Pricing & Hedging Guaranteed Minimum Withdrawal Benefits under a General Lévy Framework using the COS Method

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Overview

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What are Variable Annuities?

First of all, what is an annuity?

- Policyholders pay a lump sum (usually at retirement),
- and get a regular income until their death.
- Typically, it is either fixed or increases with inflation.

Then, what is a variable annuity?

- Similar to a classical annuity,
- but payments are **linked** to an investment fund.

Variable Annuities

Variable Annuities (VAs) were first introduced in the early 1950s and various 'GMxBs' have become available since [Ledlie et al., 2008]:

- Guaranteed Minimum Death Benefit introduced in 1980s.
- Guaranteed Minimum Living Benefits introduced in late 1990s.
    - GMAB - Accumulation
    - GMIB - Income
    - GMWB - Withdrawal
    - (GLWB - Lifelong form of GMWB)

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Pricing

- General pricing concept: finding the actuarially fair premium that corresponds to certain contingent claims.
- Pricing in the VA context: find the **regular** fair fee, as a percentage of the underlying fund, that covers the guarantees
  $\Rightarrow$ It is a fee that is paid **while** the retirement income is paid (not before or at retirement - lump sum-).

Pricing Guaranteed Minimum Withdrawal Benefits

- A challenging task due to the product's complexity.
  - Long-term stream of cash flows.
    - Withdrawals received by the policyholder
    - Fees received by the insurer
  - Sensitive to policyholder behaviour.
- Early pricing methods based on finite difference techniques.
  - Bifurcate into a Quanto Asian Put and term-certain annuity [Milevsky and Salisbury, 2006]
  - Singular stochastic control problem [Dai et al., 2008]
  - Impulse stochastic control problem [Chen and Forsyth, 2008]

Pricing Guaranteed Minimum Withdrawal Benefits

- More complicated asset return models are considered:
  - First paper to consider jumps [Chen et al., 2008].
  - Stochastic interest rates model [Peng et al., 2012].
  - Lévy processes considered [Bacinello et al., 2014].
  - Regime-switching model [Ignatieva et al., 2016].
- Efficient valuation frameworks:
  - Luo and Shevchenko [2014] use 'GHQC' in GBM framework.

Hedging

- Reducing the risk of adverse price movements by purchasing stocks and options.
- *Example*: Purchasing a put option to protect yourself from downside trends of an asset you hold.

Hedging Guaranteed Minimum Benefits

- Milevsky and Salisbury [2006] outline how GMWBs can be delta-hedged.
- Coleman et al. [2007] use risk-minimisation methods to hedge GMDBs in a GBM framework.
  - Kélani and Quittard-Pinon [2015] extend to Lévy framework.
- Kolkiewicz and Liu [2012] use risk-minimisation methods to hedge GMWB:
  $\rightarrow$ Monte Carlo simulation, static behaviour case.
- Spike in interest:
  - Bernard and Kwak [2016]; Carr et al. [2016]; Feng et al. [2016]; Goudenege et al. [2016]; Ignatieva et al. [2016]

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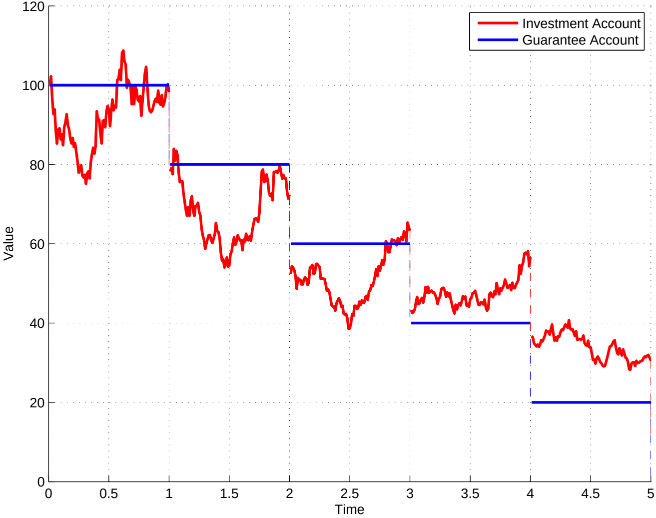
Guaranteed Minimum Withdrawal Benefits

Interest in GMWBs:

- VA industry is large and still expanding:
  - US$1.35 trillion in the U.S. as of 2008 [Condron, 2008].
  - US$1.98 trillion as of 2015 [IRI, 2015].
- GMWBs are embedded in a large portion of these VAs:
  - 40% of VAs sold had a GMWB attached in the first half of 2007 [Ledlie et al., 2008].
- Insurers take on large down-side market risks.
- The recent financial crisis caused huge losses in VA portfolios.

Account Dynamics



Guaranteed Minimum Withdrawal Benefit

Variable annuities with an embedded GMWB provide:

- A guaranteed stream of income.
- Exposure to equity market with downside protection.
- Longevity protection.
- Flexibility.

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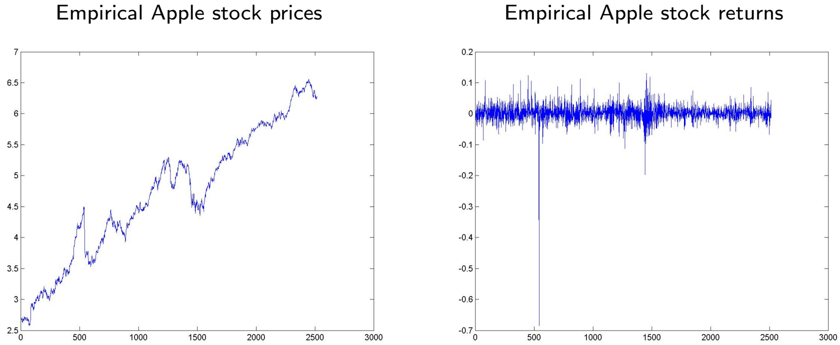
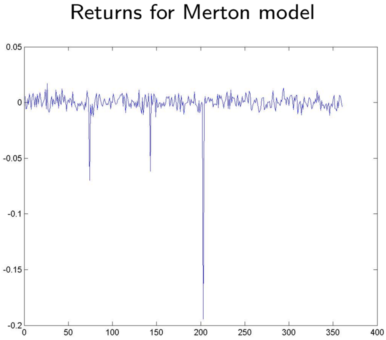
Lévy Processes - Empirical Evidence

Figure: Empirical asset returns (Apple shares) *Source:* Monteiro [2013].



Lévy Processes - Empirical Evidence

Figure: Simulated asset returns (Apple shares) *Source:* Monteiro [2013].



Lévy Processes

Interest in Lévy processes:

- Good match to empirical asset return distributions.
  - Heavy tails.
  - Can fit skewness and kurtosis.
- Able to incorporate jump and diffusion terms.
- Overall they are very flexible.
- Relatively tractable - ties in with COS method.
  - Characteristic function known in closed form.

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The COS Method

Interest in the COS method:

- An efficient numerical method.
- Has been used to price:
  - European-style options [Fang and Oosterlee, 2008].
  - Early-exercise options [Fang and Oosterlee, 2009].
  - Rainbow options [Ruijter and Oosterlee, 2012].
- Has been used in optimal control theory [Ruijter et al., 2013].
- Appears in the actuarial literature.
  $\rightarrow$ [Chau et al., 2015a,b; Deng et al., 2014]

Research Aims

Pricing the GMWB contract

- Use realistic account dynamics.
- Allow for different policyholder behaviours.
- Devise an efficient valuation framework using the COS method.

Hedging the GMWB liabilities

- Construct a hedging framework using the COS method.
- Compare different local risk-minimisation hedging strategies.

Overview

Asset dynamics

- Denote S_t the underlying asset price at time t.

$$S_t = S_s \cdot e^{L_{t-s}},$$

where L_{t-s} has Lévy triplet (μ, σ^2, ν).

- Lévy triplet under risk-neutral dynamics is

$$\left(r - \frac{\sigma^2}{2} - \int_{\mathbb{R}} (e^x - 1 - x)\nu(dx),\ \sigma^2,\ \nu \right),$$

- Characteristic function of the return known in closed-form through the Lévy-Khintchine formula:

$$\phi(u) = \exp\left[i\mu u - \frac{u^2\sigma^2}{2} + \int_{\mathbb{R}} (e^{iux} - 1 - iux\mathbf{1}_{\{|x|<1\}})\nu(dx) \right].$$

[Papapantoleon, 2008]

Valuation

- There are penalties for withdrawing over the guaranteed rate.

$$C(\gamma_{t_m}) = \begin{cases} \gamma_{t_m} & \text{if } 0 \le \gamma_{t_m} \le G, \\ G + (1-\kappa)\cdot(\gamma_{t_m} - G) & \text{if } \gamma_{t_m} > G. \end{cases}$$

- The VAs value is the expected present value of cash flows

$$V_{t_m}(W_{t_m^-}, A_{t_m^-}) = \sup_{\gamma} \Bigg[\mathbb{E}^{\mathbb{Q}} \Big[e^{-r(t_M - t_m)} \max[W_{t_M}, C(A_{t_M})] + \sum_{j=m}^{M-1} e^{-r(t_j - t_m)} C(\gamma_{t_j}) \Big| W_{t_m^-}, A_{t_m^-}, \gamma_{t_m} \Big] \Bigg],$$

- γ is restricted based on the assumed withdrawal behaviour.

Recursive Formula

- Express the valuation formula recursively:

$$V_{t_m}\left(W_{t_m^-}, A_{t_m^-}\right) = \sup_{\gamma}\left[\mathbb{E}^{\mathbb{Q}}\left[C(\gamma_{t_m}) + e^{-r(t_{m+1}-t_m)} V_{t_{m+1}}\left(W_{t_{m+1}^-}, A_{t_{m+1}^-}; \gamma_{t_m}\right) \Big| W_{t_m^-}, A_{t_m^-}, \gamma_{t_m}\right]\right]$$

$$= \sup_{\gamma}\left[C(\gamma_{t_m}) + e^{-r(t_{m+1}-t_m)} \underbrace{\mathbb{E}^{\mathbb{Q}}\left[V_{t_{m+1}}\left(W_{t_{m+1}^-}, A_{t_{m+1}^-}; \gamma_{t_m}\right) \Big| W_{t_m^-}, A_{t_m^-}, \gamma_{t_m}\right]}_{\zeta}\right]$$

- Use the COS method to approximate ζ, with terminal condition:

$$V_{t_M}(W_{t_M}, A_{t_M}) = \max\left[W_{t_M}, C\left(A_{t_M}\right)\right].$$

Re-expressing the Expectation Term

Change variable and truncate the integral form of the Q-expectation:

$$\zeta = \int_{-\infty}^{\infty} V_{t_{m+1}} \left(w_{t_{m+1}^-}, A_{t_{m+1}^-}; \gamma_{t_m} \right) g^{\mathbb{Q}}(w_{t_{m+1}^-} | W_{t_m^-}, \gamma_{t_m}) dw_{t_{m+1}^-}$$

$$\zeta \approx \zeta_1 = \int_{a}^{b} V_{t_{m+1}} \left(\max \left[W_{t_m^+},\ 0 \right] \cdot e^y, A_{t_m^+}; \gamma_{t_m} \right) f^{\mathbb{Q}}(y) dy,$$

where

$$y = \ln \left(\frac{S_{t_{m+1}}}{S_{t_m}} \right)$$ and a and b are based on the cumulants.

Use COS method approximation for the density function and rearrange terms:

$$\zeta \approx \zeta_2 = \sum_{k=0}^{N-1}{}' \mathrm{Re} \left\{ \phi^{\mathbb{Q}} \left(\frac{k\pi}{b-a} \right) \cdot \exp \left(-i \frac{ka\pi}{b-a} \right) \right\} \cdot U_k \left(W_{t_m^+}, A_{t_m^+} \right).$$

Overview

Local risk-minimisation

Hedging strategies that minimize a certain risk measure under real-world probabilities.

- Minimise some uncertainty, $H(\cdot)$ with risk measure, $\rho(\cdot)$:

$$H^{\rho}_{t_{m+1}}\left(S_{t_{m+1}}\middle|S_{t_m}, V_{t_m}\left(W_{t_m^+}\right), \vec{\theta}\right) = H_{t_{m+1}}\left(S_{t_{m+1}}\middle|S_{t_m}, V_{t_m}\left(W_{t_m^+}\right)\right) - \sum_{j=1}^{n} \theta_j \cdot F\left(S_{t_{m+1}}, K_j\right).$$

- Using a portfolio of European options.
- Trade options such that

$$\vec{\theta}^{\rho} = \inf_{\vec{\theta}}\left[\rho\left(H^{\rho}_{t_{m+1}}\left(S_{t_{m+1}}\middle|V_{t_m}\left(W_{t_m^+}\right), S_{t_m}, \vec{\theta}\right)\right)\right],$$

Moments-based Risk Measures

- An application of the COS method:

$$E[H^\rho(Y)^n] = \int_{-\infty}^{\infty} H^\rho(y|\vec{\theta})^n \cdot f(y)dy$$

$$\approx \sum_{k=0}^{N-1}{}' \mathrm{Re}\left\{\phi\left(\frac{k\pi}{b-a}\right) e^{\frac{-ika\pi}{b-a}}\right\} \cdot U_k\left(W_{t_m^+}, A_{t_m^+}|\vec{\theta}\right),$$

where

$$U_k\left(W_{t_m^+}, A_{t_m^+}|\vec{\theta}\right) = \frac{2}{b-a}\int_a^b \left(H^\rho_{t_{m+1}}(y|\vec{\theta})\right)^n \cdot \cos\left(k\pi\frac{y-a}{b-a}\right) dy.$$

- Option payoffs/values known in closed form.
- GMWB payoffs/value available from the valuation framework.

Quantile-based Risk Measures

- $W_{t_m^+}$ accumulates until the next period into a discrete number of $W_{t_{m+1}^-}$ considered.
- Each corresponds to a different realised one-period asset return, y.
- $f(y)$ is estimated using the COS method.
- Hedging loss densities are proportional to the return distribution densities, such that:

$$f^h(H(S_{t_{m+1}}|y^{(j)})) \propto \sum_{\text{all } i} f(y^{(i)}) \cdot \mathbf{1}_{\{H(S_{t_{m+1}}|y^{(i)})=H(S_{t_{m+1}}|y^{(j)})\}}.$$

- Hedging loss quantiles are found using the approximated hedging loss density function.

Overview

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Numerical Comparisons

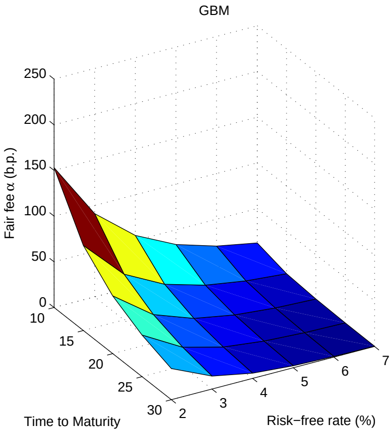
Table: Comparison to Luo and Shevchenko (2014) fair fees with varying T - GBM, quarterly withdrawals, continuous fees

T	10	12.5	20	25
COS	95.87	67.05	28.23	17.49
GHQC	95.81	66.99	28.33	17.59
FD	95.78	66.93	28.30	17.79

- Fees in the dynamic case have a comparable accuracy.

Sensitivity

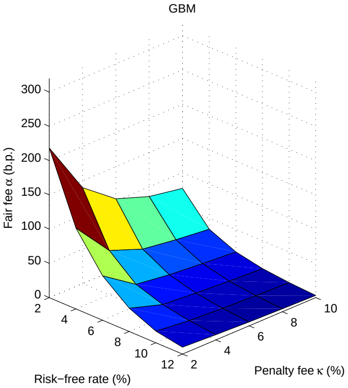
Figure: Comparison of fair fees for varying T and r



- Annual withdrawals and discretely charged fees.

Sensitivity

Figure: Comparison of fair fees for varying r and κ with dynamic withdrawals



- Twenty annual withdrawals and continuously charged fees.

Convergence in COS Parameters: GBM

Table: Comparison to Bacinello et al. [2014], with average time of five trials (sec.)

J \ N	16	32	64	128
25	125.38 (0.014)	125.38 (0.015)	125.38 (0.016)	125.38 (0.019)
50	99.92 (0.027)	99.92 (0.029)	99.92 (0.035)	99.92 (0.044)
250	100.00 (0.181)	100.00 (0.294)	100.00 (0.317)	100.00 (0.449)
1000	100.00 (1.560)	100.00 (1.951)	100.00 (3.762)	100.00 (5.768)
Bac.	110.73 (0.491)	102.3 (1.023)	100.49 (2.057)	100.06 (4.030)

- GBM asset dynamics with twenty annual withdrawals and discretely charged fees.

Convergence in COS Parameters: VG

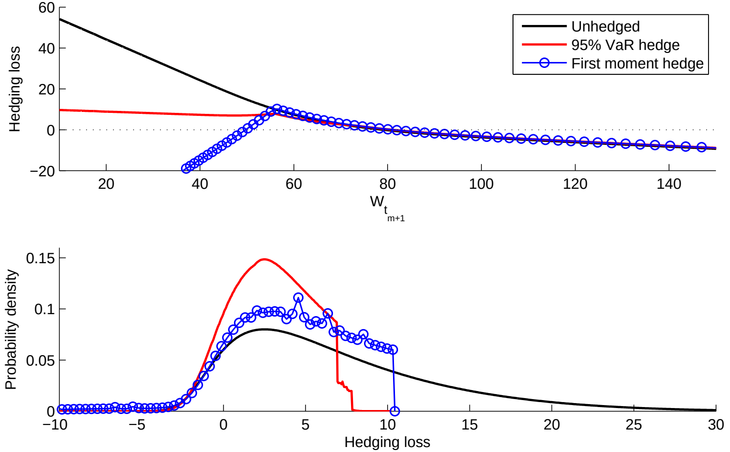
Table: Comparison to Bacinello et al. [2014], with average time of five trials (sec.)

J \ N	16	32	64	128
25	83.35 (0.013)	62.01 (0.014)	61.88 (0.017)	61.88 (0.020)
50	99.51 (0.027)	99.88 (0.032)	99.28 (0.039)	99.28 (0.053)
250	100.05 (0.204)	100.02 (0.285)	100.01 (0.444)	100.01 (0.815)
1000	100.04 (1.965)	100.01 (4.716)	100.01 (6.242)	100.01 (9.408)
Bac.	110.51 (0.496)	102.39 (1.002)	100.5 (1.977)	100.07 (3.960)

- VG asset dynamics with twenty annual withdrawals and discretely charged fees.

Static case: VaR vs First Moment risk-minimisation

Figure: Example of hedging the 95% VaR and first moment of the hedging error with VG asset dynamics.



- Hedging loss = $\Delta Liab. - \Delta Assets$, no shorting allowed, VG asset dynamics, budget

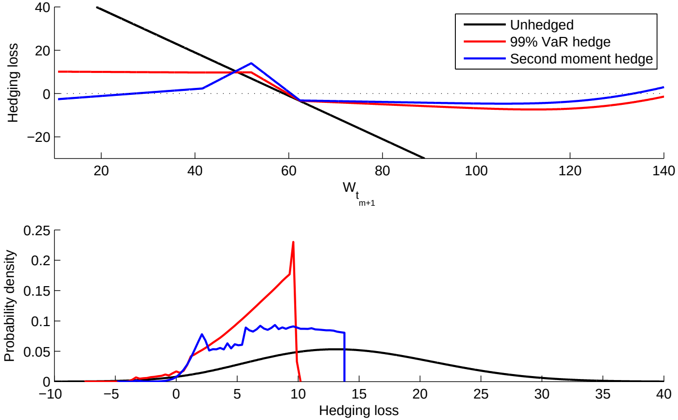
Static case: VaR vs First Moment risk-minimisation

Table: Comparison of the effectiveness of hedging the net liabilityfor the static case under VG asset dynamics, annual withdrawals and no short-selling.

Hedge	Cost	1^st mom.	Var	95% VaR	99.5% VaR	90% TVaR
Unhedged	-	5.29	36.09	19.25	28.80	20.41
95% VaR	0.45	3.56	7.68	6.59	7.51	6.65
First moment	0.45	2.79	23.71	9.63	10.36	9.62

Dynamic case: VaR vs Second Moment risk-minimisation

Figure: Comparison of the effectiveness of hedging the net liabilityfor the dynamic case under VG asset dynamics and annual withdrawals



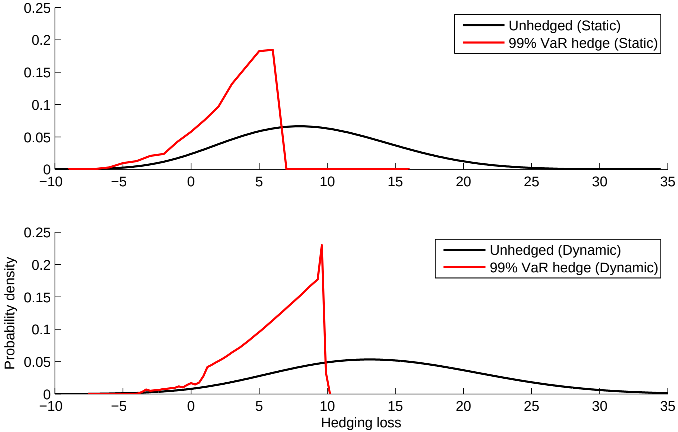
Dynamic case: VaR vs Second Moment risk-minimisation

Table: Comparison of the effectiveness of hedging the net liabilityfor the dynamic case under VG asset dynamics and annual withdrawals

Hedge	Cost	1^{st} mom.	Var	95% VaR	99.5% VaR	90% TVaR
Unhedged	-	15.429	57.924	26.714	33.882	27.546
99% VaR	0.136	9.204	3.405	9.576	9.793	9.559
2^{nd} moment	0.260	5.375	17.957	13.295	13.896	13.306

Static versus Dynamic policyholder behavior

Figure: Comparison of the effectiveness of hedging the net liability: static (top), dynamic (bottom)for VG asset dynamics and annual withdrawals



Summary of hedging results

- Confirm the suitability of a local risk minimisation strategy for GMWB.
- This strategy outperforms Greeks gain (however, gain is less big for quarterly withdrawal).
- Examples in the paper suggest that 95% and second moment are comparable for the static case (and the first moment is the least performing).
- However, for the dynamic case ($\uparrow$ variability) choice should not only be based on cost and moments of the losses but also higher quantiles.

Overview

Contributions

An efficient GMWB valuation framework:

- Extended the COS Method to the valuation of GMWBs.
- Can incorporate:
  - both static and dynamic policyholder behaviour;
  - the reset provision;
  - continuous or discretely charged fees;
- A flexible hedging framework;
  - can incorporate short-selling and budgeting constraints;
  - possible to hedge the insurer's fee received as well as the change in net liabilities.

Next steps:

- Incorporate the effect of tax benefits on policyholder behaviour.
- Extend to incorporate stochastic volatility.
- Extend analysis of local risk-minimisation to the hedging loss across multiple periods.

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Thanks

Thanks for your attention

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