An individual risk model for premium calculation based on quantile:

a comparison between Generalized Linear Models and Quantile Regression

Fabio Baione, Davide Biancalana

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AFIR ERM 19 Florence

IAA AAI AFIR-ERM Finance, Investment & ERM

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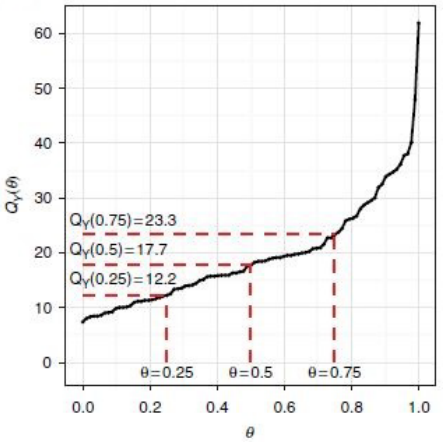
Agenda

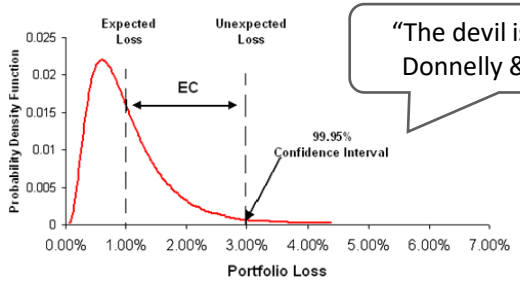
- A brief introduction on…myself!
- Motivation
  - Quantile
  - Regression
- References on QR in Actuarial context
- Risk Model
- Premium Principles
- Ratemaking
  - GLM
  - QR
- Pros and cons
- Further researches and conclusions

Motivation….



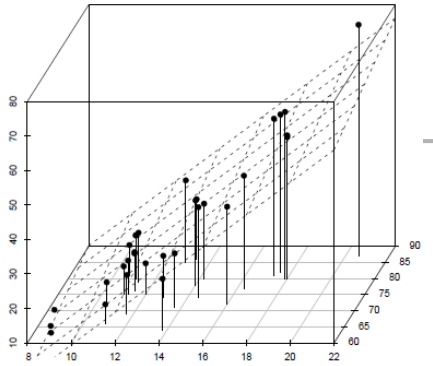
- $Q_Y(\theta) = F_Y^{-1}(\theta) = inf\{y : F(y) > \theta\}$, with $\theta \epsilon [0,1]$
- Why Actuaries use Quantile?
- Because we live in a quantile world….
  - Solvency II Directive: Article 101 states:
    - " The Solvency Capital Requirement (SCR) shall correspond to the Value-at-Risk (VaR) of the basic own funds of an insurance or reinsu undertaking subject to a confidence level of 99.5% over a one-year period. "
  - Basel III – Paragraph 186 Minimum capital requirements for market risk states:
    - "Default risk must be measured using a VaR model …must be done weekly and be based on a one-year time horizon at a one-tai percentile confidence level"
  - IFRS 17 - Risk adjustment for non-financial risk – paragraph 119 states:
    - "An entity shall disclose the confidence level used to determine the risk adjustment for non-financial risk»
- When Actuaries use Quantile?
  - Risk Assessment
    - Premium principles
    - Reserves, especially in non-life sector.
  - Risk Theory
    - Ruin Probability ⟶ Solvency Capital Requirement
  - Enterprise (Quantitative) Risk Management
    - Extreme Value Theory
    - Risk Measure: VaR, ES, Expectile

The compensation an entity requires for bearing the uncertainty about the amount and timing of the cash flows that arises from non-financial risk as the entity fulfils **insurance contracts**.



"The devil is in the tails"
Donnelly & Embrechts

Motivation….



- $Y_i = \beta_0 + \sum_{j=1}^{m} \beta_j \cdot x_{i,j} + \epsilon_i$

- Why Actuaries use Regression?
  - Because a large part of insured risks are related to risk factors which characterize their risk profiles (e.g. demographic, economic or geographical factors).
- When Actuaries use Regression?
  - A priori ratemaking (GLMs for Insurance Rating CAS)
  - Survival Models, Mortality Laws (Gompertz, Perks, Cox (1972). "Regression Models and Life-Tables". Journal of the Royal Statistical Society, Series B. 34 (2): 187–220).
  - Claim Reserving (e.g. Chain ladder method, Claims Count Process with ODP)
  - Least Square Monte Carlo (as a proxy for SCR)
- Which regression models are applied by Actuariaes?
  - Ordinary Linear or Polynomial Regression
  - Generalized Linear Models (GLMs) are the most widely accepted method in a priori classification (extensions GAMs, HGLMs,GMMLs)
  - Other Non Linear Models.

References of QR applied to the Actuarial context

- Widely used in statistics and econometrics, while there are very few applications in actuarial field.
- To the best of our knowledge:
  - Ratemaking
    - (2009) Kudryavtsev, A. A., 2009. Using quantile regression for ratemaking. Insurance: Mathematics and Economics 45, 296-304
    - (2010) Fu, L. and Cheng-sheng Wu, P. Applications of Quantile Regression in Commercial Underwriting Models https://www.casact.org/education/rpm/2010/handouts/CL1-Fu.pdf
    - (2014) Wolny-Dominiak, A., Ornat-Acedaska, A., Trzpiot G., 2012. Insurance portfolios rate making: Quantile regression approach, Proceedings of 30th International Conference Mathematical Methods in Economics
    - (2018) Heras A. Moreno I. Vilar-Zanon J.L. An application of two-stage quantile regression to insurance ratemaking, Scandinavian Actuarial Journal, 2018 (9) pp.753-769
    - (2019) Baione, F., Biancalana D.,. An individual risk model for premium calculation based on quantile: a comparison between Generalized Linear Models and Quantile Regression, to be pusblished in North American Actuarial Journal
  - Solvency 2
    - Pitselis, G., Solvency supervision based on a total balance sheet approach, Journal of Computational and Applied Mathematics 233(1): 83-96
  - Credibility Theory
    - (2013) Pitselis, G. «Quantile credibility models» Insurance: Mathematics and Economics Volume 52, Issue 3, May 2013, Pages 477-489
  - Claims Reserving
    - (2014) Risk Margin Quantile Function Via Parametric and Non-Parametric Bayesian Quantile Regression, arXiv:1402.2492.

Quantile Regression
Koenker and Bassett (1978)

- The (un)conditional Mean solves

$$\mu = \operatorname{argmin} E(Y - c)^2$$

$$\mathrm{E}(\mathrm{Y}|\boldsymbol{x}_\mathrm{i}) \overset{c}{=} \boldsymbol{x}_\mathrm{i}\boldsymbol{\beta}$$

- The (un)conditional Median solves

$$Me = \operatorname{argmin} E|Y - c|$$

$$\mathrm{Me}(\mathrm{Y}|\boldsymbol{x}_\mathrm{i}) \overset{c}{=} \boldsymbol{x}_\mathrm{i}\boldsymbol{\beta}_{\mathrm{Me}}$$

- The (un)conditional Quantile solves

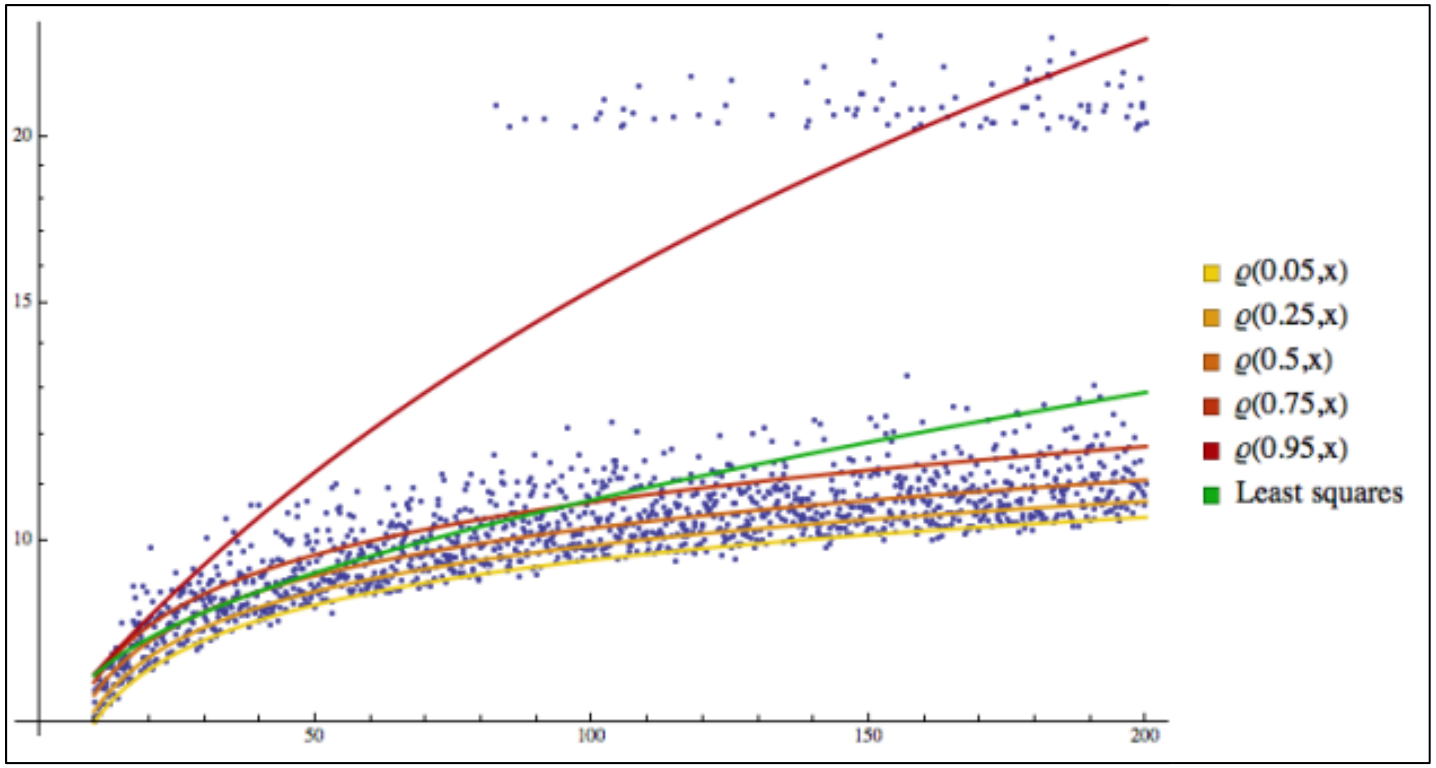
$$\mathrm{Q}(\theta) = \operatorname{argmin} E[\rho_\theta(Y - c)]$$

$$\mathrm{Q}_\theta(\mathrm{Y}|\boldsymbol{x}_\mathrm{i}) \overset{c}{=} \boldsymbol{x}_\mathrm{i}\boldsymbol{\beta}_{\boldsymbol{\theta}}$$

  - With an asymmetric absolute loss (or score) function

$$\rho_\theta = [\theta - I(y < 0)] \cdot y = [(1 - \theta) \cdot I(y \leq 0) + \theta \cdot I(y > 0)] \cdot |y|$$

QR is robust to outliers: an example



QR Equivariance properties

- Let $\boldsymbol{\beta}^*_{\boldsymbol{\theta}}(y, x)$ be the vector of solutions of a θ-Regression quantile based on observation (x_i, y_i):
  - Scale Equivariance:

$$\beta^*_\theta(\alpha y, x) = \alpha\beta^*_\theta(y, x),$$
$$\beta^*_\theta(-\alpha y, x) = \alpha\beta^*_{1-\theta}(y, x) \qquad \text{with } \alpha > 0$$

  - Regression Shift:

$$\beta^*_\theta(y + x\lambda, x) = \beta^*_\theta(y, x) + \lambda \text{ with } \lambda \in \mathbb{R}^m$$

  - Reparameterization of Design:

$$\beta^*_\theta(y, xA) = A^{-1}\beta^*_\theta(y, x) \quad \text{for any } |A| \neq 0$$

  - **Equivariance to Monotone Transformations**
    - For any monotone function g( ), conditional quantile functions $Q(Y|x_i)$ are equivariant in the sense that (Koenker 2005)

$$Q_\theta(g(Y)) = F^{-1}_{g(Y)}(\theta) = g(F_Y^{-1}(\theta)) = g(Q_\theta(Y))$$

While for conditional mean functions for which, generally,

$$E\big(g(y)\big) \neq g\big(E(y)\big)$$

    - Examples: $g(Y) = min(0, y)$ or $\boldsymbol{g(Y) = log(y)}$

Basic Premium Principles

- Let:
  - Υ be a set of r.v.s in $(\Omega, \mathfrak{F}, \boldsymbol{\mathcal{P}})$
  - $Y \epsilon \Upsilon, with\ Y \geq 0$
  - For each $\omega \epsilon \Omega, Y(\omega) > 0$ then we will consider this to be a loss.
  - Risk measure π: $\pi(Y)$: $\Upsilon \rightarrow \mathbb{R}$

Table 1. Basic Premium Principles

Name	$\pi(Y)$	*Premium Loading*	
Equivalence Premium	$P = E(Y)$	0	
Expected value	$P_E = (1+\alpha)\cdot E(Y)$	$\alpha \cdot E(Y)$	*with* $\alpha > 0$
Variance	$P_{Var} = E(Y) + \alpha \cdot Var(Y)$	$\alpha \cdot Var(Y)$	*with* $\alpha > 0$
Standard Deviation	$P_{SD} = E(Y) + \alpha \cdot \sigma(Y)$	$\alpha \cdot \sigma(Y)$	*with* $\alpha > 0$
Semi Variance	$P_{SV} = E(Y) + \alpha \cdot E\left\{[Y - E(Y)]^+\right\}^2$	$\alpha \cdot E\left\{[Y - E(Y)]^+\right\}^2$	*with* $\alpha > 0$
Exponential premium	$P_{Exp} = \frac{1}{\alpha} \cdot logE(e^{\alpha Y})$	$P_{Exp} - E(Y)$	*with* $\alpha > 0$
Quantile or VaR premium	$P_{VaR} = \inf\{y \in \mathbb{R} : \alpha \leq F_Y(y)\}$	$P_{VaR} - E(Y)$	*with* $\alpha \in [0,1]$

- Premium Loading: $\mathrm{L}(Y) = \pi(Y) - E(Y)$
- **Risk margin function: $\boldsymbol{\rho(Y) = \frac{\pi(Y)}{E(Y)}}$**
- Remark: the assignment of a real number through the mapping $\pi(Y)$ can be expressed using the inverse of the cdf of Y, i.e. the quantile function, if exists (i.e. $\pi(Y) = Q_Y(\theta), \theta \epsilon [0,1]$

Two-part models for insurance ratemaking

Collective risk model - The total loss of the portfolio

$$Y = \sum_{i=1}^{r} Y^{(i)} = \sum_{k=1}^{N} Z_k = \sum_{i=1}^{r} \sum_{j=1}^{N^{(i)}} Z_j^{(i)}$$

Individual Risk Model 1 (**RM1**)

Let

- $Y^{(i)} = \mathbb{I}_{N^{(i)}} \cdot \tilde{Y}^{(i)}$

$$F_{Y^{(i)}}(y) = Pr\left(Y^{(i)} < y, \mathbb{I}_{N^{(i)}} = 0\right) + Pr\left(Y^{(i)} < y, \mathbb{I}_{N^{(i)}} = 1\right)$$

$$\Downarrow$$

$$F_{Y^{(i)}}(y) = p_i + (1 - p_i) \cdot Pr\left(\tilde{Y}^{(i)} < y\right)$$

- Where
  - $\tilde{Y}^{(i)} = \left(\sum_{j=1}^{N^{(i)}} Z_j^{(i)} \middle| N^{(i)} > 0\right)$
  - $p_i = Pr\left(\mathbb{I}_{N^{(i)}} = 0\right)$

Individual Risk Model 2 (**RM2**)

- Let
- $Y^{(i)} = \sum_{j=1}^{N^{(i)}} Z_j^{(i)}$

$$F_{Y^{(i)}}(y) = \sum_{n=0}^{\infty} Pr\left(Z_1^{(i)} + \cdots + Z_n^{(i)} \middle| N^{(i)} = n\right) \cdot Pr\left(N^{(i)} = n\right)$$

$$\Downarrow Z_j^{(i)}\ i.i.d.$$

$$\sum_{n=1}^{\infty} F_{Z^{(i)}}^{n*}(y) \cdot Pr\left(N^{(i)} = n\right)$$

A quantile premium principle based on a two-part model

- RM1

$$F_{Y^{(i)}}\left(Q_{\tilde{\theta}_i}\left[\tilde{Y}^{(i)}\right]\right) = p_i + (1-p_i)\cdot F_{\tilde{Y}^{(i)}}\left(Q_{\tilde{\theta}_i}\left[\tilde{Y}^{(i)}\right]\right)$$

- By setting $F_{Y^{(i)}}\left(Q_{\tilde{\theta}_i}\left[\tilde{Y}^{(i)}\right]\right) = \theta_i$

$$\theta_i = p_i + (1-p_i)\cdot\tilde{\theta}_i \Leftrightarrow \tilde{\theta}_i = \frac{\theta_i - p_i}{1-p_i}$$

$$\Downarrow$$

$$Q_{\tilde{\theta}_i}\left[\tilde{Y}^{(i)}\right] = Q_{\theta_i}\left[Y^{(i)}\right]$$

- RM2

$$F_{Y^{(i)}}\left(Q_{\tilde{\theta}_i}\left[Z^{(i)}\right]\right)$$
$$= Pr\left(N^{(i)}=0\right) + Pr\left(N^{(i)}=1\right)\cdot F_{Z^{(i)}}\left(Q_{\tilde{\theta}_i}\left[Z^{(i)}\right]\right)$$
$$+ \sum_{n=2}^{\infty} F_{Z^{(i)}}^{n*}\left(Q_{\tilde{\theta}_i}\left[Z^{(i)}\right]\right)\cdot Pr\left(N^{(i)}=n\right)$$

- By setting $F_{Y^{(i)}}\left(Q_{\tilde{\theta}_i}\left[\tilde{Z}^{(i)}\right]\right) = \theta_i^{\#}$

$$\theta_i^{\#} =$$
$$= Pr\left(N^{(i)}=0\right) + Pr\left(N^{(i)}=1\right)\cdot\tilde{\theta}_i + \sum_{n=2}^{\infty} F_{Z^{(i)}}^{n*}\left(Q_{\tilde{\theta}_i}\left[Z^{(i)}\right]\right)$$
$$\cdot Pr\left(N^{(i)}=n\right)$$

RM1

Equivalence Premium:

$$E\left[Y^{(i)}\right] = \mathrm{E}\left[\mathbb{I}_{N^{(i)}}\right]\cdot\mathrm{E}\left[\tilde{Y}^{(i)}\right]$$

Net Premium:

$$P_{Q_{RM1}} = \mathrm{E}\left[\mathbb{I}_{N^{(i)}}\right]\cdot Q_{\tilde{\theta}_i}\left[\tilde{Y}^{(i)}\right]$$

RM2

Equivalence Premium:

$$E\left[Y^{(i)}\right] = \mathrm{E}\left[N^{(i)}\right]\cdot\mathrm{E}\left[Z^{(i)}\right]$$

Net Premium

$$P_{Q_{RM2}} = \mathrm{E}\left[N^{(i)}\right]\cdot Q_{\tilde{\theta}_i}\left[Z^{(i)}\right]$$

Generalized Linear Models (GLMs)
Nelder e Weddenburn,1972

- GLMs are a flexible generalization of OL Regression:
  - The response variables that have error distribution models other than a normal distribution.
  - The dependent variable has a distribution from the exponential family with probability density function (normal, binomial, Poisson and gamma distribution etc.)
- $f(y,\kappa,\varphi) = exp\left\{\frac{y\kappa - b(\kappa)}{a(\varphi)} + c(y,\varphi)\right\}$
- where a, b, c are known functions and κ,φ are unknown canonical and dispersion parameters.

Table 2. Characteristics of some common univariate distributions in the exponential family[1]

	Name	*Notation*	*Density function* $f(y,\kappa,\varphi)$	*Dispersion parameter* φ	*Canonical parameter* κ	*Mean Value* $\mu(\kappa)$	*Variance Function* $Var(\mu)$
$N^{(i)}$	*Poisson*	$P(\lambda)$	$\frac{\lambda^y \cdot e^{-\lambda}}{y!}$	1	$log(\lambda)$	e^{κ}	λ
$\mathbb{I}_{N^{(i)}}$	*Binomial*	$B(m,\pi)/m$	$\binom{m}{y}\pi^y(1-\pi)^{m-y}$	$1/m$	$logit(\pi)$	$e^{\kappa}/(1+e^{\kappa})$	$\mu(1-\mu)$
$\tilde{Y}_i$	*Gamma*	$G(\nu,\eta)$	$\frac{1}{\Gamma(\nu)y}(\frac{y}{\eta})^{\nu}e^{-\frac{y}{\eta}}$	$\frac{1}{\nu}$	$-\frac{1}{\eta\nu}$	$-\frac{1}{\kappa}$	μ^2

- The function $a(\varphi)$ is commonly of the form $a(\varphi) = \varphi/\omega$ where φ is constant over observation, and ω is a known prior weight that can be variable for each observation.
- The magnitude of the variance of each measurement to be a function of its predicted value $Var(Y) = a(\varphi) \cdot V(\mu)$
- The dependency is described by a link function g which is strictly monotonous and twice differentiable $E(Y|\boldsymbol{x}_i) = E(Y_i) = g^{-1}(\boldsymbol{x}_i\boldsymbol{\beta})$

- An approximately unbiased estimator φ is

$$\hat{\varphi}_X = \frac{\varphi X^2}{n-r} = \frac{1}{n-r}\sum_i \omega_i \frac{(y_i - \hat{\mu}_i)^2}{V(\mu_i)}$$

The most widely adopted is the logarithm

Risk margin estimation via GLMs:
the case of Gamma distribution with unitary prior weights

Assuming that

1) $\tilde{Y}_i \sim \mathrm{G}(\nu_i, \eta_i)$

2) prior weights $\omega_i \equiv 1$ for each $i = 1,2,\cdots,r$.

- Proposition 1
- The Coefficient of Variation (CV) of each individual loss per claimant is constant and equal to $\sqrt{\varphi}$. Then, $\tilde{Y}_i \sim \mathrm{G}(\nu, \eta_i)$ $\forall i = 1,2,\cdots,r$

Proof.

By 1)

$$E(\tilde{Y}_i) = \nu_i \cdot \eta_i \ \ and \ Var(\tilde{Y}_i) = \nu_i \cdot \eta_i^2$$
$$\Downarrow$$
$$CV(\tilde{Y}_i) = 1/\sqrt{\nu_i}$$

By 2)

$$a(\varphi) = \varphi/\omega \equiv \varphi$$
$$\Downarrow$$
$$Var(\tilde{Y}_i) = \varphi \cdot V(\mu_i) = \varphi \cdot \mu_i^2$$
$$\Downarrow$$
$$\mathrm{CV}(\tilde{Y}_i) = \sqrt{\varphi} \Rightarrow \nu_i \equiv \nu, \forall i = 1,2,\cdots,r$$
$$\Downarrow$$
$$\tilde{Y}_i \sim \mathrm{G}(\nu, \eta_i)$$

- Proposition 2
- The $\mathrm{Q}_{\tilde{\theta}}(\tilde{Y}_i)$ is obtained by multiplying the expected value $E(\tilde{Y}_i)$ by a constant depending exclusively by level $\tilde{\theta}$:

$$\mathrm{Q}_{\tilde{\theta}}(\tilde{Y}_i) = \rho_{\tilde{\theta}} \cdot E(\tilde{Y}_i) = \rho_{\tilde{\theta}} \cdot \nu \cdot \eta_i$$

Proof.

By proposition 1 we show :

$$\tilde{\theta} = \int_0^{\rho_{\tilde{\theta}} \cdot \nu \cdot \eta_1} \frac{1}{\Gamma(\nu)\eta_1^{\nu}} x^{\nu-1} e^{-\frac{x}{\eta_1}} dx = \int_0^{\rho_{\tilde{\theta}} \cdot \nu \cdot \eta_2} \frac{1}{\Gamma(\nu)\eta_2^{\nu}} x^{\nu-1} e^{-\frac{x}{\eta_2}} dx$$

Net Premium:

$$P_{Q_{RM1}} = \mathrm{E}[\mathbb{I}_{N^{(i)}}] \cdot Q_{\tilde{\theta}_i}[\tilde{Y}^{(i)}]$$
$$P_{Q_{RM1}}^{GLM} = (1 - p_i) \cdot \rho_{\tilde{\theta}} \cdot E(\tilde{Y}_i)$$

Risk margin function:

$$\rho(Y_i) = \frac{E(Y_i)}{P_{Q_{RM1}}^{GLM}} = \rho_{\tilde{\theta}}, \ \forall i = 1,2,\cdots,r$$

Risk margin estimation with QR

- Remember that:

$$P_{Q_{RM1}} = \mathrm{E}\left[\mathbb{I}_{N^{(i)}}\right] \cdot Q_{\tilde{\theta}_i}\left[\tilde{Y}^{(i)}\right]$$

  - By the equivariance to monotone transformation

$$Q_{\tilde{\theta}_i}\left(log\left(\tilde{Y}^{(i)}\right)\right) = log\left(Q_{\tilde{\theta}_i}\left(\tilde{Y}^{(i)}\right)\right) = \boldsymbol{x}_i\boldsymbol{\beta}_{\tilde{\theta}_i} \rightarrow Q_{\tilde{\theta}_i}\left(\tilde{Y}^{(i)}\right) = e^{\boldsymbol{x}_i\boldsymbol{\beta}_{\tilde{\theta}_i}}$$

  Net Premium:

$$P_{Q_{RM1}}^{QR} = (1 - p_i) \cdot e^{\boldsymbol{x}_i\boldsymbol{\beta}_{\tilde{\theta}_i}}$$

  - Risk margin function:

$$\rho(Y_i) = \frac{E(Y_i)}{P_{Q_{RM1}}^{QR}} = \frac{(1 - p_i) \cdot e^{\boldsymbol{x}_i\boldsymbol{\beta}}}{(1 - p_i) \cdot e^{\boldsymbol{x}_i\boldsymbol{\beta}_{\tilde{\theta}_i}}} = e^{\boldsymbol{x}_i\left(\boldsymbol{\beta} - \boldsymbol{\beta}_{\tilde{\theta}_i}\right)}$$

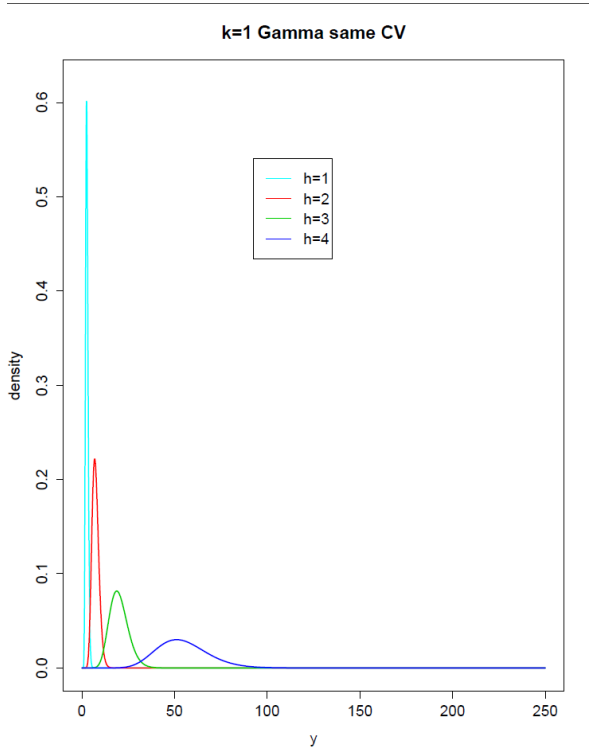
Numerical Investigation

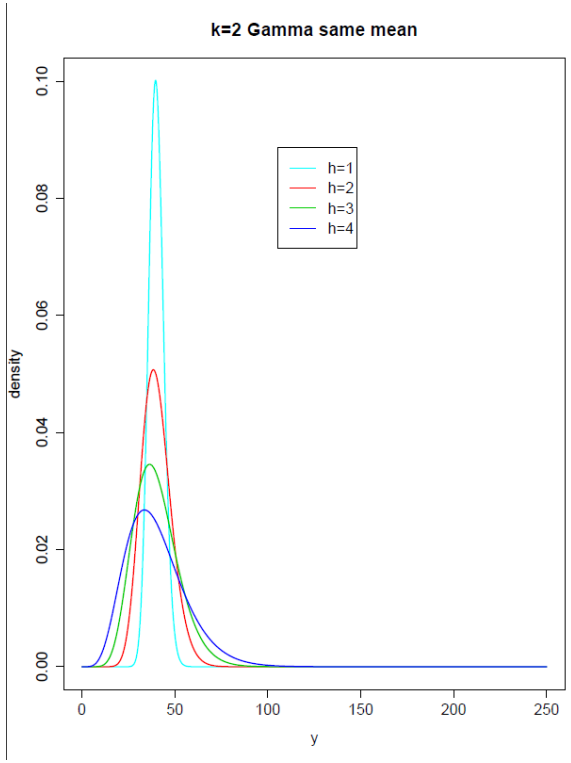
- Data sets are characterized by the same total exposure and number of claimants. We consider two rating factors (e.g. sex and age-class) each one with two levels (e.g. "Male" and "Female" and "High" and "Low" respectively). Then we have four risk profiles, briefly "M-H", "F-H", "M-L" and "F-L". We assume that each risk profile has a different exposure but the same number of claimants.
- The main difference between data sets is the assumptions adopted for the distribution for the total claim amount per claimants
- Let:
- k = 1;2;3 index the number of data set.
- r_i = 100;000 be the number of insured/exposure

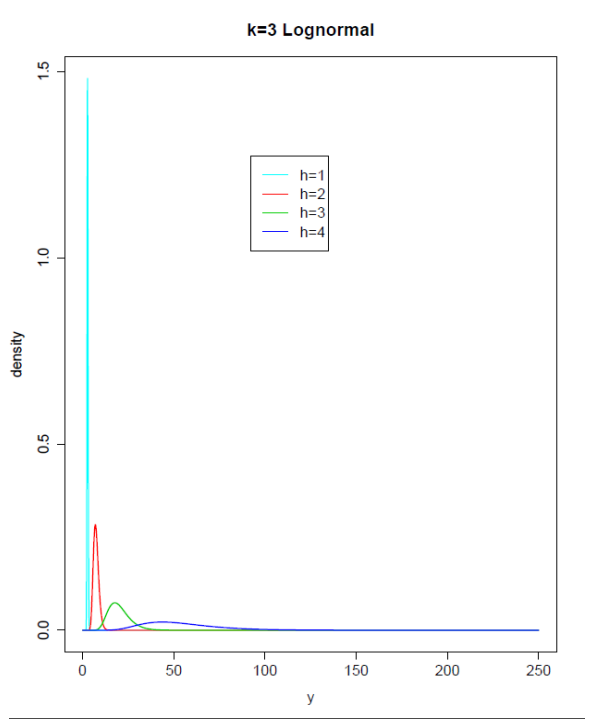
Numerical Investigation

Data Set k	Risk Profile h	Truncated loss distribution $\widetilde{Y}_k^{(h)}$	I parameter	II parameter	Mean	Variance	CV
k=1	h=1=MH	Gamma	v=16.00	η_1=0.17	2,72	0,68	25%
	h=2=FH	Gamma	v=16.00	η_2=0.46	7,39	1,85	25%
	h=3=ML	Gamma	v=16.00	η_3=1.25	20,09	5,02	25%
	h=4=FL	Gamma	v=16.00	η_4=3.41	54,60	13,65	25%
k=2	h=1=MH	Gamma	v_1=99.99	η_1=0.40	40,00	4,00	10%
	h=2=FH	Gamma	v_2=25.00	η_2=1.60	40,00	8,00	20%
	h=3=ML	Gamma	v_3=11.11	η_3=3.60	40,00	12,00	30%
	h=4=FL	Gamma	v_4=6.25	η_4=6.40	40,00	16,00	40%
k=3	h=1=MH	Lognormal	μ_1=0.99	σ_1=0.09	2,72	0,27	10%
	h=2=FH	Lognormal	μ_2=1.98	σ_2=0.20	7,39	1,48	20%
	h=3=ML	Lognormal	μ_3=2.96	σ_3=0.29	20,09	6,03	30%
	h=4=FL	Lognormal	μ_4=3.93	σ_4=0.38	54,60	21,84	40%

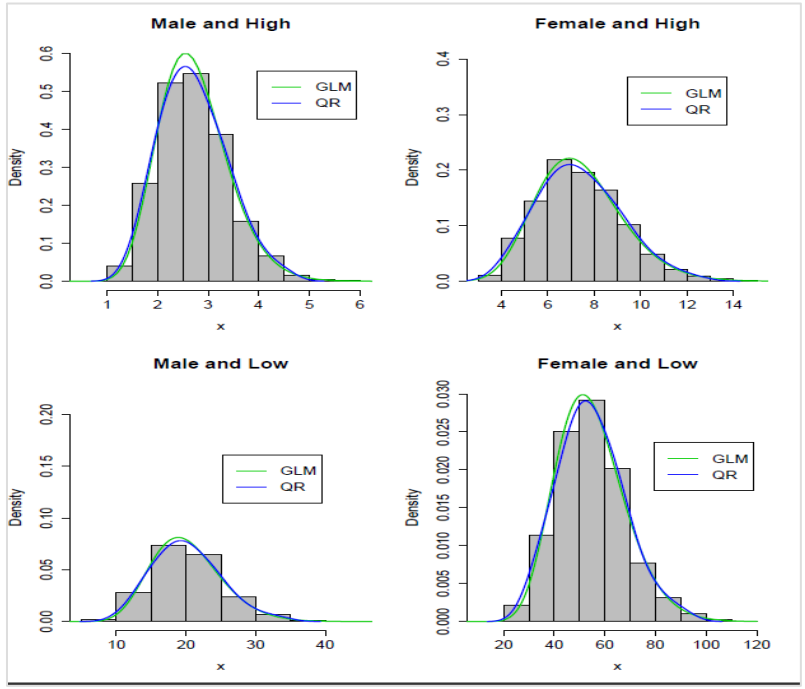
Theoretical Distributions

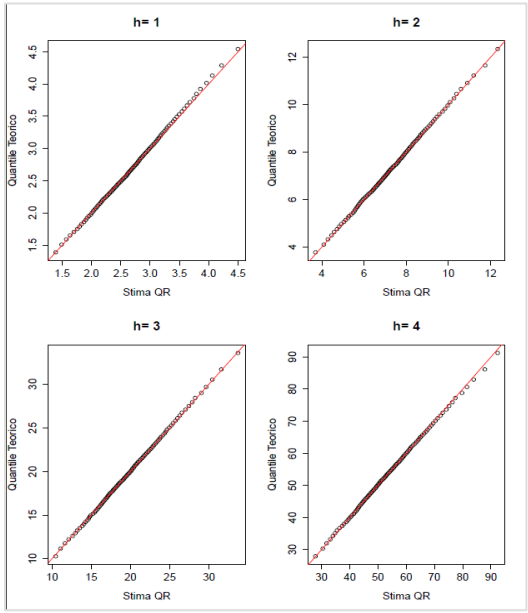


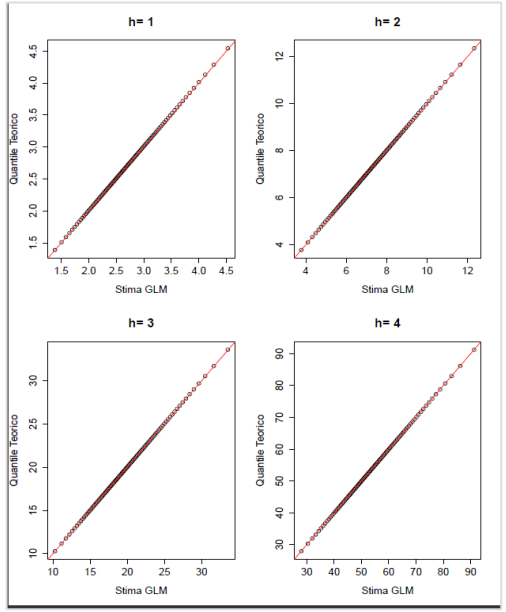




Distribution Fitting Analysis:
Gamma same CV - k=1

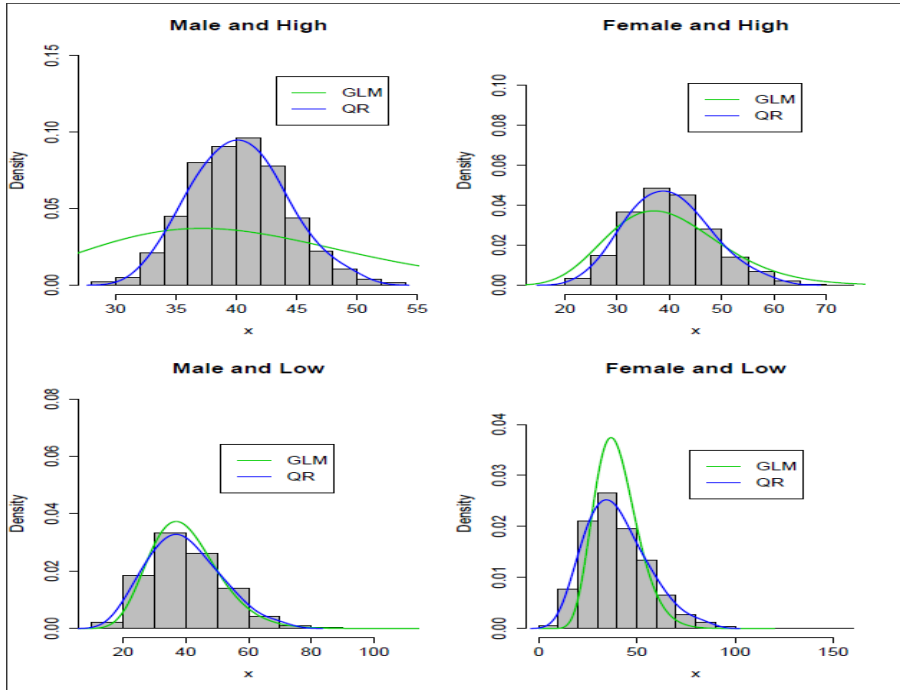


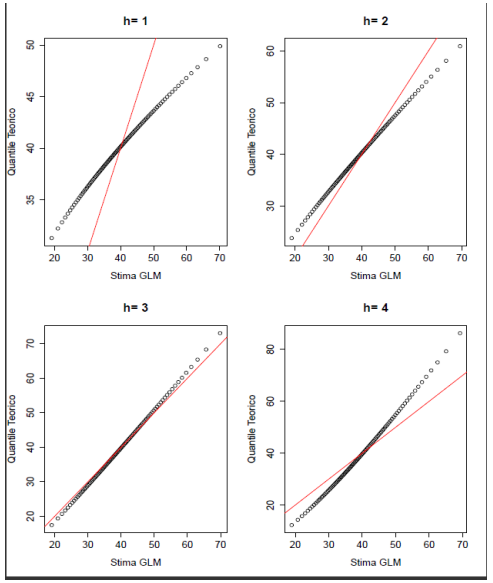


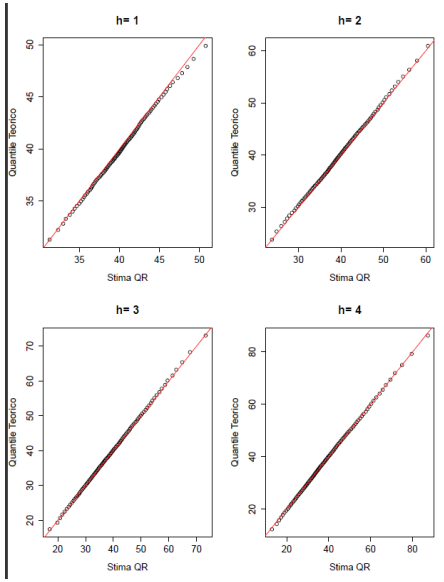


GoF measures	h=1 M-H		h=2 F-H		h=3 M-L		h=4 F-L	
	QR	GLM	QR	GLM	QR	GLM	QR	GLM
RMSE [0,+inf)	0,0176	0,0123	0,0363	0,0298	0,1174	0,0347	0,2332	0,1028
RSR[0,+inf)	0,0268	0,0188	0,0204	0,0167	0,0243	0,0072	0,0177	0,0078
NSE(-inf,1]	0,9993	0,9996	0,9996	0,9997	0,9994	0,9999	0,9997	0,9999

Distribution Fitting Analysis:
Gamma same Mean - k=2

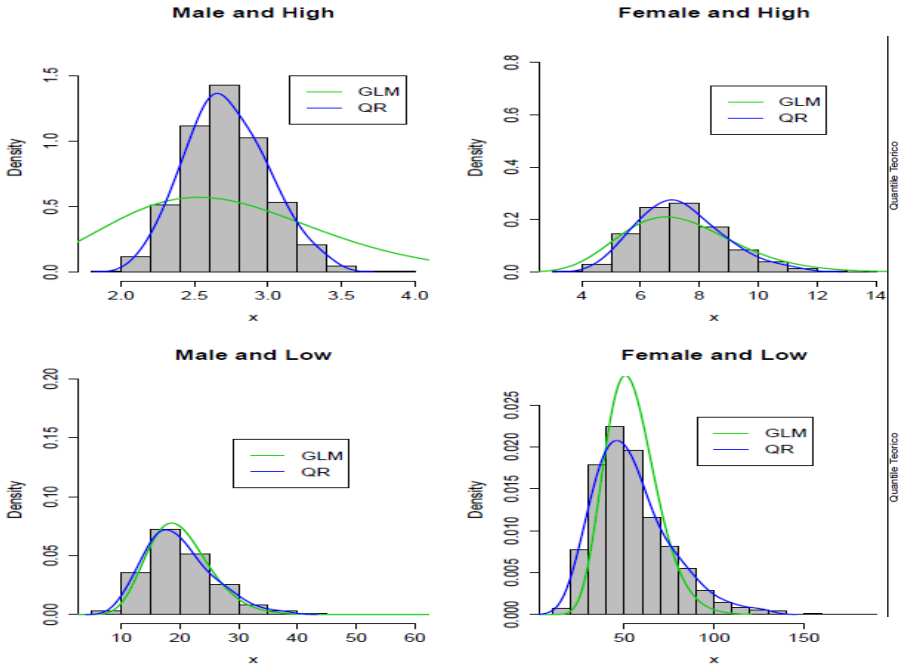


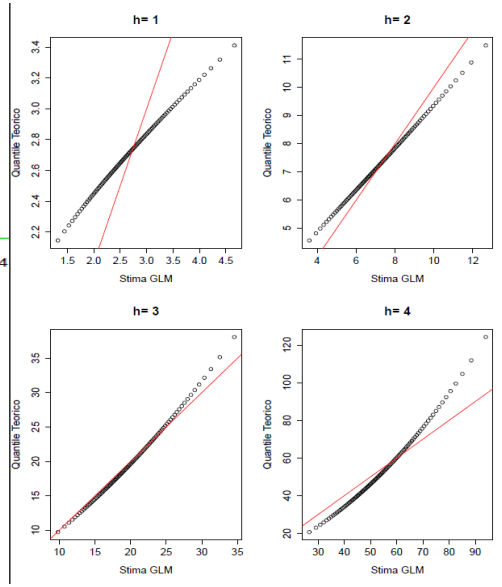


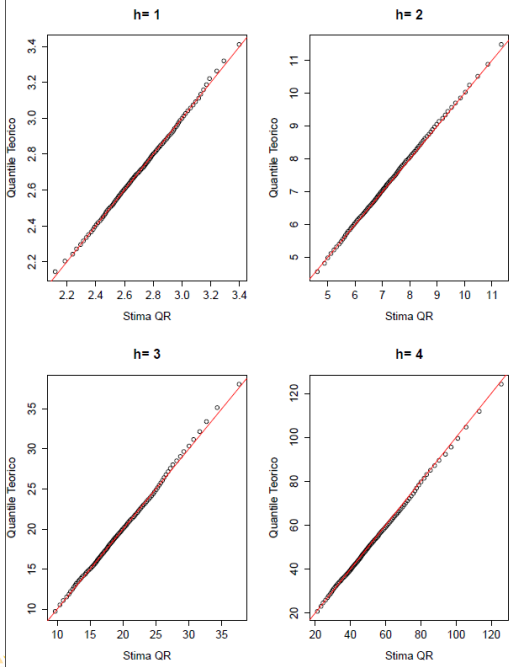


GoF measures	h=1 M-H		h=2 F-H		h=3 M-L		h=4 F-L	
	QR	GLM	QR	GLM	QR	GLM	QR	GLM
RMSE [0,+inf)	0,1949	6,6120	0,4663	2,6936	0,5798	1,1764	0,5205	5,1028
RSR[0,+inf)	0,0505	1,7134	0,0605	0,3494	0,0503	0,1020	0,0339	0,3327
NSE(-inf,1]	0,9974	- 1,9656	0,9963	0,8766	0,9974	0,9895	0,9988	0,8882

Distribution Fitting Analysis:
Lognmormal - k=3

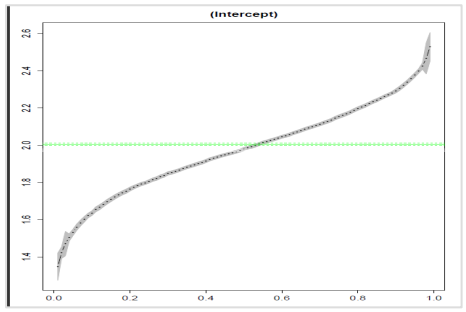


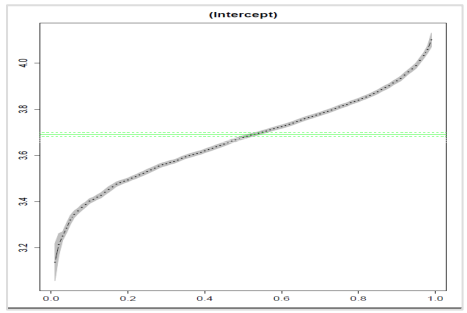


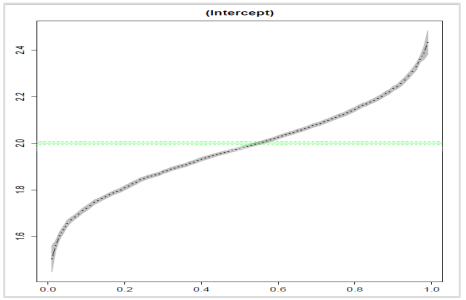


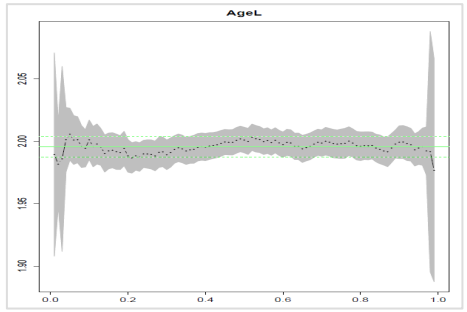
GoF measures	h=1 M-H		h=2 F-H		h=3 M-L		h=4 F-L	
	QR	GLM	QR	GLM	QR	GLM	QR	GLM
RMSE [0,+inf)	0,0079	0,4406	0,0584	0,4982	0,1881	0,6312	1,4305	6,6414
RSR[0,+inf)	0,0302	1,6818	0,0412	0,3515	0,0328	0,1100	0,0695	0,3229
NSE(-inf,1]	0,9991	- 1,8574	0,9983	0,8752	0,9989	0,9878	0,9951	0,8947

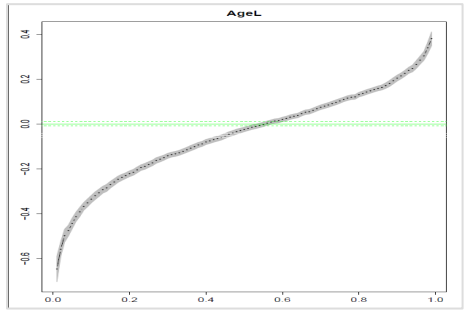
Estimated Parameter: QR vs GLMs

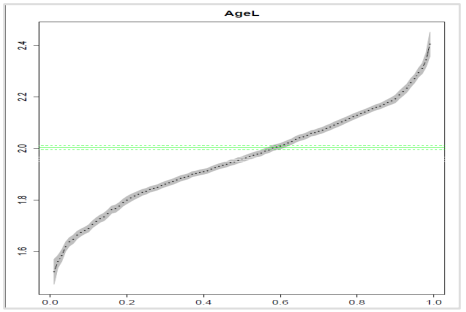


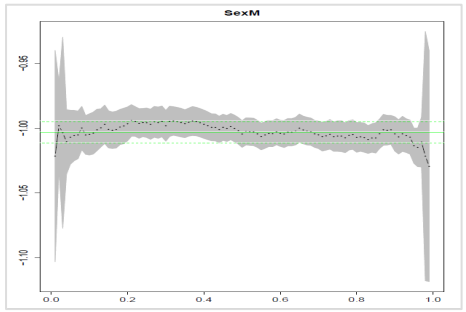


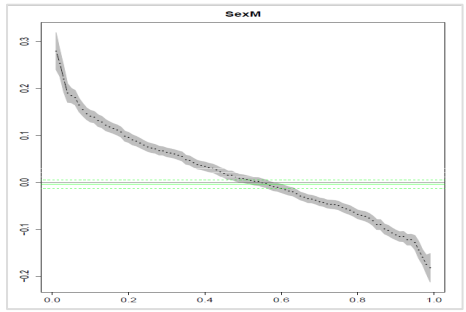


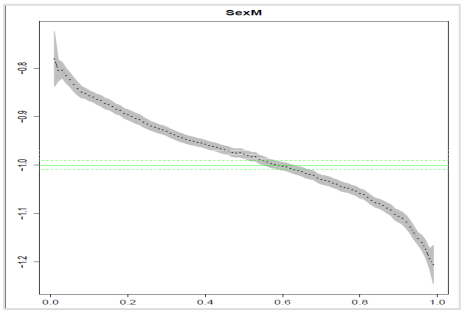












Premium Estimates

k=1 Gamma same CV

Risk Profile h	$E(\tilde{Y}^{(h)})$		$Q_{0.85}(\tilde{Y}^{(h)})$			Diff %	
	Theoretical	GLM	Theoretical	GLM	QR	GLM	QR
h=1=MH	2,72	2,72	3,42	3,42	3,46	0,04%	-1,26%
h=2=FH	7,39	7,36	9,30	9,26	9,29	0,39%	0,03%
h=3=ML	20,09	20,03	25,27	25,20	25,41	0,28%	-0,57%
h=4=FL	54,60	54,27	68,68	68,26	68,18	0,63%	0,73%

Risk Profile h	Number of Exposure	$E(Y^{(h)})$	P_Q		ρ_{θ_h}		θ_h
		GLM	GLM	QR	GLM	QR	
h=1=MH	7.500	0,96	1,21	1,23	125,77%	127,43%	94,68%
h=2=FH	22.500	0,77	0,96	0,97	125,77%	126,21%	98,44%
h=3=ML	12.500	3,75	4,75	4,75	125,77%	126,85%	97,19%
h=4=FL	57.500	2,51	3,16	3,16	125,77%	125,64%	99,31%

k=2 Gamma same Mean

Risk Profile h	$E(\tilde{Y}^{(h)})$		$Q_{0.85}(\tilde{Y}^{(h)})$			Diff %	
	Theoretical	GLM	Theoretical	GLM	QR	GLM	QR
h=1=MH	40,00	39,94	44,15	51,17	44,19	-13,73%	-0,08%
h=2=FH	40,00	40,01	48,28	51,26	48,52	-5,82%	-0,51%
h=3=ML	40,00	40,03	52,34	51,29	52,49	2,06%	-0,27%
h=4=FL	40,00	40,10	56,31	51,38	57,64	9,60%	-2,30%

Risk Profile h	Number of Exposure	$E(Y^{(h)})$	P_Q		ρ_{θ_h}		θ_h
		GLM	GLM	QR	GLM	QR	
h=1=MH	7.500	14,17	18,16	15,68	128,13%	110,64%	94,68%
h=2=FH	22.500	4,16	5,33	5,04	128,13%	121,29%	98,44%
h=3=ML	12.500	7,49	9,59	9,82	128,13%	131,13%	97,19%
h=4=FL	57.500	1,86	2,38	2,67	128,13%	143,75%	99,31%

k=3 LogN

Risk Profile h	$E(\tilde{Y}^{(h)})$		$Q_{0.85}(\tilde{Y}^{(h)})$			Diff %	
	Theoretical	GLM	Theoretical	GLM	QR	GLM	QR
h=1=MH	2,72	2,71	3,00	3,44	3,01	-12,70%	-0,26%
h=2=FH	7,39	7,44	8,90	9,44	8,87	-5,77%	0,35%
h=3=ML	20,09	20,24	26,08	25,68	25,39	1,55%	2,74%
h=4=FL	54,60	55,61	75,57	70,57	74,84	7,09%	0,98%

Risk Profile h	Number of Exposure	$E(Y^{(h)})$	P_Q		ρ_{θ_h}		θ_h
		GLM	GLM	QR	GLM	QR	
h=1=MH	7.500	0,96	1,22	1,07	126,90%	111,07%	94,68%
h=2=FH	22.500	0,77	0,98	0,92	126,90%	119,17%	98,44%
h=3=ML	12.500	3,79	4,75	4,75	126,90%	125,44%	97,19%
h=4=FL	57.500	2,57	3,27	3,46	126,90%	134,58%	99,31%

Conclusions and Further research

- Conclusions
  - The well known drawbacks GLMs are confirmed (if necessary) by our analysis.
  - The most widely adopted method used in MTPL ratemaking based on a two-part model (frequency-severity) for net premium involves a traditional expected value principle with a limited view on the variability on the risk. However it is easy to compute and to be understood.
  - The solution based on a QR on the severity is more appropriate to catch the individual variability. However, the product of the mean of frequency and the quantile of the severity is not a canonical statistical measure.
  - The adoption of a separate QR model for frequency and severity does not solve this issue.
- Further research
  - Estensione ai collective models
  - Estensione a famiglia esponenziale per severity Inverse Gaussian e Tweedie
  - Quantile su variabili discrete
  - ALD

Thank you!