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Pricing pension buy-outs under stochastic interest and mortality rates

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ABSTRACT
Pension buy-out is a special financial asset issued to offload the pension liabilities holistically in exchange for an upfront premium. In this paper, we concentrate on the pricing of pension buy-outs under dependence between interest and mortality rates risks with an explicit correlation structure in a continuous time framework. Change of measure technique is invoked to simplify the valuation. We also present how to obtain the buy-out price for a hypothetical benefit pension scheme using stochastic models to govern the dynamics of interest and mortality rates. Besides employing a non-mean reverting specification of the Ornstein–Uhlenbeck process and a continuous version of Lee–Carter setting for modeling mortality rates, we prefer Vasicek and Cox–Ingersoll–Ross models for short rates. We provide numerical results under various scenarios along with the confidence intervals using Monte Carlo simulations.

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1. Introduction

The deaths of the individual policy holders are usually supposed to be independent from the development of the financial market due to some practical matters. However, insurance market is argued to be dependent on the development of the economy in literature. Such kind of dependency is exemplified with Russian and German populations in Hoem et al. (2009) and Neyer et al. (2013). While Hoem et al. (2009) demonstrate a clear dependence between the political events in Russia and the expected life time of Russian population in the last four decades, Neyer et al. (2013) show a convergence of one year death probability of Eastern German population towards one year death probability of Western German population right after the unification of Germany.

Nicolini (2004) analyzes the increase in the agricultural production per acre in pre-industrial England at the end of 17th century by using the adult life expectancy. He adapts the mortality rates as an explanatory variable. The *life cycle hypothesis* is utilized in that study. Miltersen & Persson (2006) point out that the market price of mortality risk would be different from zero when the forward force of mortality rate depends on the development of the financial market and the future interest rates. The model suggested by Miltersen & Persson (2006) is inspired by the seminal model of Heath et al. (1992), which focuses on pricing of interest rate derivatives.

The dependency between interest and mortality rates in the long-run is also discussed in the work of Jalen & Mamon (2009). It is noted that a catastrophic event such as major natural disasters and pandemics can mutually affect not only the size of the population but also the interest rates in the short term. They investigate pricing of the basic life insurance contracts when the dependency

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between interest and mortality rates is taken into account. The method of change of measure and the Bayes' rule for conditional expectations are employed in this context.

Dhaene et al. (2013) show the necessary conditions to preserve the independence assumption between financial and insurance markets and note that such kind of assumption is naturally violated in pricing contracts.

Since the general tendency is to follow the independence assumption between financial and insurance markets, only few studies are available which take into account the potential dependency between interest and mortality rates for pricing of mortality linked derivatives. We believe that the dependency between these two markets can have a significant impact on the prices of insurance linked derivatives.

The European Union's Solvency II Regulations recommend using stochastic models for valuation of the contracts. In order to enhance the pricing model offered by Lin et al. (2017), we build a continuous time framework and make use of stochastic interest and mortality rates models under the dependence assumption between interest and mortality rates risks in this paper. To do so, the change of measure technique is utilized.

The outline of the paper is as follows. In Section 2 we provide the necessary theoretical background to implement the suggested pricing model. In Section 3 we develop the formulation of the pricing framework under the dependence assumption. In Section 4 we provide some numerical examples illustrating the applicability of the proposed model in this paper. In Section 5 we discuss our main findings and conclude the paper.

2. Integrated model framework

In order to price an annuity contract which is an important factor to define the liability process of a pension scheme, we need to deal with interest and mortality rates risks. In this section we define our miscellaneous modeling framework by giving general descriptions for short rate and mortality rate risks based on four main scenarios. Firstly, we assume affine dynamics for both risks to take advantage of the tractability of these models. Then, we improve our model assumptions by changing the dynamics of short rates and mortality rates.

2.1. Ornstein–Uhlenbeck process and Vasicek model

For illustrative purposes, we choose an affine structure for short rates and mortality rates, respectively, as follows:

$$\mathrm{d}r(t) = a^r(b^r - r(t))\mathrm{d}t + c^r\mathrm{d}W^r(t) \tag{1}$$

$$\mathrm{d}\mu(t) = a^\mu\mu(t)\mathrm{d}t + c^\mu\mathrm{d}W^\mu(t), \tag{2}$$

where a^r, b^r, c^r, a^μ and c^μ are positive constants. Also, we assume that

$$\mathrm{d}W^\mu(t) = \rho\mathrm{d}W^r(t) + \sqrt{1-\rho^2}\mathrm{d}W(t), \tag{3}$$

where $\rho \in (-1, 1)$ is a correlation coefficient. Here, W^r and W are independent Wiener processes under the risk neutral measure $\mathbb{Q}$.

It is easy to show that W^μ in (3) becomes a Wiener process for a given ρ by Lévy's theorem. Such an idea is presented by Liu et al. (2014), and then Gao et al. (2015), in order to define the dependency between short rates and mortality rates. In practice the (Pearson) correlation coefficient is determined via the historical data. Or, it can be achieved via model calibration techniques (e.g. least-squares or maximum likelihood, etc.).

The interest rate model given by (1) is the well-known Vasicek model. Although the model has an important shortcoming due to the possibility of generating negative interest rates, we promote this model for two reasons: the first reason of applying this model is its tractability; the second one is that negative interest rates being observed in some of the improved economies such as in European countries as well as in Japan. So, we believe it is important to keep such a model assumption which better represents a real-world scenario.

Since our main aim is to show the applicability of the suggested pricing framework for pension buy-outs, we employ a non-mean reverting specification of the Ornstein–Uhlenbeck (OU) process to demonstrate mortality rate dynamics as described in the study of Luciano & Vigna (2005) by following (2). Luciano & Vigna (2005) also show that the possibility of obtaining negative mortality rates is negligible under suitable parameter choice.

To begin with, zero coupon bond prices at time t with maturities t_i under Vasicek model are,

$$B(t, t_i, r(t)) = \exp\left\{-A^r(t, t_i)r(t) + B^r(t, t_i)\right\},$$

for $t \in [0, t_i]$ where $t_i = t + i$ and $i = 1, \ldots, M$ where

$$A^r(t, t_i) = \frac{1 - e^{-a^r(t_i - t)}}{a^r},$$
$$B^r(t, t_i) = \left(b^r - \frac{c^{2r}}{2a^{2r}}\right)[A^r(t, t_i) - (t_i - t)] - \frac{c^{2r}A^r(t, t_i)^2}{4a^r}.$$

See Vasicek (1977) for details. Therefore, in this setting, the dynamics of the short rate under the forward measure $\mathbb{P}^{t_i}$ is

$$\mathrm{d}r(t) = (a^r b^r - a^r r(t) - (c^r)^2 A^r(t, t_i))\mathrm{d}t + c^r \mathrm{d}\tilde{W}^r(t), \tag{4}$$

where $\mathrm{d}\tilde{W}^r(t) = \mathrm{d}W^r(t) + c^r A^r(t, t_i)\mathrm{d}t$ for all $t_i = t + i$ with $i = 1, \ldots, M$. For more detailed construction we refer readers to Mamon (2004). The dynamics of r under $\mathbb{P}^{t_i}$ satisfies the necessary conditions of affine theory stated in Liu et al. (2014). So, it is possible to obtain the price of a zero coupon bond in a functional form by solving the relevant Riccati equations.

In order to obtain the liability process, we should also calculate $(t_i - t)$-year survival probabilities $\tilde{p}(t, t_i, x)$ for age x under the forward measure $\mathbb{P}^{t_i}$. The behavior of (2) under the forward measure $\mathbb{P}^{t_i}$ is easily obtained to give

$$\mathrm{d}\mu(t) = (a^\mu \mu(t) - c^\mu \rho A^r(t, t_i)c^r)\mathrm{d}t + c^\mu \mathrm{d}\tilde{W}^\mu(t), \tag{5}$$

where

$$\mathrm{d}\tilde{W}^\mu(t) = \rho \mathrm{d}\tilde{W}^r(t) + \sqrt{1 - \rho^2}\mathrm{d}\tilde{W}(t). \tag{6}$$

Here, $\mathrm{d}\tilde{W}(t) = \mathrm{d}W(t)$ for all $t \in [0, t_M]$. Since the dynamics of $\mu(t)$ under $\mathbb{P}^{t_i}$ is in accordance with the theory of affine models, it is possible, in this case, to express survival rates $\tilde{p}(t, t_i, x)$ under an affine structure as

$$\tilde{p}(t, t_i, x) = \exp\left\{-G^\mu(t, t_i)\mu(t) + H^\mu(t, t_i)\right\}.$$

See Liu et al. (2014) for the details of the proof.

2.2. Ornstein–Uhlenbeck process and Cox–Ingersoll–Ross model

We keep the mortality model as the OU process and improve the short rate model, as a second scenario, using Cox–Ingersoll–Ross (CIR) model. Cox et al. (1985) describe the behavior of the instantaneous rates as

$$dr(t) = a^r(b^r - r(t))dt + \sigma^r\sqrt{r(t)}dW^r(t), \tag{7}$$

while the mortality rate dynamics is described as

$$d\mu(t) = a^\mu \mu(t)dt + c^\mu dW^\mu(t),$$

where $dW^\mu(t)$ is as given in (3).

The price of a zero coupon bond under the risk neutral measure $\mathbb{Q}$ at time t with maturity $t_i = t+i$, is

$$B(t, t_i, r(t)) = \exp\left\{-A^r(t_i - t)r(t) + B^r(t_i - t)\right\}$$

for $i = 1, \ldots, M$ where

$$A^r(t_i - t) = \frac{2(e^{\gamma(t_i-t)} - 1)}{(\gamma + a^r)(e^{\gamma(t_i-t)} - 1) + 2\gamma},$$
$$B^r(t_i - t) = \frac{2a^r b^r}{\sigma^{2r}} \log\left[\frac{2\gamma e^{(a^r+\gamma)\frac{(t_i-t)}{2}}}{(\gamma + a^r)(e^{\gamma(t_i-t)} - 1) + 2\gamma}\right],$$

and $\gamma = \sqrt{a^{2r} + 2\sigma^{2r}}$. See Björk (2009) for more explicit derivation of these terms.

Proposition 2.1 (CIR dynamics under measure $\mathbb{P}^{t_i}$): *Assume that the short rates follow a CIR model (7). Then, the dynamics of the short rate under the forward measure $\mathbb{P}^{t_i}$ is*

$$dr(t) = (a^r b^r - a^r r(t) - (\sigma^r)^2 r(t)A^r(t_i - t))dt + \sigma^r\sqrt{r(t)}d\tilde{W}^r(t), \tag{8}$$

where $d\tilde{W}^r(t) = dW^r(t) + \sigma^r\sqrt{r(t)}A^r(t_i - t)dt$ for all $t_i = t + i$ with $i = 1, \ldots, M$.

For the derivation of $d\tilde{W}^r(t)$, we refer readers to Mamon (2004). The dynamics of r under the forward measure $\mathbb{P}^{t_i}$ is compatible with the affine theory. The term structure of $r(t)$ is obtained as a function of time t by following a similar procedure to solve the Riccati equations as described in Rouah (2013).

Proposition 2.2 (The OU dynamics using CIR model as a short rate model under measure $\mathbb{P}^{t_i}$): *Assume that the force of mortality rate dynamics satisfies the OU process (2) under $\mathbb{Q}$. If the short rate dynamics is described by (7), then the mortality rate dynamics is*

$$d\mu(t) = [a^\mu \mu(t) - c^\mu \rho\sigma^r\sqrt{r(t)}A^r(t_i - t)]dt + c^\mu d\tilde{W}^\mu(t), \tag{9}$$

under the forward measure $\mathbb{P}^{t_i}$, where $d\tilde{W}^\mu(t)$ is as given in (6); and $d\tilde{W}^r(t) = dW^r(t) + \sigma^r\sqrt{r(t)}$ $A^r(t_i - t)dt$ for all $t_i = t + i$ with $i = 1, \ldots, M$.

The dynamics of μ under measure $\mathbb{P}^{t_i}$ given by (9) does not admit necessary conditions within the framework of affine theory. Consequently, the survival probabilities $\tilde{p}(t, t_i, x)$ do not satisfy a functional form.

2.3. Lee–Carter method and Vasicek model

This part of the study provides the necessary theoretical background in order to apply one of the commonly used actuarial methods for mortality modeling to the proposed pricing framework. We keep Vasicek model as the short rate model while we move one step further and choose Lee–Carter (LC) model for the mortality rate dynamics. LC model is preferred due to its nice fitting of the empirical data (Liu et al. 2014). We utilize the LC specification improved by Biffis et al. (2010), where the respective dynamics of r and μ under $\mathbb{Q}$ are given, respectively, by

$$\mathrm{d}r(t) = a^r(b^r - r(t))\mathrm{d}t + c^r \mathrm{d}W^r(t)$$

and

$$\mathrm{d}\mu^j(t) = \mu^j(t)(\tilde{\delta}^\mu(t)\mathrm{d}t + \sigma^\mu(t)\mathrm{d}W^\mu(t)), \tag{10}$$

where $\tilde{\delta}^\mu(t) = \delta^\mu(t) - \eta\sigma^\mu(t)$ for a constant η. Here, j represents the number of insureds and $j \in \{1, 2, \ldots, m\}$ for a fixed value of m.

Proposition 2.3 (The LC dynamics using Vasicek Model as a short rate model under measure $\mathbb{P}^{t_i}$): *Assume that μ has the dynamics (10) under $\mathbb{Q}$. If the short rate dynamics is governed by the Vasicek model (1), then under the forward measure $\mathbb{P}^{t_i}$ we have*

$$\mathrm{d}\mu^j(t) = \mu^j(t)[(\tilde{\delta}^\mu(t) - \sigma^\mu(t)\rho A^r(t, t_i)c^r)\mathrm{d}t + \sigma^\mu(t)\mathrm{d}\tilde{W}^\mu(t)]$$

for $j = 1, \ldots, m$ and $i = 1, \ldots, M$, with $t_i = t + i$, where $\mathrm{d}\tilde{W}^\mu(t)$ is as described in (6), and $\mathrm{d}\tilde{W}^r(t) = \mathrm{d}W^r(t) + c^r A^r(t, t_i)\mathrm{d}t$.

As before, since the dynamics of μ under $\mathbb{P}^{t_i}$ does not satisfy the affine structure, it is not possible to write the survival probabilities $\tilde{p}(t, t_i, x)$ in a functional form.

2.4. Lee–Carter method and Cox–Ingersoll–Ross model

In this part of the study, we improve the short rate model assumption in the previous analysis by using CIR model. The respective dynamics under $\mathbb{Q}$ of r and μ are given as follows:

$$\mathrm{d}r(t) = a^r(b^r - r(t))\mathrm{d}t + \sigma^r\sqrt{r(t)}\mathrm{d}W^r(t)$$
$$\mathrm{d}\mu^j(t) = \mu^j(t)(\tilde{\delta}^\mu(t)\mathrm{d}t + \sigma^\mu(t)\mathrm{d}W^\mu(t)).$$

The dynamics of r under $\mathbb{P}^{t_i}$ is given by Proposition 2.1; however, the dynamics of the LC model under the forward measure $\mathbb{P}^{t_i}$ is now given in the following proposition.

Proposition 2.4 (The LC dynamics using CIR Model as a short rate model under measure $\mathbb{P}^{t_i}$): *Assume μ has the dynamics described by (10) under $\mathbb{Q}$. If the interest rate dynamics is given in (7), then we have as follows:*

$$\mathrm{d}\mu^j(t) = \mu^j(t)[(\tilde{\delta}^\mu(t) - \sigma^\mu(t)\rho A^r(t_i - t)\sigma^r\sqrt{r(t)})\mathrm{d}t + \sigma^\mu(t)\mathrm{d}\tilde{W}^\mu(t)],$$

under the forward measure $\mathbb{P}^{t_i}$ where $\mathrm{d}\tilde{W}^\mu(t)$ is as given in (6) and $\mathrm{d}\tilde{W}^r(t) = \mathrm{d}W^r(t) + \sigma^r\sqrt{r(t)}\, A^r(t_i - t)\mathrm{d}t$ for $j = 1, \ldots, m$ and $i = 1, \ldots, M$ with $t_i = t + i$.

We remark again that the dynamics of μ under the forward measure $\mathbb{P}^{t_i}$ does not admit an affine form. Thus, Monte Carlo (MC) simulations for pricing purposes are unavoidable.

3. The price calculation

In order to explain the necessary theoretical background to price pension buy-outs in a continuous time framework, we first present the financial market model to describe a synthetic pension portfolio for a hypothetical pension scheme. Then, we provide the proposed pricing model which is based on the asset and liability processes of the pension scheme.

3.1. Financial market model

Let $(\Omega^{(1)}, \mathcal{F}^{(1)}, \{\mathcal{F}_t^{(1)}\}, \mathbb{P}^{(1)})$ be the given filtered probability space for modeling the financial market where $\mathcal{F}_t^{(1)}$ denotes the relevant filtration (information) up to time t. We suppose that the hypothetical pension plan is invested in three different assets in two different ways. The value of these assets is represented by $A_1(t)$, $A_2(t)$ and $A_3(t)$ with $t \in [0, T]$ for a fixed value of $T = t_M$. We also assume these assets follow geometric Wiener process as

$$\mathrm{d}A_k(t) = A_k(t)[\alpha_k \,\mathrm{d}t + \sigma_k \,\mathrm{d}W_k(t)], \quad k = 1, 2, 3. \tag{11}$$

where α_k is the drift term and σ_k is the instantaneous volatility for the kth asset. The $W_k(t)$s are the correlated Wiener processes such that the covariances are

$$\mathrm{Cov}(W_k(t), W_l(t)) = \rho_{kl}\, t, \quad k = 1, 2, 3; \quad l = 1, 2, 3,$$

where ρ_{kl} is the correlation coefficient between assets k and l. This yields to the correlation of the assets, and we then have

$$\mathrm{Cov}(A_k(t), A_l(t)) = \rho_{kl}\sigma_k\sigma_l t, \quad k = 1, 2, 3; \quad l = 1, 2, 3.$$

Now, let $PA(t)$ denote the value of the pension portfolio at time t. By investing in the existing assets, $\log(PA)$ follows

$$d\log(PA)(t) = \left(\sum_{k=1}^{3} \pi_k(t)\left(\alpha_k - \frac{1}{2}\sigma_k^2\right) + \gamma_\pi^*(t)\right)\mathrm{d}t + \sum_{k=1}^{3} \pi_k(t)\sigma_k \mathrm{d}W_k(t)$$

between annuity payment dates for $t \in (t_{i-1}, t_i)$ for $i = 1, \ldots, M$ with an initial value $\log(PA)(0)$ at time zero. Moreover, initial values of the asset portfolio $PA(t_i^+)$ for each subsequent year will be defined later (see Remark 1). Here, $t_i = t + i$ for $i = 1, \ldots, M$, and $t_M = \inf\{t : N(t) = 0\}$ where $N(t)$ is the number of survivors at time t. Furthermore, $\pi(t) = [\pi_1(t), \pi_2(t), \pi_3(t)]$ is the vector of weights of the assets in the portfolio and $\gamma_\pi^*(t)$ is the growth rate of the pension assets at time t; $\gamma_\pi^*(t)$ is described in Fernholz (2002) as follows:

$$\gamma_\pi^*(t) = \frac{1}{2}\left[\sum_{k=1}^{3} \pi_k(t)\sigma_k^2 - \sum_{k,l=1}^{3} \pi_k(t)\pi_l(t)\rho_{kl}\sigma_k\sigma_l\right].$$

Under the risk neutral measure $\mathbb{Q}$, the dynamics of the synthetic pension portfolio process turns into

$$d\log PA(t) = \left(r - \frac{1}{2}\sigma_W^2\right)\mathrm{d}t + \sum_{k=1}^{3} \pi_k(t)\sigma_k \mathrm{d}W_k^{\mathbb{Q}}(t), \tag{12}$$

where r is the risk free rate and $\sigma_W^2 = \sum_{k,l=1}^{3} \pi_k(t)\pi_l(t)\rho_{kl}\sigma_k\sigma_l$. In addition, we remark that

$$\mathrm{d}W_k^{\mathbb{P}^{(1)}}(t) = \mathrm{d}W_k^{\mathbb{Q}}(t) - \left(\frac{\sum_1^3 \pi_k(t)\alpha_k - r}{\sum_1^3 \pi_k(t)\sigma_k}\right)\mathrm{d}t.$$

Remark 1 (The value of the pension portfolio after annuity payments): Let $PA(t_i^+)$ represent the value of the pension portfolio right after possible adjustments, such as any potential funding contributions in the case of underfunded event or the annuity payments at the end of time t_i. Then,

$$PA(t_i^+) = \max\{PA(t_i) - N(t_i) \times C,\ L(t_i)\}.$$

Here, $N(t_i)$ denotes the number of survivors at time t_i and is obtained according to the force of mortality rates (Lin et al. 2017); $L(t_i)$ denotes the liability process at time t_i and

$$L(t) = N(t) \times a(t, x),$$

where $a(t, x)$ is given in Proposition 3.1 below.

3.2. Pension buy-out and its valuation

Now, we consider a combined modeling framework that covers the evolution of both mortality and interest rates processes with a filtered probability space $(\Omega, \mathcal{I}, \{\mathcal{I}_t\}, \mathbb{P})$. The filtrations generated by the mortality process μ and the short rate process r up to time t, and are represented by $\mathcal{M}_t \subset \mathcal{I}_t$ and $\mathcal{F}_t \subset \mathcal{I}_t$, respectively. The filtration $\mathcal{I}_t$ is the smallest σ-algebra generated by $\mathcal{M}_t$ and $\mathcal{F}_t$.

We state the main results of this paper with the propositions below.

Proposition 3.1: *Suppose $r(t)$ denote the stochastic short rate and $\mu(x, t)$ represent the force of mortality rate dynamics for an individual aged x at time $t \in [0, T]$. The fair price of an immediate life annuity contract at time t which guarantees to pay survival benefits C at the end of each year (under the dependence assumption) is*

$$a(t, x) = \sum_{t_i = t_1}^{t_M} B_S(t, t_i, x, C), \tag{13}$$

where

$$B_S(t, t_i, x, C) = \mathbb{1}_{\tau(x) > t}\, C\, B(t, t_i, r(t))\, \tilde{p}(t, t_i, x).$$

Here

- *$\tau(x)$ indicates the future lifetime of an individual aged x.*
- *$B_S(t, t_i, x, C)$ represents the fair price of a pure endowment contract which guarantees a constant survival benefit C to an individual aged x at time t in case of survival over the period $(t_i - t)$ where $t_i = t + i$ for $i = 1, \dots, M$ and $t_M = \inf\{t : N(t) = 0\}$.*
- *$B(t, t_i, r(t))$ is the price of a zero coupon bond at time t with maturity t_i under the risk neutral measure $\mathbb{Q}$; particularly, $B(t, t_i, r(t)) = E^{\mathbb{Q}}\left[\exp\left\{-\int_t^{t_i} r(s)\mathrm{d}s\right\} \middle| \mathcal{I}_t\right]$.*
- *$\tilde{p}(t, t_i, x)$ is the $(t_i - t)$-year survival probability of an x-year old individual at time t under the forward measure $\mathbb{P}^{t_i}$; that is, $\tilde{p}(t, t_i, x) = E^{t_i}\left[\exp\left\{-\int_t^{t_i} \mu(s, x + s)\mathrm{d}s\right\} \middle| \mathcal{I}_t\right]$.*

The basic idea of the proof is based on the study of Jalen & Mamon (2009), and a complete proof can be found in Arık (2016).

Since the responsibility of an insurer is to maintain the pension plan as fully funded as possible under the pension rules, this funding guarantee option can be considered as *a series of one-year put option spreads* on the pension scheme (Lin et al. 2017). The exercise prices should then be defined by taking into account the balance between pension assets and liabilities on the valuation dates. Hence, the fair price of a buy-out deal is given by the following proposition.

Proposition 3.2 (The fair price of the buy-out deal): *Under the dependence assumption between interest and mortality rates risks, the fair price of a buy-out deal at time t, conditioning on the filtration $\mathcal{I}_t$, under the risk neutral measure $\mathbb{Q}$, is*

$$\begin{aligned} P_{\text{buyout}}(t) &= \tfrac{1}{L(t)} \sum_{t_i > t}^{t_M} E^{\mathbb{Q}} \left[e^{-\int_t^{t_i} r(s)\mathrm{d}s} \max\{PA(t_i^+) - (PA(t_i) - N(t_i)C), 0\} | \mathcal{I}_t \right] \\ &= \tfrac{1}{L(t)} \sum_{t_i > t}^{t_M} B(t, t_i, r(t)) E^{t_i} \left[\max\{PA(t_i^+) - (PA(t_i) - N(t_i)C), 0\} | \mathcal{I}_t \right]. \end{aligned} \tag{14}$$

***Proof*:** The main idea of the proof is to derive the fair price of the buy-out deal as a product of conditional expectations under different measures using a similar technique as suggested by Jalen & Mamon (2009). Firstly, we rewrite (14) as follows:

$$P_{\text{buyout}}(t) = \frac{PV_{\text{buyout}}(t)}{L(t)}. \tag{15}$$

Here,

$$PV_{\text{buyout}}(t) = \sum_{t_i > t}^{t_M} E^{\mathbb{Q}} \left[e^{-\int_t^{t_i} r(s)\mathrm{d}s} H(t_i) \middle| \mathcal{I}_t \right], \tag{16}$$

where $H(t_i) = \max\{PA(t_i^+) - (PA(t_i) - N(t_i)C), 0\}$. Since $L(t)$ is a realized value, we focus on $PV_{\text{buyout}}(t)$. The use of the Radon–Nikodym derivative of measure $\mathbb{P}^{t_i}$ with respect to the measure $\mathbb{Q}$ yields

$$\Lambda_{0,t_i} := \frac{\mathrm{d}\mathbb{P}^{t_i}}{\mathrm{d}\mathbb{Q}}\bigg|_{\mathcal{I}_{t_i}} = \frac{\exp\left\{-\int_0^{t_i} r(s)\mathrm{d}s\right\} B(t_i, t_i, r(t))}{B(0, t_i, r(t))},$$

where $B(t, t_i, r(t))$ is as defined in Proposition 3.1 and $B(t_i, t_i, r(t)) = 1$. Since Λ_{0,t_i} is a martingale, for $t \leq t_i$, it implies that

$$\Lambda_{0,t} = E^{\mathbb{Q}}[\Lambda_{0,t_i} | \mathcal{I}_t] = \frac{\exp\left\{-\int_0^{t} r(s)\mathrm{d}s\right\} B(t, t_i, r(t))}{B(0, t_i, r(t))}. \tag{17}$$

Now, defining $E^{t_i}[H(t_i)|\mathcal{I}_t]$ by applying Bayes' rule as

$$E^{t_i}[H(t_i)|\mathcal{I}_t] = \frac{E^{\mathbb{Q}}[\Lambda_{0,t_i} H(t_i) | \mathcal{I}_t]}{E^{\mathbb{Q}}[\Lambda_{0,t_i} | \mathcal{I}_t]} \quad \textit{for} \quad t < t_i, \tag{18}$$

and substituting (17) into (18), we deduce from (16) that

$$\begin{aligned} PV_{\text{buyout}}(t) &= \sum_{t_i > t}^{t_M} E^{t_i}[H(t_i)|\mathcal{I}_t] B(t, t_i, r(t)), \\ &= \sum_{t_i > t}^{t_M} E^{t_i}[\max\{PA(t_i^+) - (PA(t_i) - N(t_i).C), 0\} | \mathcal{I}_t] B(t, t_i, r(t)). \end{aligned}$$

Table 1. Model assumptions for the numerical illustrations.

	Mortality model	Short rate model
Scenario I	Ornstein–Uhlenbeck	Vasicek
Scenario II	Ornstein–Uhlenbeck	Cox–Ingersoll–Ross
Scenario III	Lee–Carter	Vasicek
Scenario IV	Lee–Carter	Cox–Ingersoll–Ross

Table 2. Estimated parameter values.

Parameter set for the application	
Contract detail	$C = 60000, N(0) = 10000, PA(0) = L(0), x = 65$
Vasicek model	$a^r = 0.045398, b^r = 0.090070, c^r = 0.003789$
CIR model	$a^r = 0.2, b^r = 0.04, \sigma^r = 0.1, r(0) = 0.04$
OU process	$a^\mu = 0.078282, c^\mu = 0.002271, \mu(0) = 0.01820$
LC model 1	$\delta = -0.355590, \sigma = 1.457266, \eta = 0.0943$
LC model 2	$\delta = -0.355590, \sigma = 1.457266, \eta = 1.197179$

Therefore, rewriting (15) as

$$P_{\text{buyout}}(t) = \frac{\sum_{t_i>t}^{t_M} E^{t_i}[\max\{PA(t_i^+) - (PA(t_i) - N(t_i).C), 0\}|\mathcal{I}_t]B(t, t_i, r(t))}{L(t)},$$

completes the proof. □

4. Numerical illustrations

We show how to implement the proposed pricing framework under various stochastic models for interest and mortality rates. Table 1 presents the main scenarios we consider. Besides, we also examine the sensitivity of buy-out price under Scenario III and Scenario IV with respect to parameter η which is essential for the LC model under measure changes.

Meanwhile, values of the parameters related to the mortality and short rates dynamics applied for simulation purposes are shown in Table 2. Firstly, the suggested estimators for the short rate models are based on the UK financial market (Jalen & Mamon 2009; Dowd et al. 2011). Secondly, the estimated values for the mortality rate dynamics satisfy a non-mean reverting OU process under Scenario I (Jalen & Mamon 2009). Finally, there are two sets of parameters for mortality modeling under LC case. The parameters are calibrated from the UK male mortality rates for ages 65 and 110 from 1928 to 2013. Since the SDE of LC model is based on parameter η as $\tilde{W}(t) = W(t) + \int_0^t \eta_s \mathrm{d}s$ under measure $\mathbb{Q}$, we explore the impact of this parameter on the buy-out price. Indeed, there are two calculated values in literature; $\eta = 0.0943$ is calculated by Lin et al. (2017) as the market price of longevity risk using the longevity-risk security market, and $\eta = 1.197179$ is obtained by Biffis et al. (2010) based on the utilization of the Italian annuitant data.

Assume that a hypothetical pension scheme in the UK considers to purchase a pension buy-out in order to transfer its pension risks to a buy-out insurer. The insurer is assumed active in both the UK and the US Pension Bulk Annuity Markets. Moreover, suppose that annual pension benefits are to be made to the insureds at the end of each year as long as they survive; and the insureds are assumed to attain the retirement age $x = 65$ at time zero. Herewith, we assume that the last age w is 110 so that $t_M = \inf\{t : N(t) = 0\} = 46$.

We perform the analysis under three extreme cases: a strong negative correlation, no correlation and a strong positive correlation between interest and mortality rates. We derive the fair price of buy-out deal according to Proposition 3.2 when the correlation coefficient ρ is equal to -0.9, zero and 0.9, respectively. We display the buy-out prices depending on various number of simulations between 100 and 50,000 with the confidence intervals and error margins. In this respect, we examine if the outputs converge to a reasonable value with an acceptable error margin when we increase the number of simulations.

In order to calculate a confidence interval for the fair price of the buy-out deal $P_{\text{buyout}}(0)$ for a given number of simulations, say N, we first consider (16), that is,

$$PV_{\text{buyout}}(0) = \sum_{t_i=1}^{t_M} PV_{\text{payoff}}(t_i),$$

where $PV_{\text{payoff}}(t_i) = E^{\mathbb{Q}}\left[e^{-\int_0^{t_i} r(s)\mathrm{d}s} H(t_i)\right]$. Then, we determine 95% confidence interval for each $PV_{\text{payoff}}(t_i)$ for $i = 1, 2, \ldots, M$ as

$$PV^{\pm}_{\text{payoff}}(t_i) = \mu_{\text{payoff}}(t_i) \pm 1.96[\sigma_{\text{payoff}}(t_i)/\sqrt{N}], \tag{19}$$

where $\mu_{\text{payoff}}(t_i)$ and $\sigma_{\text{payoff}}(t_i)$ are the mean and the standard deviation of the discounted payoff $PV_{\text{payoff}}(t_i)$ respectively. Here, $PV^{-}_{\text{payoff}}(t_i)$ and $PV^{+}_{\text{payoff}}(t_i)$ represent the lower and upper bounds of the confidence interval, respectively, depending on the relevant sample size. The term $[1.96(\sigma_{\text{payoff}}(t_i)/\sqrt{N})]$ is the *error margin* for $PV_{\text{payoff}}(t_i)$ (Uğur 2009). Hence, the lower and upper bounds of confidence interval for $P_{\text{buyout}}(0)$ are calculated to be

$$P^{\pm}_{\text{buyout}}(0) = \frac{1}{L(0)} \sum_{t_i=1}^{t_M} PV^{\pm}_{\text{payoff}}(t_i). \tag{20}$$

The error margin

$$\frac{1}{L(0)} \sum_{t_i=1}^{t_M} 1.96\, \sigma_{\text{payoff}}(t_i)/\sqrt{N}, \tag{21}$$

for $P_{\text{buyout}}(0)$ is used.

4.1. Financial markets

We assume that investment will be made in two different ways as follows: (i) we accept that the insurer invests 10% of its assets in the S&P UK stock total return index, 85% in the Merrill Lynch UK Sterling corporate bond total return index and 5% in the 3-month UK cash total return index. (ii) we suppose that the buy-out insurer invests 4.22% of its pension assets in the S&P 500 index, 92.94% in the US treasury 10-year bond and 2.84% in the 3-month T-bill in the US financial market. Values of the assets are represented by $A_1(t)$, $A_2(t)$ and $A_3(t)$, respectively for each investment strategy.

4.1.1. Investment Strategy I

Let us assume the investment in the UK financial market, with the portfolio suggested by Lin et al. (2017). The correlation coefficients used for the simulation of the synthetic asset portfolio $PA(t)$ are

$$\rho^{(1)} = \begin{bmatrix} 1 & \rho_{12} & \rho_{13} \\ \rho_{21} & 1 & \rho_{23} \\ \rho_{31} & \rho_{32} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0.3483 & -0.1002 \\ 0.3483 & 1 & -0.1772 \\ -0.1002 & -0.1772 & 1 \end{bmatrix}$$

Table 3. The estimated parameter values for Investment Strategy I.

Parameter	Estimate	Parameter	Estimate
α_1	0.0448	σ_1	0.1600
α_2	0.0215	σ_2	0.0716
α_3	0.0001	σ_3	0.0077

Table 4. The estimated parameter values for Investment Strategy II.

Parameter	Estimate	Parameter	Estimate
α_1	0.1126	σ_1	0.1988
α_2	0.0538	σ_2	0.0738
α_3	0.0361	σ_3	0.0303

The correlation matrix states that while the S&P UK stock total return index and the Merrill Lynch UK Sterling corporate bond total return index are positively correlated, the 3-month UK cash total return index is negatively correlated with the stock and corporate indexes.

Table 3 represents the estimated values of the parameters derived from (11) according to a maximum likelihood estimation. The stock index has a higher expected log return ($\alpha_1 = 0.0448$) than those of the corporate bond index ($\alpha_2 = 0.0215$) and the cash index ($\alpha_3 = 0.0001$). The asset weights are determined as regards the weights of a European insurer (Lin et al. 2017).

4.1.2. Investment Strategy II

Now, let us assume that investment in the US financial market. We estimate the model parameters for the S&P 500 index, the US treasury 10-year bond and the 3-month T-bill according to (11) by using annual data from 1928 to 2013. The data sets are provided by FRED at Federal Reserve Bank of St. Louis (Damodaran 2016). The estimated correlation coefficients are

$$\rho^{(2)} = \begin{bmatrix} 1 & \rho_{12} & \rho_{13} \\ \rho_{21} & 1 & \rho_{23} \\ \rho_{31} & \rho_{32} & 1 \end{bmatrix} = \begin{bmatrix} 1 & -0.0076 & -0.0176 \\ -0.0076 & 1 & 0.2815 \\ -0.0176 & 0.2815 & 1 \end{bmatrix}$$

and the estimated drift and volatility components for the assets are presented in Table 4.

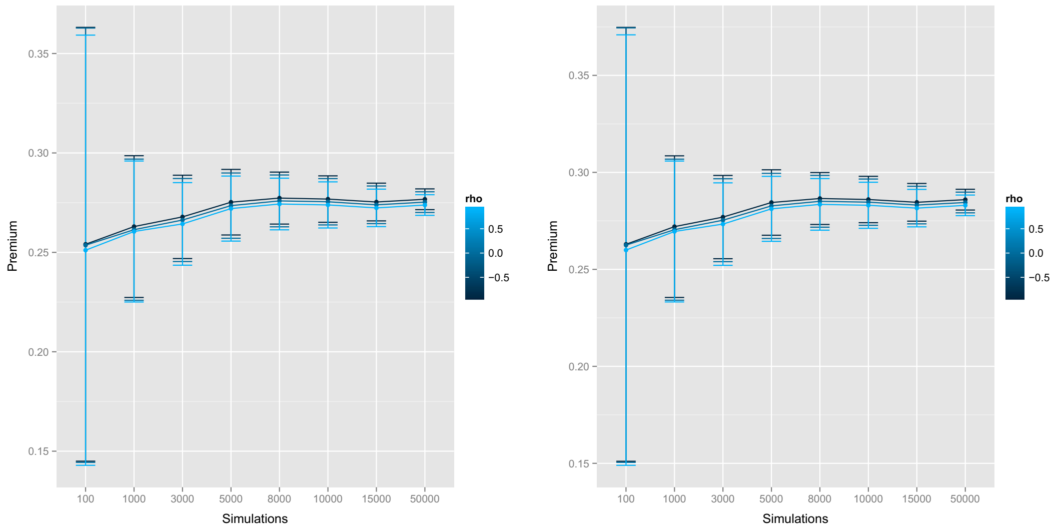
Table 4 indicates that the S&P 500 index has a higher expected annual log return ($\alpha_1 = 0.1126$) than those of the US treasury 10-year bond ($\alpha_2 = 0.0538$) and the 3-month T-bill ($\alpha_3 = 0.0361$). In addition, the correlation matrix points out that the US treasury 10-year bond and the 3-month T-bill are positively correlated. On the other hand, the S&P 500 index is negatively correlated with the other two assets. The asset weights that we accept for this case are akin to the ones declared by MetLife Assurance Limited in 2004 which is an active player in the UK and the US Pension Bulk Annuity Markets.

4.2. Scenario I and Scenario II: Pricing

Here, we illustrate how to price pension buy-outs under Scenario I and Scenario II described in Section 3.

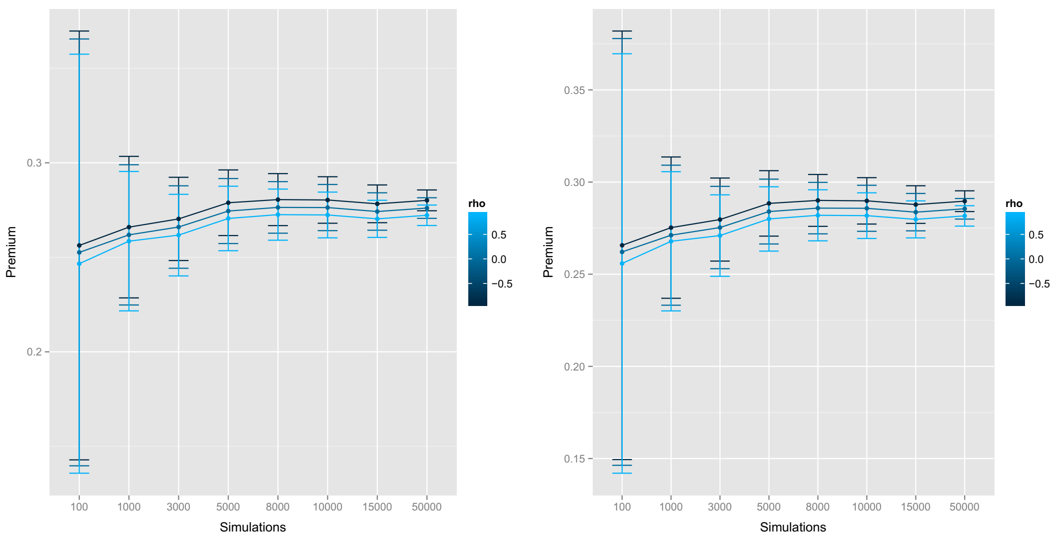
After we confirm the consistency of the results by applying the analytical solutions to both equations in Proposition 3.2 for Scenario I, we apply MC simulation technique to generate all sample paths depending on the first equation in Proposition 3.2 under $\mathbb{Q}$ for both scenarios. We use Euler approximation for the discretization of the governing SDEs for interest and mortality rates.

In the figures, the color code is based on the correlation coefficient and it gets darker when ρ becomes negative. Hence, the results given by light blue points and lines show the premiums and the corresponding confidence intervals respectively for $\rho = 0.9$.



(a) Investment Strategy I (b) Investment Strategy II

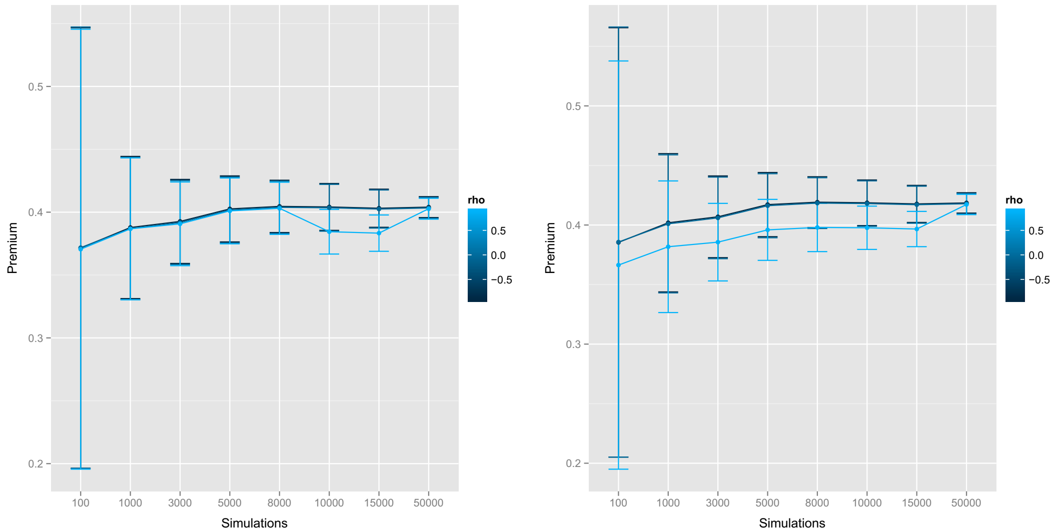
Figure 1. Buy-out premiums with confidence intervals for all ρ based on Scenario I.



(a) Investment Strategy I (b) Investment Strategy II

Figure 2. Buy-out premiums with confidence intervals for all ρ based on Scenario II.

Figures 1(a) and 2(a) display the buy-out prices with the corresponding confidence intervals for each MC simulation under the assumption of investing in the UK financial market based on Scenario I and Scenario II, respectively. The figures state that the buy-out price starts to converge to a certain value for each scenario after 5000 MC samples. Confidence intervals and error margins are derived based on (20) and (21) respectively. The confidence intervals shrink when we increase the iteration numbers.



(a) Investment Strategy I (b) Investment Strategy II

Figure 3. Buy-out premiums with confidence intervals for all ρ based on Scenario III according to the calibration of the UK data when $\eta = 0.0943$.

As a second case, we carry out the same analysis under the assumption of investing in the US financial market for both scenarios. Figures 1(b) and 2(b) depict the obtained buy-out prices under Investment Strategy II with the corresponding confidence intervals for Scenario I and Scenario II, respectively. As the number of MC samples is increased, it is possible to observe the convergence of the results.

Furthermore, we observe a decrement in the price when the correlation coefficient moves from negative to positive in both scenarios. On the contrary, the differences between the calculated buy-out prices under Scenario II are larger than the ones obtained in Scenario I for each correlation coefficient.

4.3. Scenario III and Scenario IV: Pricing

We illustrate how buy-out prices change when we apply LC model to represent the dynamics of mortality rates.

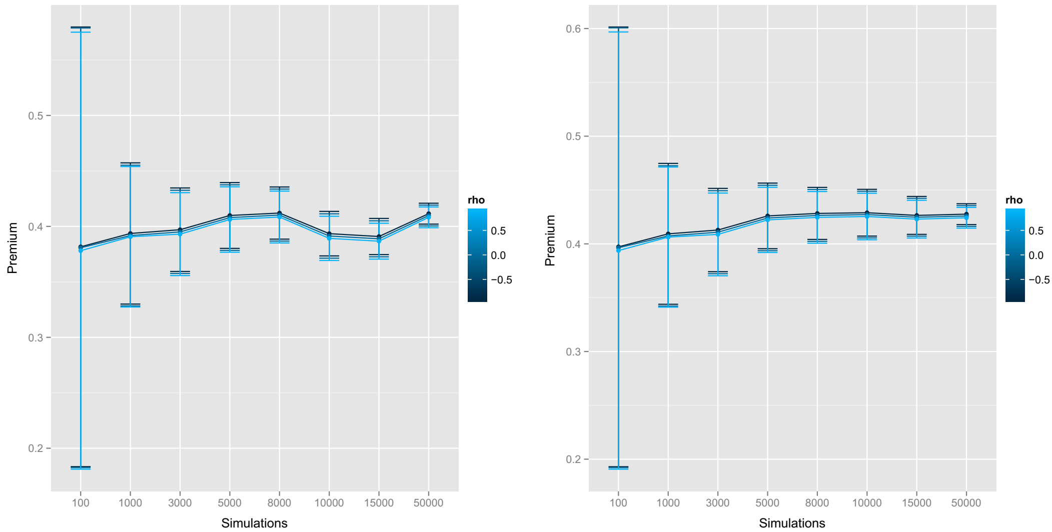
Here, we follow a similar procedure suggested by Lee & Carter (1992) to estimate parameters $\alpha(x)$, $\beta(x)$ and $\kappa(t)$ in the LC model. General mortality index $\kappa(t)$ is modeled by using the SDE

$$d\kappa(t) = \delta dt + \sigma dW(t),$$

and we calibrate the estimated $\kappa(t)$ in order to obtain δ and σ parameters. For simplicity, we also assume that all insureds aged 65 have the same stopping time τ (Biffis & Denuit 2006). Hence, we suppose that the death time of each insured is the same. Therefore, let us drop the indicator j on the force of mortality rate and the relevant functions. Here, parameter $\beta(x)$ is essential for modeling μ due to the drift term under $\mathbb{Q}$, that is

$$\tilde{\delta}^{\mu}(t) = \beta(x+t)\delta - \eta\beta(x+t)\sigma.$$

For the numerical simulations, we generate sample paths for r and μ under $\mathbb{Q}$, respectively, for each time $t \in [t_{i-1}, t_i]$.



(a) Investment Strategy I (b) Investment Strategy II

Figure 4. Buy-out premiums with confidence intervals for all ρ based on Scenario IV according to the calibration of the UK data when $\eta = 0.0943$.

We first assume that investment is in the UK financial market. Figures 3(a) and 4(a) present the fair prices of the buy-out deal based on Scenario III and Scenario IV, respectively, for different correlation coefficients when $\eta = 0.0943$. The corresponding confidence intervals for the MC simulations are depicted in the figures: the convergence of MC simulations is observed.

We repeat the whole pricing process by using Investment Strategy II. As it is seen in Figures 3(b) and 4(b), the buy-out prices are higher when one prefers to make investment in the US market.

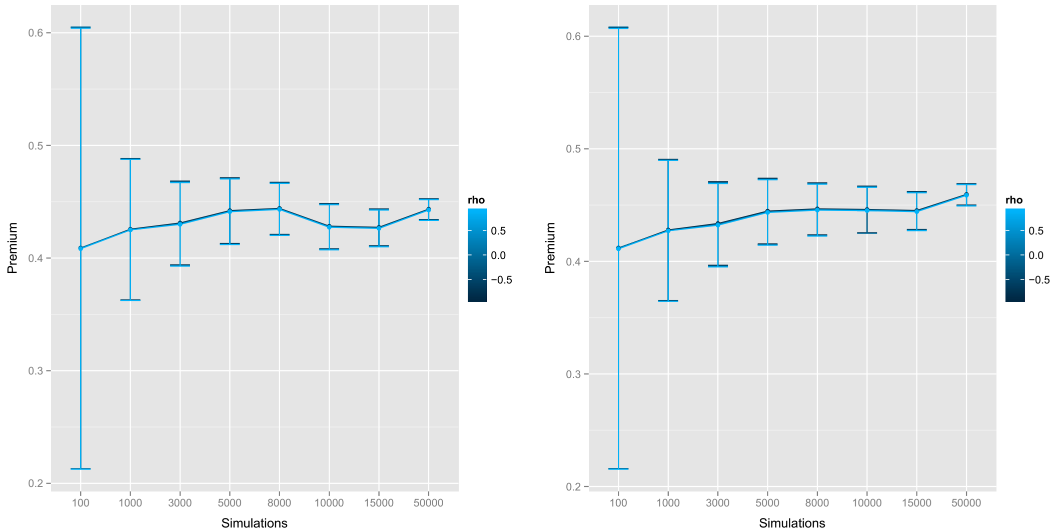
We also investigate the change of the buy-out price due to the parameter η by choosing a higher value for it. While Figures 5(a) and 6(a) display the buy-out premiums when $\eta = 1.197179$ based on Investment Strategy I, Figures 5(b) and 6(b) exhibit the buy-out premiums for the same η based on Investment Strategy II for Scenario III and Scenario IV, respectively. We see that the buy-out prices increase under both scenarios when a higher value for η is preferred.

Due to the nature of the dynamics in the model, lower risk premiums are observed when there is a positive correlation between the interest and mortality rates risks as we already figured out in Scenario I and Scenario II. Besides, the differences between buy-out prices under Scenario IV are more detectable than the ones obtained under Scenario III.

5. Discussion

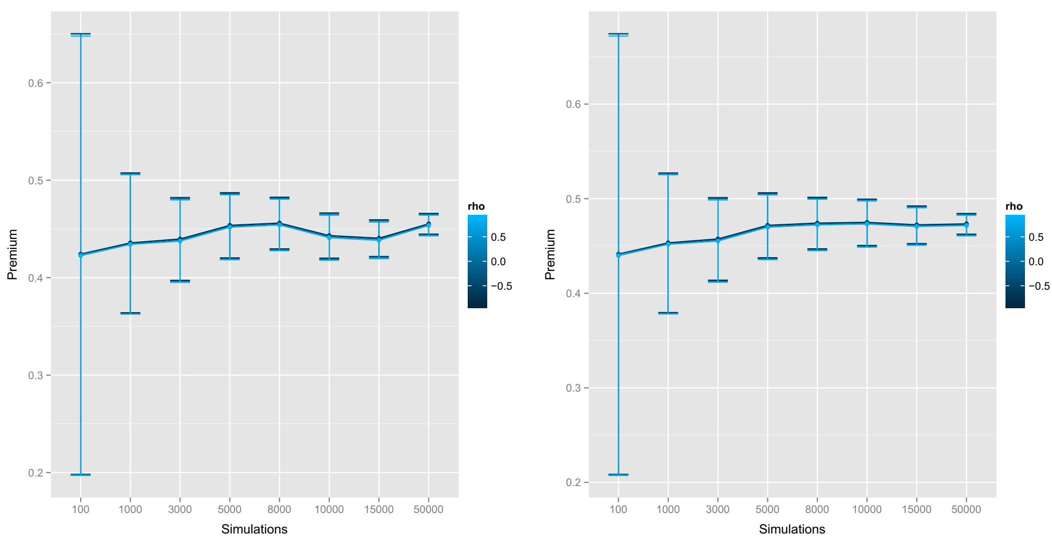
In this paper, we have suggested a new pricing model based on a continuous time framework to price pension buy-out deals under the dependence assumption between interest and mortality rates risks. The dynamics in this model is motivated by the European Union's Solvency II Directive. Solvency II advocates evaluation of the contracts using stochastic models. Moreover, it is recommended to test the capital adequacy requirements under the explicit assumption of the dependence between financial and insurance markets by taking into account the dependency between interest and mortality rates risks (Insurance & Authority 2010).

We have investigated the model under four main scenarios based on different correlation structures and different investment strategies. To summarize, the calculated buy-out prices based on



(a) Investment Strategy I (b) Investment Strategy II

Figure 5. Buy-out premiums with confidence intervals for all ρ based on Scenario III according to the calibration of the UK data when $\eta = 1.197179$.



(a) Investment Strategy I (b) Investment Strategy II

Figure 6. Buy-out premiums with confidence intervals for all ρ based on Scenario IV according to the calibration of the UK data when $\eta = 1.197179$.

50,000 MC sample paths for all scenarios are presented in Tables 5 and 6 under the assumption of investing in the UK and US financial assets, respectively.

Firstly, we have detected that there is not a significant difference between the prices under the same scenario even when the correlation coefficient is changed enormously. This might be considered as the main shortcoming of our setting. However, as we study on a continuous time framework, it is hard to observe dramatic changes in small time intervals. Secondly, the buy-out prices have changed

Table 5. Actuarial fair prices of the buy-out deal under 50000 MC samples according to different correlation coefficients for each scenario based on Investment Strategy I.

		$\rho = -0.9$	$\rho = 0$	$\rho = 0.9$
Scenario I		0.276702	0.275262	0.273804
Scenario II		0.280131	0.276051	0.272238
Scenario III	$\eta = 0.0943$	0.404026	0.403342	0.402736
	$\eta = 1.197179$	0.443413	0.442993	0.442623
Scenario IV	$\eta = 0.0943$	0.411471	0.409774	0.408163
	$\eta = 1.197179$	0.455282	0.454347	0.453450

Table 6. Actuarial fair prices of the buy-out deal under 50000 MC samples according to different correlation coefficients for each scenario based on Investment Strategy II.

		$\rho = -0.9$	$\rho = 0$	$\rho = 0.9$
Scenario I		0.285935	0.284474	0.282996
Scenario II		0.289637	0.285504	0.281642
Scenario III	$\eta = 0.0943$	0.418545	0.417848	0.417231
	$\eta = 1.197179$	0.459505	0.459076	0.458699
Scenario IV	$\eta = 0.0943$	0.427590	0.425858	0.424214
	$\eta = 1.197179$	0.473400	0.472442	0.471525

significantly when we apply LC model, instead of the OU process, to model mortality rate dynamics. Hence, an appropriate mortality model assumption is essential for pricing of the buy-out deal. For our setting, LC model yields more conservative prices. Thirdly, we applied CIR model to avoid the negative interest rates caused by Vasicek model while we keep the mortality model as the OU process or LC model. Both, the buy-out prices and the differences between the prices are relatively higher when CIR model is preferred for modeling short rates. Finally, we have examined the effect of parameter η in Scenario III and Scenario IV. To do so, we have chosen a higher level of η to quantify the changes of the buy-out premium due to this parameter. When the value of η increases, the buy-out prices increase as well.

We should also be cautious about the interpretation of how the correlation coefficient affects the buy-out price. For this aim, we have investigated the fair price of the annuity deals $a(t_i)$, which is an important determinant for the liability process $L(t_i)$ and the payoff process for each scenario. It is seen that the prices of the annuity deals increase when the correlation coefficient changes from negative to positive. This is consistent with the results of Jalen & Mamon (2009) and Liu et al. (2014). When interest and mortality rates are negatively correlated, a natural hedge appears between these two risk factors. Alternatively, a positive correlation causes this natural hedge to disappear (Liu et al. 2014). Hence, the price of the annuity deal is higher when there is a positive correlation between interest and mortality rates.

Since we assume that there is no initial gap between asset and liability processes at time zero, we choose a higher initial value for the asset portfolio $PA(t_i)$ when ρ moves from negative to positive. Moreover, the initial value of $PA(t_i)$ for each subsequent year is based on Remark 1, where the liability process $L(t_i)$ is a key factor to decide the initial value of the asset process $PA(t_i)$ at each t_i. Therefore, the payoff process is mostly realized lower when the interest and mortality rates are positively correlated. It is apparent that the natural hedge is not sufficient any more to compensate the overall uncertainty caused by the nature of the deal. That is a potential reason of the higher prices when the correlation coefficient indicates an inverse proportionality between mortality and interest rates dynamics.

Finally, we note that, when it is preferred to invest in the US financial assets for the same DB pension scheme in the UK, the pricing model suggests higher risk premiums for all scenarios and each correlation coefficient as stated in Table 6.

6. Conclusion

Having suggested a pricing model for buy-out deals depending on a clear correlation structure in this paper, between interest and mortality rates risks in a continuous time framework, we obtained a simplified valuation expression for the general pricing formula. The formula is reduced to the summation of the products of the payoff processes under the appropriate forward measure and the zero coupon bond price under the risk neutral measure. We performed a numerical analysis using various models for interest and mortality rates while we assumed investment in the UK and the US markets. We have discussed the impact of short rates and mortality rates on the buy-out price and concluded that the buy-out price is highly sensitive to the choice of mortality model. Furthermore, we have observed that a strictly positive short rate model increases the difference between buy-out prices according to different correlation coefficients under the same scenario. On the other hand, since we study on a continuous time framework, our setting may not allow us to see abrupt changes in the buy-out price due to the change in the correlation coefficient.

Fortunately, we achieved consistent results with the earlier studies in the pricing of liability values of the hypothetical pension scheme. We have also explained the decrement in the buy-out price when the interest and mortality rates are positively correlated, contrary to the increase in the liability process. We have concluded that the decrement in the price is a result of the relation between asset and liability processes. Besides, we have presented the obtained buy-out prices with the corresponding confidence intervals for various MC simulations, showing that the prices converge for each scenario. Furthermore, we have also discussed the effect of the parameter η under Scenario III and Scenario IV and concluded that the buy-out price is higher when we prefer a higher level of η. Finally, we have observed lower buy-out prices for each scenario when the investment strategy is based on the UK financial market.

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