variable denoted by $N\tilde{A}V_{t_j}$ for the period t_j. This variable shares the same first moments. Two distributions are considered to approximate the NAV: the Gaussian and the Normal Inverse Gaussian (NIG). A similar approach was implemented in Hainaut (2016) to evaluate the SCR of variable annuities. For $j = 1$ to n, the Gaussian law is identified by its mean and its standard deviation as follows

$$N\tilde{A}V_{t_j} \sim N\left(\mu_j^{gaus}, \sigma_j^{gaus}\right)$$

where $\mu_j^{gaus} := \mathbb{E}\left(NAV_{t_j}\right)$ and $\left(\sigma_j^{gaus}\right)^2 := \mathbb{E}\left(NAV_{t_j}^2\right) - \mathbb{E}\left(NAV_{t_j}\right)^2$ are calculated by proposition 7.1. The NIG approximation of the NAV is defined by four parameters

$$N\tilde{A}V_{t_j} \sim NIG\left(\mu_j^{gaus}, \alpha_j^{nig}, \beta_j^{nig}, \delta_j^{nig}\right)$$

where the parameters α_j^{nig} and β_j^{nig} must satisfy the constraint, $\alpha_j^{nig\,2}-\beta_j^{nig\,2}\geq 0$. If $\gamma_j^{nig}:=\sqrt{\alpha_j^{nig\,2}-\beta_j^{nig\,2}}$, the mean, variance, skewness and excess of kurtosis of $\tilde{NAV}_{t_j}$ are equal to:

$$
\begin{aligned}
\mathbb{E}\left(\tilde{NAV}_{t_j}\right) &= \mu_j^{nig}+\frac{\delta_j^{nig}\beta_j^{nig}}{\sqrt{\alpha_j^{nig\,2}-\beta_j^{nig\,2}}}\,, && (45)\\
\mathbb{V}\left(\tilde{NAV}_{t_j}\right) &= \frac{\delta_j^{nig}(\beta_j^{nig\,2}+\gamma_j^{nig\,2})}{\gamma_j^{nig\,3}}\,, && (46)\\
\mathbb{S}\left(\tilde{NAV}_{t_j}\right) &= 3\frac{\beta_j^{nig}}{\alpha_j^{nig}\sqrt{\delta_j^{nig}\gamma_j^{nig}}}\,, && (47)\\
\mathbb{K}(\tilde{NAV}_{t_j}) &= 3\frac{\alpha_j^{nig\,2}+4\beta_j^{nig\,2}}{\delta_j^{nig}\alpha_j^{nig\,2}\gamma_j^{nig}}-3\,. && (48)
\end{aligned}
$$

If we remember that the skewness and the kurtosis of the NAV are related to its non centered moments by the relations

$$
\mathbb{S}\left(NAV_{t_j}\right) = \frac{\mathbb{E}\left(NAV_{t_j}^3\right)-3\mathbb{E}\left(NAV_{t_j}\right)\mathbb{V}\left(NAV_{t_j}\right)-\mathbb{E}\left(NAV_{t_j}\right)^3}{\mathbb{V}(NAV_{t_j})^{\frac{3}{2}}}\,,
$$

$$
\begin{aligned}
\mathbb{K}(NAV_{t_j}) = \frac{1}{\left(\mathbb{V}(NAV_{t_j})\right)^2}&\left(\mathbb{E}\left(NAV_{t_j}^4\right)-4\mathbb{E}\left(NAV_{t_j}\right)\mathbb{E}\left(NAV_{t_j}^3\right)\right.\\
&\left.+6\mathbb{E}\left(NAV_{t_j}\right)^2\mathbb{E}\left(NAV_{t_j}^2\right)-3\mathbb{E}\left(NAV_{t_j}\right)^4\right)-3
\end{aligned}
$$

we can easily compute them by proposition 7.1 and the parameters $\mu_j^{nig},\alpha_j^{nig},\beta_j^{nig},\delta_j^{nig}$ are obtained by matching the moments of $\tilde{NAV}_{t_j}$ on these of NAV_{t_j}. The density function of $\tilde{NAV}_{t_j}$, that is denoted by $g_j(y,\,\mu_j^{nig},\alpha_j^{nig},\beta_j^{nig},\delta_j^{nig})$, has a closed form expression:

$$
\begin{aligned}
g(.) &= a(\alpha_j^{nig},\beta_j^{nig},\delta_j^{nig})q\left(\frac{y-\mu_j^{nig}}{\delta_j^{nig}}\right)^{-1} && (49)\\
&\quad\times K_1\left(\delta_j^{nig}\alpha_j^{nig}q\left(\frac{y-\mu_j^{nig}}{\delta_j^{nig}}\right)\right)e^{\beta_j^{nig}(y-\mu_j^{nig})}
\end{aligned}
$$

where $q(x)=\sqrt{1+x^2}$, $K_1(x)$ is the third order Bessel function and

$$
a(\alpha_j^{nig},\beta_j^{nig},\delta_j^{nig}) = \pi^{-1}\alpha_j^{nig}e^{\delta_j^{nig}\sqrt{(\alpha_j^{nig\,2}-\beta_j^{nig\,2})}}.
$$

Once that $\tilde{NAV}_{t_j}$ are fitted by moments matching, the current and prospective solvency capital requirements are determined by relations (30) and (33). So as to illustrate these developments, the table 5 reports the robust ratios $\frac{SCR_t}{\mathbb{E}(BE_t)}$ and $\frac{\mathbb{E}(NAV_t)}{SCR_t}$ computed with Gaussian and NIG approximations, for the participating contract specified in table 4. These ratios may respectively be interpreted as a measure of risk and of profitability. We observe that for the first 5 years, the NIG model produces higher robust SCR's and lower NAV's than the Gaussian approximation. From year 5 to 10, the trend is inverted.

The figure 5 compares robust and non robust ratios calculated with the actuarial assumption that $r=\mu_0$ and $\upsilon=0$. The relative expected NAV on SCR are convex functions of time with a local minimum between 2 and 4 years, depending upon the model. The non-robust NAV ratios dominate the robust ones. Whereas the non-robust SCR is above the robust one and is an increasing concave function of expiry, whatsoever the considered approximation. This is an interesting and surprising feature: adopting a robust method does not cause an increase of the SCR. The reason is that introducing robustness leads to a prudent estimate of the NAV. The standard deviation of the NAV is then lower in absolute terms than if computed with a non-robust approach. As the SCR is proportional to this standard deviation, the capital requirement is reduced.

	NIG approximation		Gaussian approximation	
t	$\frac{SCR_t}{\mathbb{E}(BE_t)}(\%)$	$\frac{\mathbb{E}(NAV_t)}{SCR_t}(\%)$	$\frac{SCR_t}{\mathbb{E}(BE_t)}(\%)$	$\frac{\mathbb{E}(NAV_t)}{SCR_t}(\%)$
1	20.89	56.83	16.71	71.04
2	25.21	56.11	21.90	64.61
3	28.02	58.81	25.70	64.12
4	30.20	62.43	28.87	65.30
5	32.08	66.31	31.69	67.12
6	33.82	70.22	34.31	69.30
7	35.48	74.04	36.81	71.36
8	37.11	77.72	39.25	73.50
9	38.75	81.23	41.65	75.56
10	40.41	84.51	44.07	77.50

Table 5: Robust SCR and expected NAV (in %) computed with the normal and NIG approximations for the insurance contract with specifications reported in table 4.

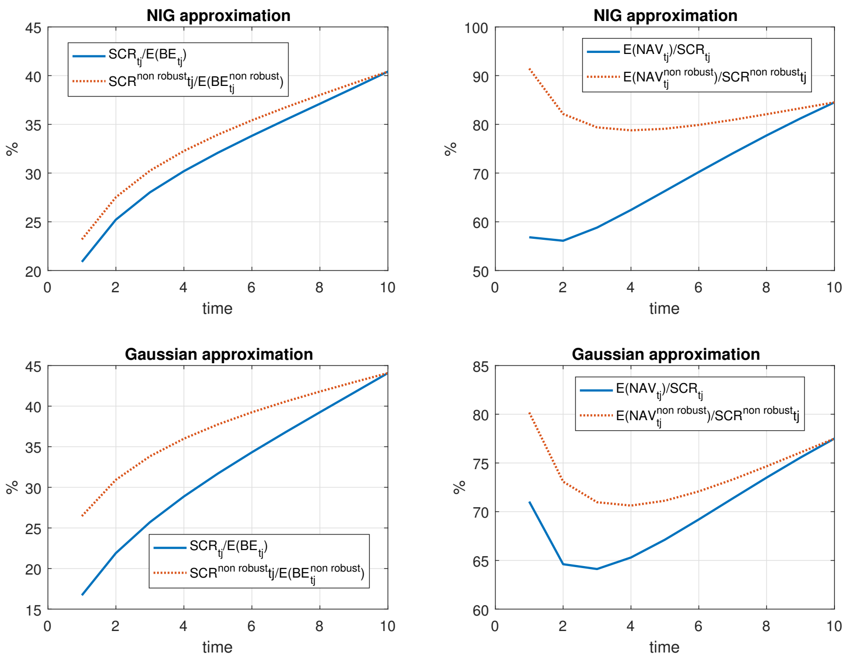

Figure 5: Current and forward robust solvency capital requirements and NAV's for the participating contract with specifications of table 4.

To conclude this section, we show that our robust framework may also be used to optimize the asset allocation. To illustrate this point, we draw in figure 6 the efficient frontiers of investment strategies in the space of performances, $\frac{\mathbb{E}(NAV_t)}{SCR_t}$, and risks, $\frac{SCR_t}{\mathbb{E}(BE_t)}$, for $t = 1$ and 5 years. Each point plotted in this plan corresponds to a policy of investment and the percentage indicates the proportion, θ_2, of the total asset invested in the second security. Robust and non robust curves are similar but the robust efficient frontiers are translated to an area which corresponds to lower NAV and SCR levels. As reported in table 6, an insurer who aims to minimize the ratio SCR/BE should invest between 6 and 8% of the total asset into the riskier asset. With this strategy, the average performance measured by the ratio $\frac{\mathbb{E}(NAV_1)}{SCR_1}$ is close to its maximum.

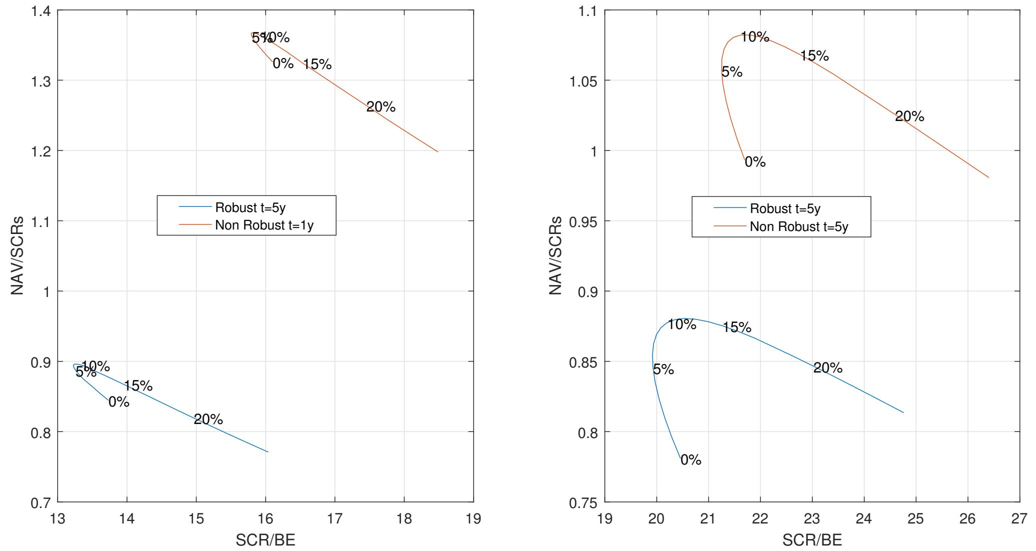

Figure 6: Ratios$\left(\frac{SCR_t}{\mathbb{E}(BE_t)}, \frac{\mathbb{E}(NAV_t)}{SCR_t}\right)_{t=1,5}$ for different strategies of investment.

	% Stocks	$\frac{SCR_1}{\mathbb{E}(BE_1)}$	$\frac{\mathbb{E}(NAV_1)}{SCR_1}$
Robust	8%	13.23%	0.89
Non Robust	7%	15.79%	1.36
	% Stocks	$\frac{SCR_5}{\mathbb{E}(BE_5)}$	$\frac{\mathbb{E}(NAV_5)}{SCR_5}$
Robust	7%	19.92%	0.85
Non Robust	6%	21.25%	1.05

Table 6: Investment strategy that minimizes the ratio $\frac{SCR_t}{\mathbb{E}(BE_t)}$ for $t = 1$ year and $t = 5$ years (NIG approximation).

9 Uncertainty about P

In previous developments we take into account the model ambiguity related to the choice of a risk neutral measure Q. But we ignore the potential misspecifications under the real measure. The importance of the uncertainty about parameters and the potential weakness of the modeling approach under P should not be underestimated as the solvency capital is the value at risk of the NAV under the real measure. In this section, we search to integrate in our framework the preference for robustness both under P and Q. With this preference, the agent treats the dynamics (1) and (2) under the real measure as an approximate model towards the unknown true state evolution of S_t and S_t^M. We consider that the true real measure, denoted by $\tilde{P}$, is unknown but somewhere in the neighborhood of P. To delimit this neighborhood, we bound the entropy of the change of measure from $\tilde{P}$ to P, defined by the following Radon-Nikodym derivative:

$$\left.\frac{d\tilde{P}}{dP}\right|_t = \exp\left(-\frac{1}{2}\int_0^t \Gamma^\top \Gamma ds - \int_0^t \Gamma^\top dB_s\right).$$

If $\tilde{\mu}_S \in \mathbb{R}^d$ is the vector of assets drifts under $\tilde{P}$ and $\tilde{\upsilon} \in \mathbb{R}$, then we define $\Gamma^\top := \begin{pmatrix} \Sigma_S^{-1}(\mu_S - \tilde{\mu}_S) \\ \tilde{\upsilon} \end{pmatrix}$. Under $\tilde{P}$, the mortality account, S_t^M, has a drift equal to $\mu_M(s) = \mu_d(s) + \sigma_d^\top \Gamma$. To summarize, under the equivalent measure, the joint dynamics of the model is given by the SDE:

$$\frac{dS_t}{S_t} = \begin{pmatrix} \tilde{\mu}_S \\ \mu_M(s) \end{pmatrix} dt + \Sigma dW_t.$$

As we wish to bound the entropy of this change of measure, a constraint of the form:

$$\mathbb{E}^{\tilde{P}}\left(\ln \left.\frac{d\tilde{P}}{dP}\right|_t \mathcal{F}_0\right) \leq \frac{1}{2}U_P^2 t\,,$$

is added, where U_P is a constant. If we develop the left hand term in this last equation, the entropic constraint is rewritten as follows:

$$\frac{1}{2}\int_0^t \Gamma^\top \Gamma ds \quad \leq \quad \frac{1}{2}U_P^2 t\,.$$

or after developments,

$$\left(\mu_S - \tilde{\mu}_S\right)^\top \Sigma_S^{-1\,\top}\Sigma_S^{-1}\left(\mu_S - \tilde{\mu}_S\right) + \left(\upsilon\right)^2 \quad \leq \quad U_P^2 \tag{50}$$

that defines an elliptic domain for eligible $(\tilde{\mu}_S, \upsilon)^\top$ in $\mathbb{R}^{d+1}$. However, this single constraint is not sufficient to delimit the set of admissible equivalent real measures. Indeed, the vector $(\tilde{\mu}_S, \upsilon)^\top$ defining $\tilde{P}$ must also satisfy a constraint similar to the equation (8). To clarify this point, let us denote by U_Q , the constant that delimits the set of admissible risk neutral measures and that was previously noted U. It delimits the boundary on the entropy of the change of measure from $\tilde{P}$ to Q as follows:

$$\mathbb{E}^{Q}\left(\ln \left.\frac{dQ}{d\tilde{P}}\right|_t \mathcal{F}_0\right) \leq \frac{1}{2}U_Q^2 t\,. \tag{51}$$

As shown in the proof of proposition 5.1, for a given U_Q, the next constraint must be satisfied by $\tilde{\mu}_S$

$$U_Q^2 \geq \tilde{\mu}_S^\top \left(\Sigma_S\Sigma_S^\top\right)^{-1}\tilde{\mu}_S - \frac{\left(\tilde{\mu}_S^\top \left(\Sigma_S\Sigma_S^\top\right)^{-1}\mathbf{1}\right)^2}{\mathbf{1}^\top\left(\Sigma_S\Sigma_S^\top\right)^{-1}\mathbf{1}}$$

to ensure that the entropic distance between $\tilde{P}$ and Q is lower or equal to $\frac{1}{2}U_Q^2 t$. The set of parameters $(\tilde{\mu}_S, \upsilon)^\top$ defining an eligible equivalent real measure is defined as follows:

$$\tilde{\mathcal{A}} = \left\{ (\upsilon, \tilde{\mu}_S) \in \mathbb{R}^d \;\middle|\; \begin{array}{c} \left(\mu_S - \tilde{\mu}_S\right)^\top \Sigma_S^{-1\,\top}\Sigma_S^{-1}\left(\mu_S - \tilde{\mu}_S\right) + \left(\upsilon\right)^2 \leq U_P^2 \\ U_Q^2 \geq \tilde{\mu}_S^\top \left(\Sigma_S\Sigma_S^\top\right)^{-1}\tilde{\mu}_S - \frac{\left(\tilde{\mu}_S^\top \left(\Sigma_S\Sigma_S^\top\right)^{-1}\mathbf{1}\right)^2}{\mathbf{1}^\top\left(\Sigma_S\Sigma_S^\top\right)^{-1}\mathbf{1}} \end{array} \right\}.$$

Whereas we consider that the solvency capital requirement is in an interval

$$SCR \quad \in \quad \left[\min_{(\upsilon,\tilde{\mu}_S)\in\tilde{\mathcal{A}}} SCR(\upsilon,\tilde{\mu}_S),\ \max_{(\upsilon,\tilde{\mu}_S)\in\tilde{\mathcal{A}}} SCR(\upsilon,\tilde{\mu}_S)\right]$$

where $SCR(\upsilon,\tilde{\mu}_S)$ is the Q-robust SCR computed by the procedure developed in section 8. The size of the interval measures here the uncertainty about parameters and the model under the real and risk neutral measures. What we call the robust $P-Q$ SCR is precisely the maximum value attained over the set of eligible equivalent measures:

$$\text{Robust } P-Q \text{ SCR} \quad = \quad \max_{(\upsilon,\tilde{\mu}_S)\in\tilde{\mathcal{A}}} SCR(\upsilon,\tilde{\mu}_S)\,.$$

The calculation of this robust $P-Q$ SCR is more computationally intensive as it entails a maximization on $\tilde{\mathcal{A}}$ of a quantity that is itself a maximum over $\mathcal{A}$. The table 7 presents the robust $P-Q$ SCR for the participating contract with specifications reported in table 4. The investment strategy is set to$(\theta_1, \theta_2) = (92\%,, 8\%)$, which is the asset allocation that minimizes the Q-robust SCR for a time horizon of one year. To limit the computation time, we use the Gaussian model. The parameter defining the bound on the entropy of $\frac{d\tilde{P}}{dP}$ is equal to $U_P = 0.60$. The robust $P-Q$ SCR increases from 12% to 31%. The second and third columns of table 7 contains the parameters defining $\tilde{P}$ for years 1 to 10. The first three years, the worst case scenario in $\tilde{A}$ corresponds to negative average returns for all assets. After three years, the worst case scenario totally changes: the worst average returns are positive and high. Whatsoever the maturity, the mortality risk premium is small and negative.

t	$\frac{SCR_t}{\mathbb{E}(BE_t)}$(%)	$\tilde{\mu}_S(1)$ (%)	$\tilde{\mu}_S(2)$ (%)	$\tilde{\upsilon}$	Mortality Risk Premium
1	12.22	-1.35	4.81	0.0000	0.0001
2	15.36	-1.34	4.54	0.0000	0.0001
3	17.24	**-1.33**	**4.36**	-0.0000	0.0001
4	19.01	**3.35**	**5.14**	0.0000	-0.0001
5	21.11	3.35	5.19	0.0000	-0.0001
6	23.12	3.35	5.20	0.0000	-0.0001
7	25.08	3.35	5.20	0.0000	-0.0001
8	27.03	3.35	5.20	0.0000	-0.0001
9	28.99	3.35	5.20	0.0000	-0.0001
10	31.00	3.35	5.20	0.0000	-0.0001

Table 7: $P-Q$ robust solvency capital requirements for the contract with specification of table 4.

10 Conclusions

This article proposes a flexible analytical tool to evaluate the net asset value and the solvency capital requirement of a participating life insurance. The model also addresses the issues related to parameters misspecifications and incompleteness of the market. A preference for robustness is introduced in the valuation framework by considering a set of equivalent measures, in the neighborhood of the real measure, delimited by a constraint on the entropy. This constraint on entropy may eventually be calibrated so as to match BE and SCR estimates yield by our model with these obtained with a more complex internal model. The relative simplicity of the model allows us to obtain closed form expressions for most of quantities of interest as BE and NAV moments. On the other hand, the potential shortcomings induced by the Brownian dynamics are partly compensated by the robustness of the procedure. Our tool may also serve to optimize the asset allocation strategy.

We draw several interesting conclusions from numerical illustrations. Firstly, the robust BE is not necessarily calculated with the lower eligible drift under Q. In particular, if death benefits are significantly higher than provisions, the robust BE are evaluated with a mortality risk premium. Secondly, using A robust model leads to a prudent estimate of the NAV. However, this does not systematically increase the solvency capital requirement. Finally, when we consider the ambiguity under the real measure P, the worst case scenario used to evaluate the robust SCR may vary with the time horizon.

Appendix A, mortality assumptions

In the examples presented in this article, the real mortality rates $\mu_d(t)$ are assumed to follow a Gompertz Makeham distribution. The chosen parameters are those defined by the Belgian regulator ("Arrêté Vie 2003") for the pricing of life annuities purchased by males. For an individual of age x, the mortality rate is given by:

$$\mu_d(t) = a_\mu + b_\mu c_\mu^{x+t} \qquad a_\mu = -\ln(s_\mu) \qquad b_\mu = -\ln(g_\mu)\ln(c_\mu)$$

where the parameters s_μ, g_μ, c_μ take the values given in Table 8.

Table 8: Belgian legal parameters for modeling mortality rates, for life insurance products, targeting a male population.

s_μ:	0.999441703848
g_μ:	0.999733441115
c_μ :	1.101077536030

Acknowledgment

The author thank Frederic Planchet for their fruitful comments. We also acknowledge for its financial support the Chair "Data Analytics and Models for insurance" of BNP Paribas Cardiff, hosted by ISFA

(Université Claude Bernard, Lyon France). The authors also thank the two anonymous referees and the editor, Rob Kaas, for their careful reading and suggestions.

References

[1] Anderson, E., Hansen, L.P., Sargent, T.J., 2003. A quartet of semi-groups for model specification, robustness, prices of risk, and model detection. Journal of the European Economic Association 1, 68–123

[2] Bauer D., Börger M., Russ J. 2010, On the pricing of longevity-linked securities. Insurance: Mathematics and Economics 46, 139–149

[3] Balter A., Pelsser A. 2015. Pricing and hedging in incomplete markets with model ambiguity. Netspat Academic series.

[4] Bjork T. and I. Slinko. Towards a general theory of good-deal bounds. Review of Finance, 10:221–260, 2006.

[5] Bonnin F., Juillard M., Planchet F. 2014. Best estimate calculations of savings contracts by closed formulas - Application to the ORSA. European Actuarial Journal, Vol. 4 (1), 181-196.

[6] Cochrane J., Saa Requejo J. 2010. Beyond arbitrage: good-deal asset price bounds in incomplete markets. Journal of Political Economy 108, 79-119.

[7] Combes F., Planchet F. Tammar M. 2016. Pilotage de la participation aux bénéfices et calcul de l'option de revalorisation. Bulletin Français d'Actuariat, vol. 16, n°31.

[8] Deelstra G., Grasselli M., Van Weverberg C. 2016. The role of the dependence between mortality and interest rates when pricing Guaranteed Annuity Options. Insurance: Mathematics and economics. 71, 205-219.

[9] Donnelly C. 2011. Good-deal bounds in a regime-switching diffusion market. Applied Mathematical Finance. 18 (6), 491-515.

[10] Floryszczak A., Le Courtois O., Majri M. 2016. Inside the Solvency 2 Black Box: Net Asset Values and Solvency Capital Requirements with a least-squares Monte-Carlo approach. Insurance: Mathematics and economics. 71, 15-26.

[11] Guidolin M., Rinaldi F. 2010. Ambiguity in Asset Pricing and Portfolio Choice: A Review of the Literature. Working paper 2010-028A Federal Reserve Bank of St. Louis.

[12] Hainaut D. 2016. Impact of volatility clustering on equity indexed annuities. Insurance: Mathematics and economics. 71, 367-381.

[13] Hansen, L. , T. Sargent, 1995, Discounted Linear Exponential Quadratic Gaussian Control, IEEE Transactions on Automatic Control, 40, 968-971.

[14] Hansen, L., Sargent, T. 2001. Robust control and model uncertainty. American Economic Review, 91(2):60–66. 3

[15] Hansen, L., Sargent, T. 2007. Robustness. Princeton University Press Princeton, NJ. 1, 3, 6

[16] Luciano, E. and E. Vigna, 2008. Mortality risk via affine stochastic intensities: calibration and empirical relevance. Belgian Actuarial Bulletin 8 (1), 5–16.

[17] Luciano E., Regis L., Vigna E. 2017. Single and cross-generation natural hedging of longevity and financial risk. The Journal of Risk and Insurance, 84(3), 961-986.

[18] Maenhout, P.J., 2004. Robust portfolio rules and asset pricing. Review of Financial Studies 17, 951–983.

[19] Merton, R. C., 1969. Lifetime Portfolio Selection Under Uncertainty: The Continuous Time Case. Review of Economics and Statistics, 51, 247-257.

[20] Maenhout, P.J., 2006. Robust portfolio rules and dectection-error probabilities for a mean-reverting risk premium. Journal of Economic Theory 128, 136–163.