The Evolution of Yield Curves in 2 Factor Models

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About the speaker

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- **Fraunhofer ITWM**, Financial Mathematics
- Consulting for financial economy and energy industry
- Competence in statistical methods and their applications

Agenda

Introduction to yield curves

Yield curves in a Vasicek model

Yield curves in a two-dimensional Vasicek model

Yield curves in a two-dimensional Hull-White model

Introduction of the yield curve

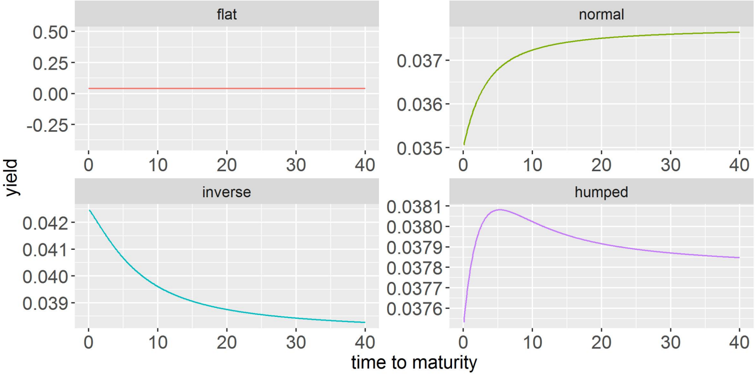
What is the yield curve?

The yield curve shows the yield of zero-coupon bonds as a function of their time to maturity.

Why is the yield curve important for life insurances?

The yield curve contains the complete information on the possible returns of today’s bond investment over "all" durations.

Which shapes of yield curves exist?



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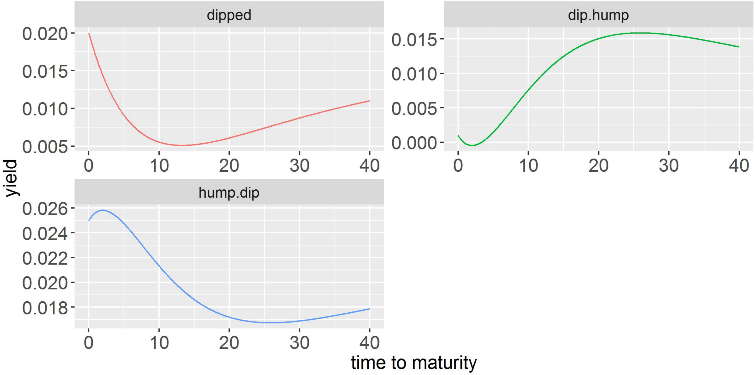


Vasicek model

Vasicek model

Dynamics of the short-rate r in the Vasicek model follows the process under the risk-neutral measure:

$$dr(t) = a\,(\theta - r(t))\,dt + \sigma dW(t), \qquad r(0) = r_0 \tag{1}$$

with r_0, a, θ, σ positive constants, $W(t)$ a one dimensional Brownian motion.

The short-rate $r(t)$ is normally distributed with:

$$r(t) \sim \mathcal{N}\left(r_0 e^{-at} + \theta(1 - e^{-at}), \frac{\sigma^2}{2a}(1 - e^{-2at})\right) \tag{2}$$

Vasicek model

The yield of a zero-coupon bond with time to maturity x at time t is given by

$$y(t,x) = \theta - \frac{\sigma^2}{2a^2} + \frac{(1-e^{-ax})}{ax}\left(r(t) - \theta + \frac{\sigma^2}{2a^2} + \frac{\sigma^2}{4a^2}\left(1-e^{-ax}\right)\right) \tag{3}$$

It directly follows

$$\lim_{x\to 0} y(t,x) = r(t), \qquad \lim_{x\to\infty} y(t,x) = \theta - \frac{\sigma^2}{2a^2} \tag{4}$$

Yield curves

The shape of the yield curve is determined only by r(t) (see Desmettre, Korn (2018), Keller-Ressel (2018), Vasicek (1977)).

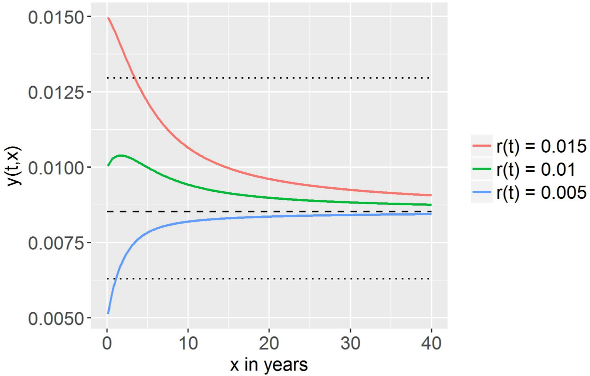
Theorem

Let $a, \sigma > 0$. Then the following holds:

$$\begin{aligned} y(t,x) \text{ is normal} &\iff r(t) \leq \theta - \frac{3\sigma^2}{4a^2} \\ y(t,x) \text{ is humped} &\iff \theta - \frac{3\sigma^2}{4a^2} < r(t) < \theta \\ y(t,x) \text{ is inverse} &\iff r(t) \geq \theta \end{aligned}$$

Yield curves

$a = 0.401$, $\sigma = 0.0378$, $\theta = 0.01297$



Evolution of yield curves in time

We have

$$\mathbb{E}(r(t)) = \theta + (r(0) - \theta)e^{-at} \xrightarrow{t\to\infty} \theta \tag{5}$$

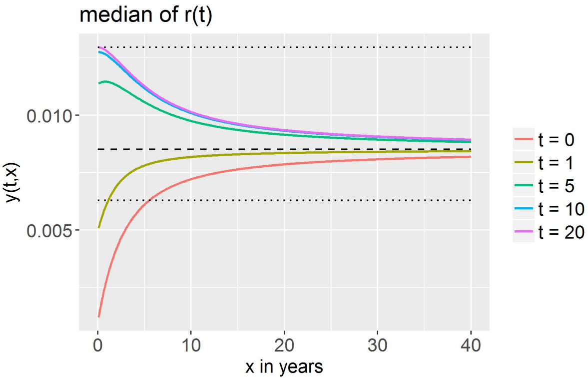
Furthermore, for very long term bonds we obtain

$$y(t,x) \xrightarrow{x\to\infty} \theta - \frac{\sigma^2}{2a^2} \tag{6}$$

It follows that more humped and inverse yield curves are observed than normal ones in the Vasicek model with the passing of time.

Evolution of yield curves in time

$a = 0.401,\ \sigma = 0.0378,\ \theta = 0.01297,\ r(0) = 0.001$



Yield curves in one-dimensional affine-linear models

- Examples for one-dimensional affine-linear models: Vasicek, Cox-Ingersoll-Ross and Gamma model.
- In literature there exist different statements on what kind of shapes of yield curves can be produced by the Cox-Ingersoll-Ross model.
- Keller-Ressel and Steiner (2008): in any one-dimensional affine-linear model yield curves can only be inverse, normal and humped (see also Keller-Ressel (2018)).
- Only the level of the Short Rate $r(t)$ determines the shape.

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Two-dimensional Vasicek model

Two-dimensional Vasicek model

The Short-Rate $r(t)$ is given by

$$r(t) = \psi(t) + X(t) + Y(t) \tag{7}$$

where $\psi(t)$ is the deterministic function

$$\psi(t) = r_0 e^{-at} + \theta(1 - e^{-at}) \tag{8}$$

with r_0, $\theta \in \mathbb{R}$

Two-dimensional Vasicek model

and $X(t)$ and $Y(t)$ are stochastic processes with

$$dX(t) = -aX(t)dt + \sigma dW_1(t), \qquad X(0) = 0 \tag{9}$$

$$dY(t) = -bY(t)dt + \eta dW_2(t), \qquad Y(0) = 0 \tag{10}$$

with a, b, σ, $\eta > 0$, W_1 and W_2 are one dimensional Brownian motions with correlation $-1 \leq \rho \leq 1$.

Two-dimensional Vasicek model

The yield of a zero-coupon bond with duration x at time t is given by

$$\begin{aligned} y(t,x) &= \frac{1}{x}\int_t^{t+x} \psi(u)\,\mathrm{d}u + \frac{1-e^{-ax}}{ax}X(t) + \frac{1-e^{-bx}}{bx}Y(t) - \frac{1}{2x}V(x) \\ &= \theta + (\psi(t)-\theta)\frac{1-e^{-ax}}{ax} + \frac{1-e^{-ax}}{ax}X(t) + \frac{1-e^{-bx}}{bx}Y(t) - \frac{1}{2x}V(x) \end{aligned} \tag{11}$$

Two-dimensional Vasicek model

where

$$
\begin{aligned}
V(x) = {} & \frac{\sigma^2}{a^2}\left(x - 2\frac{1-e^{-ax}}{a} + \frac{1-e^{-2ax}}{2a}\right) \\
& + \frac{\eta^2}{b^2}\left(x - 2\frac{1-e^{-bx}}{b} + \frac{1-e^{-2bx}}{2b}\right) \\
& + 2\rho\frac{\sigma\eta}{ab}\left(x - \frac{1-e^{-ax}}{a} - \frac{1-e^{-bx}}{b} + \frac{1-e^{-(a+b)x}}{a+b}\right)
\end{aligned}
\tag{12}
$$

Yield curves

$y(t,x)$ can be transformed to

$$\begin{aligned} y(t,x) =& y^{1FV}(t,x) + \frac{1-e^{-bx}}{bx} Y(t) - \frac{\eta^2}{2b^2}\left(1 - 2\frac{1-e^{-bx}}{bx} + \frac{1-e^{-2bx}}{2bx}\right) \\ & - \rho\frac{\sigma\eta}{ab}\left(1 - \frac{1-e^{-ax}}{ax} - \frac{1-e^{-bx}}{bx} + \frac{1-e^{-(a+b)x}}{(a+b)x}\right) \end{aligned} \tag{13}$$

Yield curves

Theorem

For all values of the short rate $r(t)$ *pairs of* $(X(t), Y(t))$ *exist with* $y'(t,0) > 0$ *and* $y'(t,0) < 0$*, where* $y'(t,0) = \lim_{x\to 0} \frac{\partial}{\partial x} y(t,x)$.

Proof.

$$\lim_{x\to 0} \frac{\partial}{\partial x} y(t,x) = \lim_{x\to 0} \frac{\partial}{\partial x} y^{1FV}(t,x) - bY(t)$$

□

Yield curves

Important consequences of the theorem

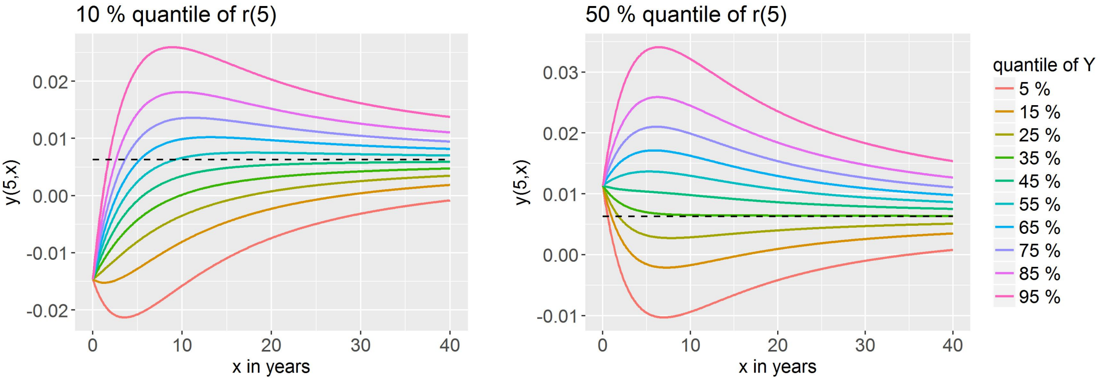
- The shape of yield curves is not determined by the level of the current short rate $r(t)$ alone (compared to the one-dimensional Vasicek).
- Two-dimensional Vasicek also produces the shapes of yield curves of the one-dimensional Vasicek.
- In addition, there exist yield curves with one dip.
- There exist yield curves with one hump for $r(t) > \theta$.

Numerical examples

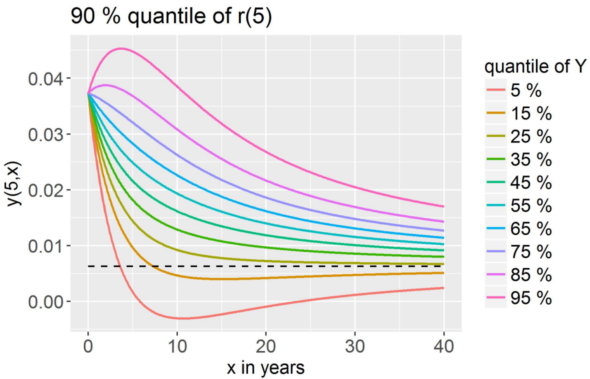
parameters:

$$a = 0.401,\ \sigma = 0.0378$$
$$b = 0.178,\ \eta = 0.0372$$
$$\rho = -0.996$$
$$\theta = 0.01297$$
$$r(0) = 0.001$$

Numerical examples



Numerical examples



Two-dimensional Hull-White model

Two-dimensional Hull-White model

Short rate and yield of a zero-coupon bond with duration x at time t are calculate analogous to the two-dimensional Vasicek. Only the deterministic function $\psi(t)$ is different:

$$\psi(t) = f_M(0,t) + \frac{\sigma^2}{2a^2}(1-e^{-at})^2 + \frac{\eta^2}{2b^2}(1-e^{-bt})^2 + \rho\frac{\sigma\eta}{ab}(1-e^{-at})(1-e^{-bt}) \quad (14)$$

where $f_M(0,t)$ are the forward rates which are produced by the market price of zero-coupon bonds.

Numerical examples

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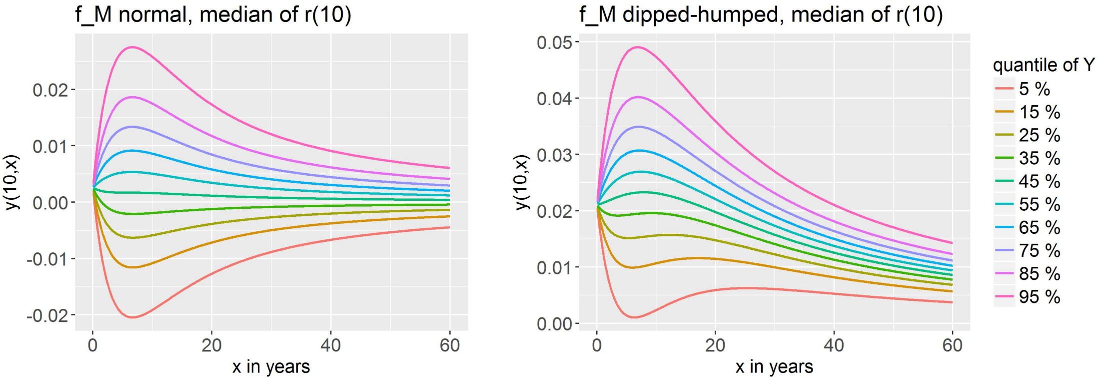
Numerical examples



Thank you very much for your attention!

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