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OPTIMAL QUADRATIC HEDGING
WITH INSURANCE LINKED SECURITIES

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THE MODEL AND THE INVESTMENT PROBLEM

The financial market model

$(\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$ satisfying usual conditions, \mathcal{F}_0 trivial.

Financial market observed over finite time period $[0, T]$:

$m + 1$ tradeable assets with prices $\mathbf{S}_t = (S_t^0, \dots, S_t^m)'$ at time t .

$\mathbf{S} = (\mathbf{S}_t)_{t \in [0, T]}$ is adapted and RCLL.

Asset No. 0 is cash account, $S_t^0 = e^{\int_0^t r_u du}$,

$$dS_t^0 = S_t^0 r_t dt, \quad S_0^0 = 1 \quad (1)$$

Remaining assets are risky.

Investment strategies

Portfolio is adapted process $\theta = (\theta_t)_{t \in [0, T]}$, $\theta_t = (\theta_t^0, \dots, \theta_t^m)'$,
 θ_t^i is number of units held in asset No. i at time t .

Value of θ at time t :

$$U_t = \theta_t' S_t = \sum_{i=0}^m \theta_t^i S_t^i \quad (2)$$

The portfolio is financed by adapted RCLL *Cost process* $C = (C_t)_{t \in [0, T]}$:
 C_t is total amount invested in $[0, t]$. C need not be increasing.

Postulate based on discrete time consideration: Dynamics of U is

$$dU_t = \theta_t' dS_t + dC_t \quad (3)$$

cash position θ^0 is adapted

exposure process $\vartheta = (\theta^1, \dots, \theta^m)'$ is predictable

(3) integrates to (use $U_0 = C_0$)

$$U_t = \int_0^t \boldsymbol{\theta}'_{\tau} d\mathbf{S}_{\tau} + C_t \quad (4)$$

(2) and (4) show that the *investment strategy* $(C, \boldsymbol{\vartheta})$ corresponds 1-to-1 with portfolio $\boldsymbol{\theta}$ for given \mathbf{S} :

Strategy $(C, \boldsymbol{\vartheta})$ determines $U = (U_t)_{t \in [0, T]}$.

Discounted values

Measure any capital value at time t in units of current value of a unit of the cash account.

Discounted asset prices and values are given by

$$\tilde{S}_t^i = \frac{S_t^i}{S_t^0} \quad \tilde{U}_t^i = \frac{U_t}{S_t^0} \quad d\tilde{C}_t = \frac{dC_t}{S_t^0}$$

$\tilde{S}_t^0 \equiv 1$ is particularly simple. Assemble discounted prices of the risky assets in

$$\tilde{\mathbf{S}}_t = (\tilde{S}_t^1, \dots, \tilde{S}_t^m)' \quad (5)$$

Rewrite $S_t^i = S_t^0 \tilde{S}_t^i$ and use $dS_t^0 = S_t^0 r_t dt$ to obtain

$$dS_t^i = S_t^0 r_t dt \tilde{S}_t^i + S_t^0 d\tilde{S}_t^i$$

$$dU_t = S_t^0 r_t dt \tilde{U}_t + S_t^0 d\tilde{U}_t$$

$$dC_t = S_t^0 d\tilde{C}_t$$

Insert these into $dU_t = \boldsymbol{\theta}'_t d\mathbf{S}_t + dC_t$ to obtain

$$d\tilde{U}_t = \boldsymbol{\vartheta}'_t d\tilde{\mathbf{S}}_t + d\tilde{C}_t \quad (6)$$

$$\tilde{U}_0 = \tilde{C}_0 \quad (7)$$

Integral form

$$\tilde{U}_t = \int_0^t \boldsymbol{\vartheta}'_\tau d\tilde{\mathbf{S}}_\tau + \tilde{C}_t \quad (8)$$

Strategy (C, ϑ) is *costless* if $C_t \equiv 0$.

Costless (C, θ) is an *arbitrage* if $\tilde{U}_T \geq 0$ and $\mathbb{P}[\tilde{U}_T > 0] > 0$.

Absence of arbitrage is equivalent to existence of *equivalent local martingale measure* (ELMM) $\tilde{\mathbb{P}}$.

Contingent claims, pricing, and hedging

Insurance policy issued at time 0 and terminating at time T generates a stream of contractual payments:

contributions (premiums) paid by the insured to the insurer

benefits (claims) paid by the insurer to the insured.

B_t is total of benefits less contributions in $[0, t]$.

$B = (B_t)_{t \in [0, T]}$ is adapted and RCLL.

(C, θ) is a *hedge* of B if

$$d\tilde{U}_t = \vartheta'_t d\tilde{\mathbf{S}}_t + d\tilde{C}_t - d\tilde{B}_t \quad (9)$$

$$\tilde{U}_T = 0 \quad (10)$$

The integral form of (9) is

$$\tilde{U}_t = \int_0^t \vartheta'_\tau d\tilde{\mathbf{S}}_\tau + \tilde{C}_t - \tilde{B}_t \quad (11)$$

hence the balance relation (10) means

$$\tilde{C}_T = \tilde{B}_T - \int_0^T \vartheta'_\tau d\tilde{\mathbf{S}}_\tau \quad (12)$$

One-off strategies and attainable claims

Investment strategy (C, θ) is *one-off* (“*self-financing*”) if $C_t = C_0$, $t \in [0, T]$.

A stream B of contractual payments is *attainable* if it admits a one-off hedge. In this case C_0 is the *price* of B at time 0. This price is unique.

If B is not attainable, one needs to formulate an objective in order to select an optimal investment strategy.

QUADRATIC HEDGING

Standing assumption. There exists equivalent martingale measure $\tilde{\mathbb{P}}$ such that discounted asset prices are martingales, sufficiently integrable to justify all operations seen in the following.

The reserve. Insurance regulation requires the company to provide a reserve that covers outstanding net liabilities in respect of the contract. The simplest notion of reserve is the *net reserve*, the expected present value of future benefits less contributions. Taking the expectation under the measure $\tilde{\mathbb{P}}$ conforms with current regulatory standards, which prescribes that assets and liabilities be booked at “market consistent embedded value”.

Then the discounted net reserve is

$$\tilde{V}_t = \tilde{\mathbb{E}}[\tilde{B}_T - \tilde{B}_t | \mathcal{F}_t] = \frac{1}{S_t^0} \tilde{\mathbb{E}} \left[\int_t^T e^{-\int_t^\tau r_u du} dB_\tau \middle| \mathcal{F}_t \right]$$

Martingale associated with the total contractual payments is

$\tilde{M}^B = (\tilde{M}_t^B)_{t \in [0, T]}$ given by

$$\tilde{M}_t^B = \tilde{\mathbb{E}}[\tilde{B}_T | \mathcal{F}_t] = \tilde{B}_t + \tilde{V}_t \quad (13)$$

$$\tilde{M}_T^B = \tilde{B}_T \quad (14)$$

$$\tilde{M}_0^B = \tilde{\mathbb{E}}[\tilde{B}_T]$$

Minimizing the hedging error: a mean-variance criterion

Cost function constrained to

$$\tilde{C}_t = \tilde{C}_0 1_{[0,\infty)}(t) + \Delta\tilde{C}_T 1_{[T,\infty)}(t) \quad (15)$$

$\tilde{C}_0 = \Delta\tilde{C}_0 \in \mathcal{F}_0$, $\Delta\tilde{C}_T \in \mathcal{F}_T$ (the hedge is one-off up to term of contract and hedging error – shortfall or surplus – is settled at term).
By (12),

$$\Delta\tilde{C}_T = \tilde{B}_T - \tilde{C}_0 - \int_0^T \vartheta'_\tau d\tilde{\mathbf{S}}_\tau \quad (16)$$

Choose \tilde{C}_0 and ϑ so as to minimize

$$Q(\tilde{C}_0, \vartheta) = \tilde{\mathbb{E}}(\Delta\tilde{C}_T)^2 = \tilde{\mathbb{E}} \left(\tilde{B}_T - \tilde{C}_0 - \int_0^T \vartheta'_\tau d\tilde{\mathbf{S}}_\tau \right)^2$$

Insert $\tilde{B}_T = \tilde{M}_T^B = \tilde{M}_0^B + \int_0^T d\tilde{M}_\tau^B$:

$$\begin{aligned}
Q(\tilde{C}_0, \boldsymbol{\vartheta}) &= \tilde{\mathbb{E}} \left(\tilde{M}_0^B - \tilde{C}_0 + \int_0^T (d\tilde{M}_\tau^B - \boldsymbol{\vartheta}'_\tau d\tilde{\mathbf{S}}_\tau) \right)^2 \\
&= (\tilde{M}_0^B - \tilde{C}_0)^2 + \tilde{\mathbb{E}} \left(\int_0^T (d\tilde{M}_\tau^B - \boldsymbol{\vartheta}'_\tau d\tilde{\mathbf{S}}_\tau) \right)^2 \quad (17)
\end{aligned}$$

First term in (17) minimized to 0 by

$$\tilde{C}_0^* = \tilde{M}_0^B = \tilde{B}_0 + \tilde{V}_0$$

Second term is

$$\begin{aligned}
Q &= \tilde{\mathbb{E}} \left[\int_0^T d\langle \tilde{M}^B - \boldsymbol{\vartheta}' \cdot \tilde{\mathbf{S}}, \tilde{M}^B - \boldsymbol{\vartheta}' \cdot \tilde{\mathbf{S}} \rangle_t \right] \\
&= \tilde{\mathbb{E}} \left[\int_0^T \left(d\langle \tilde{M}^B, \tilde{M}^B \rangle_t - 2 \boldsymbol{\vartheta}'_t d\langle \tilde{\mathbf{S}}, \tilde{M}^B \rangle_t + \boldsymbol{\vartheta}'_t d\langle \tilde{\mathbf{S}}, \tilde{\mathbf{S}}' \rangle_t \boldsymbol{\vartheta}_t \right) \right]
\end{aligned}$$

$$\begin{aligned}
d\langle \mathbf{X}, \mathbf{Y}' \rangle_t &= \text{Cov}(d\mathbf{X}_t, d\mathbf{Y}'_t | \mathcal{F}_{t-}) \\
&\text{if martingales} \\
&= \mathbb{E}(d\mathbf{X}_t d\mathbf{Y}'_t | \mathcal{F}_{t-})
\end{aligned}$$

If \mathbf{X} and \mathbf{Y} are martingales, then

$$\begin{aligned}
\mathbb{E}(\mathbf{X}_T \mathbf{Y}'_T) &= \mathbb{E}\left(\int_0^T d\mathbf{X}_s \int_0^T d\mathbf{Y}'_t\right) \\
&= \mathbb{E}\left(\int_0^T \mathbb{E}(d\mathbf{X}_t d\mathbf{Y}'_t | \mathcal{F}_{t-})\right) \\
&= \mathbb{E}\left(\int_0^T d\langle \mathbf{X}, \mathbf{Y}' \rangle_t\right)
\end{aligned}$$

$$d\langle \tilde{M}^B, \tilde{M}^B \rangle_t = \tilde{\sigma}_t^{BB} dt, \quad d\langle \tilde{\mathbf{S}}, \tilde{\mathbf{S}}' \rangle_t = \tilde{\Sigma}_t^{SS} dt, \quad d\langle \tilde{\mathbf{S}}, \tilde{M}^B \rangle_t = \tilde{\sigma}_t^{BS} dt$$

$$Q = \int_0^T \tilde{\mathbb{E}}[\tilde{\sigma}_t^{BB}] dt - \tilde{\mathbb{E}} \left[\int_0^T \left(2 \boldsymbol{\vartheta}_t' \tilde{\sigma}_t^{BS} - \boldsymbol{\vartheta}_t' \tilde{\Sigma}_t^{SS} \boldsymbol{\vartheta}_t \right) dt \right]$$

Minimizing Q means maximizing *MSE (mean square error) reduction*

$$R = \tilde{\mathbb{E}} \left[\int_0^T \left(2 \boldsymbol{\vartheta}_t' \tilde{\sigma}_t^{BS} - \boldsymbol{\vartheta}_t' \tilde{\Sigma}_t^{SS} \boldsymbol{\vartheta}_t \right) dt \right] \quad (18)$$

Solution depends on conditions placed on $\boldsymbol{\vartheta}$. Invariably, the problem reduces to

$$\max (2 \mathbf{x}' \mathbf{b} - \mathbf{x}' \mathbf{A} \mathbf{x}) = \mathbf{b}' \mathbf{A}^{-1} \mathbf{b} \quad \text{for } \mathbf{x} = \mathbf{A}^{-1} \mathbf{b}$$

Some special cases:

Unconstrained exposure process. If ϑ any F -predictable process, then R in (18) is maximized by maximizing the integrand point-wise for each t . Optimal exposure process

$$\vartheta_t^* = \left(\tilde{\Sigma}_t^{SS} \right)^{-1} \tilde{\sigma}_t^{BS}$$

and maximum of R is

$$R^* = \tilde{\mathbb{E}} \left[\int_0^T \tilde{\sigma}_t^{BS'} \left(\tilde{\Sigma}_t^{SS} \right)^{-1} \tilde{\sigma}_t^{BS} dt \right]$$

The final cost is

$$\Delta \tilde{C}_T^* = \tilde{B}_T - \tilde{C}_0^* - \int_0^T \vartheta_t^{*'} d\tilde{S}_t$$

Exposure process with measurability constraint. Let $G = (\mathcal{G}_t)_{t \in [0, T]}$ be a sub-filtration of F , and suppose ϑ is constrained to G -predictable processes.

Interpretation: the hedger has access only to partial information (asymmetry of information).

For any G -predictable ϑ (18) can be expressed as

$$R = \tilde{\mathbb{E}} \left[\int_0^T \left(2 \vartheta_t' \tilde{\mathbb{E}}[\tilde{\sigma}_t^{BS} | \mathcal{G}_{t-}] - \vartheta_t' \tilde{\mathbb{E}}[\tilde{\Sigma}_t^{SS} | \mathcal{G}_{t-}] \vartheta_t \right) dt \right]$$

Ignore the left-limits in the conditional expected values. Optimum is

$$\vartheta_t^G = \left(\tilde{\mathbb{E}}[\tilde{\Sigma}_t^{SS} | \mathcal{G}_t] \right)^{-1} \tilde{\mathbb{E}}[\tilde{\sigma}_t^{BS} | \mathcal{G}_t],$$

$$R^G = \tilde{\mathbb{E}} \left[\int_0^T \tilde{\mathbb{E}}[\tilde{\sigma}_t^{BS'} | \mathcal{G}_t] \left(\tilde{\mathbb{E}}[\tilde{\Sigma}_t^{SS} | \mathcal{G}_t] \right)^{-1} \tilde{\mathbb{E}}[\tilde{\sigma}_t^{BS} | \mathcal{G}_t] dt \right].$$

Special case: the hedger observes the market information only at certain predictable stopping times $T_1 < T_2 < \dots$. Then $\mathcal{G}_t = \mathcal{F}_{\max\{T_i; T_i \leq t\}}$.

In particular, if ϑ is deterministic, then $\mathcal{G}_t = \mathcal{F}_0$ for all t , and the optimal solution is

$$\vartheta_t^d = \left(\tilde{\mathbb{E}}[\tilde{\Sigma}_t^{SS}] \right)^{-1} \tilde{\mathbb{E}}[\tilde{\sigma}_t^{BS}]$$

$$R^d = \int_0^T \tilde{\mathbb{E}}[\tilde{\sigma}_t^{BS'}] \left(\tilde{\mathbb{E}}[\tilde{\Sigma}_t^{SS}] \right)^{-1} \tilde{\mathbb{E}}[\tilde{\sigma}_t^{BS}] dt$$

Semi-static hedging. Suppose the hedger re-balances the portfolio only at certain predictable stopping times $0 = T_0 < T_1 < T_2 < \dots$, that is, $\boldsymbol{\vartheta}_t = \boldsymbol{\vartheta}_i \in \mathcal{F}_{T_{i-1}}$ for $T_{i-1} < t \leq T_i$. Then (18) becomes

$$\begin{aligned} R &= \tilde{\mathbb{E}} \left[\sum_{i=1}^{\infty} \int_{T_{i-1} \wedge T}^{T_i \wedge T} \left(2 \boldsymbol{\vartheta}'_i \tilde{\mathbb{E}}[\tilde{\boldsymbol{\sigma}}_t^{BS} | \mathcal{F}_{T_{i-1}}] - \boldsymbol{\vartheta}'_i \tilde{\mathbb{E}}[\tilde{\boldsymbol{\Sigma}}_t^{SS} | \mathcal{F}_{T_{i-1}}] \boldsymbol{\vartheta}_i \right) dt \right] \\ &= \sum_{i=1}^{\infty} \tilde{\mathbb{E}} \left[2 \boldsymbol{\vartheta}'_i \int_{T_{i-1} \wedge T}^{T_i \wedge T} \tilde{\mathbb{E}}[\tilde{\boldsymbol{\sigma}}_t^{BS} | \mathcal{F}_{T_{i-1}}] dt - \boldsymbol{\vartheta}'_i \int_{T_{i-1} \wedge T}^{T_i \wedge T} \tilde{\mathbb{E}}[\tilde{\boldsymbol{\Sigma}}_t^{SS} | \mathcal{F}_{T_{i-1}}] dt \boldsymbol{\vartheta}_i \right] \end{aligned}$$

Optimum is

$$\boldsymbol{\vartheta}_i^{ss} = \left(\int_{T_{i-1} \wedge T}^{T_i \wedge T} \tilde{\mathbb{E}}[\tilde{\boldsymbol{\Sigma}}_t^{SS} | \mathcal{F}_{T_{i-1}}] dt \right)^{-1} \int_{T_{i-1} \wedge T}^{T_i \wedge T} \tilde{\mathbb{E}}[\tilde{\boldsymbol{\sigma}}_t^{BS} | \mathcal{F}_{T_{i-1}}] dt$$

$$R^{ss} = \tilde{\mathbb{E}} \left[\sum_{i=1}^{\infty} \int_{T_{i-1} \wedge T}^{T_i \wedge T} \tilde{\mathbb{E}}[\tilde{\boldsymbol{\sigma}}_t^{BS'} | \mathcal{F}_{T_{i-1}}] dt \left(\int_{T_{i-1} \wedge T}^{T_i \wedge T} \tilde{\mathbb{E}}[\tilde{\boldsymbol{\Sigma}}_t^{SS} | \mathcal{F}_{T_{i-1}}] dt \right)^{-1} \int_{T_{i-1} \wedge T}^{T_i \wedge T} \tilde{\mathbb{E}}[\tilde{\boldsymbol{\sigma}}_t^{BS} | \mathcal{F}_{T_{i-1}}] dt \right]$$

A special case is the buy-and-hold strategy, which means $\vartheta_t = \vartheta_0 \in \mathcal{F}_0$. The optimal solution is

$$\vartheta^{bh} = \left(\int_0^T \tilde{\mathbb{E}}[\tilde{\Sigma}_t^{SS}] dt \right)^{-1} \int_0^T \tilde{\mathbb{E}}[\tilde{\sigma}_t^{BS}] dt$$

$$R^{bh} = \int_0^T \tilde{\mathbb{E}}[\tilde{\sigma}_t^{BS'}] dt \left(\int_0^T \tilde{\mathbb{E}}[\tilde{\Sigma}_t^{SS}] dt \right)^{-1} \int_0^T \tilde{\mathbb{E}}[\tilde{\sigma}_t^{BS}] dt$$

Finite number of driving factors:

$$d\tilde{M}_t^B = \boldsymbol{\eta}'_t d\tilde{\mathbf{M}}_t, \quad \tilde{\mathbf{S}}_t = \boldsymbol{\Xi}'_t d\tilde{\mathbf{M}}_t$$

$\tilde{\mathbf{M}} = (\tilde{M}^1, \dots, \tilde{M}^n)'$ is martingale, $\boldsymbol{\eta}$ ($n \times 1$) and $\boldsymbol{\Xi}$ ($n \times m$) predictable,

$$d\langle \tilde{\mathbf{M}}, \tilde{\mathbf{M}}' \rangle_t = \boldsymbol{\Sigma}_t dt$$

$$\boldsymbol{\vartheta}_t^* = \left(\boldsymbol{\Xi}'_t \boldsymbol{\Sigma}_t \boldsymbol{\Xi}_t \right)^{-1} \boldsymbol{\Xi}'_t \boldsymbol{\Sigma}_t \boldsymbol{\eta}_t$$

$$R^* = \tilde{\mathbb{E}} \int_0^T \boldsymbol{\eta}'_t \boldsymbol{\Sigma}_t \mathbf{P}_t \boldsymbol{\eta}_t dt \quad (19)$$

\mathbf{P}_t projector onto column space of $\boldsymbol{\Xi}_t$ under inner product $\boldsymbol{x}' \boldsymbol{\Sigma}_t \boldsymbol{y}$.

Risk minimization. Instigated by Föllmer and Sondermann (1986) for simple T -claims, extended to payment streams by Møller (2001). (Worked with two assets; multi-asset extension is straightforward.)
Remaining cost after time t is

$$\begin{aligned}\tilde{C}_T - \tilde{C}_t &= \tilde{B}_T - \tilde{B}_t - \tilde{U}_t - \int_t^T \boldsymbol{\vartheta}'_{\tau} d\tilde{\mathbf{S}}_{\tau} \\ &= \tilde{V}_t - \tilde{U}_t + \int_t^T (d\tilde{M}_{\tau}^B - \boldsymbol{\vartheta}'_{\tau} d\tilde{\mathbf{S}}_{\tau})\end{aligned}\quad (20)$$

Remaining risk at time t is

$$\begin{aligned}R_t &:= \tilde{\mathbb{E}} \left[(\tilde{C}_T - \tilde{C}_t)^2 \middle| \mathcal{F}_t \right] \\ &= (\tilde{V}_t - \tilde{U}_t)^2 + \tilde{\mathbb{E}} \left[\left(\int_t^T (d\tilde{M}_{\tau}^B - \boldsymbol{\vartheta}'_{\tau} d\tilde{\mathbf{S}}_{\tau}) \right)^2 \middle| \mathcal{F}_t \right] \\ &= (\tilde{V}_t - \tilde{U}_t)^2 + \tilde{\mathbb{E}} \left[\int_t^T d\langle \tilde{M}^B - \boldsymbol{\vartheta}' \cdot \tilde{\mathbf{S}}, \tilde{M}^B - \boldsymbol{\vartheta} \cdot \tilde{\mathbf{S}} \rangle_{\tau} \middle| \mathcal{F}_t \right]\end{aligned}\quad (21)$$

First: Minimize R_t w.r.t. the investment strategy for *fixed* t .
 First term in (21) depends only on the strategy in $[0, t]$.
 Second term depends only on the exposure in $(t, T]$.
 They can therefore be minimized separately.

Second term minimized by $(\vartheta_\tau^*)_{\tau \in (t, T]}$ mean-variance optimal.
 First term minimized to 0 for any $(\vartheta_\tau)_{\tau \in [0, t]}$ in conjunction with \tilde{C}_t
 given by

$$\tilde{U}_t = \tilde{V}_t \tag{22}$$

which is obtained by taking \tilde{C}_t to satisfy

$$\tilde{V}_t = \int_0^t \vartheta'_\tau d\tilde{S}_\tau + \tilde{C}_t - \tilde{B}_t. \tag{23}$$

By (13), this means

$$\tilde{C}_t = \tilde{M}_t^B - \int_0^t \vartheta'_\tau d\tilde{\mathbf{S}}_\tau = \tilde{M}_0^B + \int_0^t (d\tilde{M}_\tau^B - \vartheta'_\tau d\tilde{\mathbf{S}}_\tau) \quad (24)$$

Solutions to optimization problems at different points of time are consistent, leaving a global solution to the problem of minimizing R_t for all $t \in [0, T]$: optimal strategy is (\tilde{C}, ϑ^*) defined by (19) and (24).

Optimal \tilde{C} is martingale, and optimal \tilde{R} can be cast as

$$\tilde{R}_t = \tilde{\mathbb{E}} \left[\int_t^T d\langle \tilde{C}, \tilde{C} \rangle_\tau \middle| \mathcal{F}_t \right]$$

Thus, risk minimization amounts to minimizing $\tilde{\mathbb{E}} \int_t^T d\langle \tilde{C}, \tilde{C} \rangle_\tau$ under the constraints that the strategy be a hedge and the cost function be a martingale.

Schweizer (2008) extended the theory of *local risk minimization* to payment streams and multiple assets.

Minimizing the risk when the portfolio value is specified

In the context of insurance (22) is known as the *principle of equivalence*, which is imposed as a capital requirement: *retrospective reserve* (assets \tilde{U}_t) should equal prospective reserve (liabilities \tilde{V}_t).

Since the solution to the unconstrained optimization problem satisfies the equivalence principle, this could have been imposed in the first place without any consequence to the solution.

This remark invites the idea of constraining the value function \tilde{U} to be some specified adapted function \tilde{V}^* , not necessarily equal to the prospective reserve. This means that, instead of (23), we require

$$\tilde{V}_t^* = \int_0^t \vartheta'_\tau d\tilde{\mathbf{S}}_\tau + \tilde{C}_t - \tilde{B}_t. \quad (25)$$

The calculations from risk minimization go through unchanged. Remaining risk is

$$R_t = (\tilde{V}_t - \tilde{V}_t^*)^2 + \tilde{\mathbb{E}} \left[\left(\int_t^T (d\tilde{M}_\tau^B - \boldsymbol{\vartheta}' d\tilde{\mathbf{S}}_\tau) \right)^2 \middle| \mathcal{F}_t \right]. \quad (26)$$

is minimized by $(\tilde{C}, \boldsymbol{\vartheta}^*)$, where $\boldsymbol{\vartheta}^*$ is mean variance optimal and the cost is adjusted to (25), which means

$$\tilde{C}_t = \tilde{M}_0^B + \int_0^t (d\tilde{M}_\tau^B - \boldsymbol{\vartheta}_\tau^{*'} d\tilde{\mathbf{S}}_\tau) + \tilde{V}_t^* - \tilde{V}_t$$

If $\tilde{V}^* \neq \tilde{V}$, then \tilde{C} is not a martingale.

This extension of risk minimization offers an approach to optimal hedging under capital requirements introduced through regulatory regimes like Basel II and Solvency II. The latter would specify that \tilde{V}_t^* should be an upper percentile in the conditional distribution of the outstanding liability $\tilde{B}_T - \tilde{B}_t$, given \mathcal{F}_t . The so-called normal power approximation to the upper ε -percentile of this distribution is

$$\tilde{V}_t^* \approx \tilde{V}_t + c_{1-\varepsilon} \sqrt{\tilde{V}_t^{(2)}} + \frac{c_{1-\varepsilon}^2 - 1}{6} \frac{\tilde{V}_t^{(3)}}{\tilde{V}_t^{(2)}}$$

\tilde{V}_t is conditional expected value of $\tilde{B}_T - \tilde{B}_t$,
 $\tilde{V}_t^{(2)}$ and $\tilde{V}_t^{(3)}$ are corresponding second and third central moments,
 $c_{1-\varepsilon}$ is upper ε -percentile of standard normal distribution.

The moments can typically be computed as solutions to differential equations.

OPTIMAL DESIGN OF DERIVATIVES

So far we have been discussing the individual hedger's optimal investment strategy in a market with given assets. In the context of insurance the relevant assets could be derivatives with payoffs related to indices for e.g. catastrophes or mortality.

Question: what should these derivatives look like?

The following idea was proposed by the author (Norberg, 2010) in a special model for mortality risk.

Suppose there is a market with m (risky) derivatives and that there are q agents (life offices, pension funds,...) who take optimal hedging positions in accordance with the mean-variance criterion.

Problem: how to design the very derivatives so as to optimally serve the objectives of the agents? A reasonable objective is to minimize the weighted average of the agents' minimized hedging errors. In view of (19), we seek to maximize

$$Q = \sum_p w_p \tilde{\mathbb{E}} \int_0^T \eta'_{p,t} \Sigma_t P_t \eta_{p,t} dt$$

for constant weights w_p , where the subscript p labels agents.

Maximize w.r.t. the projector P_t or, equivalently, the coefficients Ξ_t appearing in the defining expression (19).

Upon introducing

$$\tilde{\eta}_{p,t} = \Sigma_t^{1/2} \eta_{p,t}, \quad \tilde{\Xi}_t = \Sigma_t^{1/2} \Xi_t, \quad \tilde{P}_t = \tilde{\Xi}_t (\tilde{\Xi}_t' \tilde{\Xi}_t)^{-1} \tilde{\Xi}_t',$$

$$Q = \tilde{\mathbb{E}} \int_0^T \sum_p w_p \tilde{\boldsymbol{\eta}}'_{p,t} \tilde{\mathbf{P}}_t \tilde{\boldsymbol{\eta}}_{p,t} dt \quad (27)$$

$$= \tilde{\mathbb{E}} \left[\int_0^T \text{tr} \left(\tilde{\mathbf{P}}_t \sum_p w_p \tilde{\boldsymbol{\eta}}_{p,t} \tilde{\boldsymbol{\eta}}'_{p,t} \right) dt \right] \quad (28)$$

The problem has now been reduced to:

Maximize for each t

$$\text{tr} \left(\tilde{\mathbf{P}}_t \sum_p w_p \tilde{\boldsymbol{\eta}}_{p,t} \tilde{\boldsymbol{\eta}}'_{p,t} \right)$$

w.r.t. projector $\tilde{\mathbf{P}}_t$ or w.r.t. the columns of $\tilde{\boldsymbol{\Xi}}_t$ spanning the subspace it projects onto.

To this end use spectral representation

$$\sum_p w_p \tilde{\boldsymbol{\eta}}_{p,t} \tilde{\boldsymbol{\eta}}'_{p,t} = \sum_{j=1}^n d_{j,t} \mathbf{c}_{j,t} \mathbf{c}'_{j,t}$$

the $d_{j,t}$ are the eigenvalues, the $\mathbf{c}_{j,t}$ ($\mathbf{c}'_{j,t}$) are the corresponding right (left) eigenvectors, which are orthonormal. Best choice of $\tilde{\boldsymbol{\Xi}}_t$ of rank $\leq m$ is the matrix with columns equal to the $\mathbf{c}_{j,t}$ corresponding to the $\min(m, n)$ largest eigenvalues.

Since $\text{tr}(\mathbf{c}_{j,t} \mathbf{c}'_{j,t}) = \mathbf{c}'_{j,t} \mathbf{c}_{j,t} = 1$, the maximum is

$$\max Q = \tilde{\mathbb{E}} \int_0^T \sum_{j; \text{maximal}} d_{j,t} dt.$$

The device prescribes a rule for building a market for insurance risk in steps by supplying derivatives in their order of hedging efficiency. The first derivative has dynamics coefficients $\mathbf{c}_{j,t}$ corresponding to the largest $d_{j,t}$, the second derivative has dynamics coefficients $\mathbf{c}_{j,t}$ corresponding to the second largest $d_{j,t}$, and so on. Derivatives constructed this way may appear less transparent than e.g. survivor bonds or digital bonds and, therefore, be deemed unpractical. However, what matters from a theoretical point of view, are the random sources involved in their price dynamics.

Design of derivatives has been studied by Norberg and Savina (2012) in a shot-noise model for catastrophes, the scope being limited to just comparison of specified derivatives.

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