Evaluation of Uncertainty Risk of The Limit Life by Brownian-Bridge Mortality Model

Noriaki Yokoo *

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Abstract

Under the global circumstance that economic value-based solvency regime can contribute to recognizing the financial conditions of an insurance company appropriately and lead to enhancements in risk management by insurers, the discussion towards the introduction of such regime is being developed at organizations including the IAIS (International Association of Insurance Supervisors). Similarly in Japan, The Financial Services Agency will consider the introduction of economic value-based solvency regime. In this case, we should pay more attention to the evaluation of the longevity risks because of world top class life expectancy in Japan. And so the study of the mortality model becomes important.

In this paper, by using The Brownian bridge to have a characteristic to arrive at the specific point almost surely in specific non-negative time T, we construct the mortality model to converge to 1 at limiting age almost surely.

Furthermore, we consider the change of a longevity risk from the change of limiting age based on the mortality model in Japan. We get a result that the longevity risk as defined in the mortality model using The Brownian bridge is less than in the Solvency Ⅱ and in typical standard Lee-Carter model.

keywords:limiting age,The Brownian bridge,Lee-Carter Brownian bridge model,Longevity Risk

*The staff of Meijiyasuda Life Insurance co, Member of AFIR section of The Institute of Actuaries of Japan, Adress:postal number 100-0005 2-1-1 Marunouchi,Chiyoda-ku,Tokyo,Japan, E-mail:no-yokoo@meijiyasuda.co.jp

1 Introduction

In the situation that the risk the insurance companies are exposed to becomes complicated and diversification, it is necessary for the insurance companies to secure further financial soundness. For this purpose, it is important for the insurance companies to maintain capital under the rules on capital requirements, to disclose the financial information appropriately and to develop the system of comprehensive risk manegemant.

As a part of the development of comprehensive risk management system, the introduction of economic value-based solvency regime is considered globally, especially in Europe. Similarly, The Financial Services Agency will consider the introduction of economic value-based solvency regime in Japan. If economic value-based solvency regime is introduced, the mortality model which we can project the future mortality by is nessesary for evaluation of mortality risk and longevity risk. But under the circumstances that Japan keeps life expectation world's top level and contains the uncertainty of mortality, it is not easy to project Japanese future mortality.

It is necessary to construct the mortality model to consider the estimation of the mortality and the uncertainty of limiting age. However as existence or unexistence of the limiting age, we can not draw the conclusion.

In this paper by using the character of the Brownian bridge that has the character to arrive at the specific point almost surely in specific non-negative time T, we construct the mortality model to converge to 1 at limiting age almost surely.

We consider the effect for longevity risk by changing limiting age and comparing the longevity risk defined by this model to by SolvencyⅡ.

2 Recent mortality trend in Japan

2.1 changes in life expectancy in Japan

The life expectancy of 2011 in Japan is male 79.44 years old, female 85.90 years old. In most recent three years, the life expectancy have been decreasing, but this is world's top level.

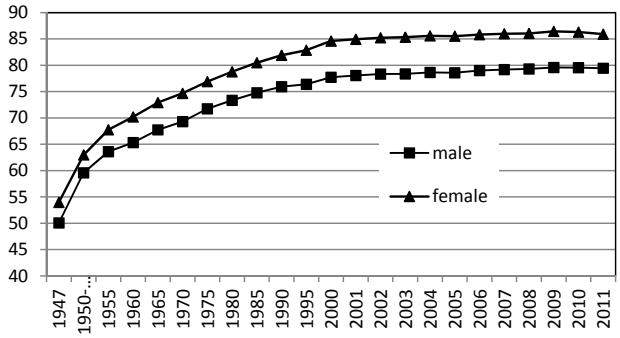

Figure 1: The trend of life expectancy

2.2 Limit of Life

The life expectancy of the oldest-old is considered with various views. Masaaki Shibuya,and Nobutane Hanayama(2004) show limiting age of Japanese males has the boundary with assumpion of limiting age by extreme value theory.

On the other hand, Ishii(2008) shows the extension of life expectancy in Japan is caused by the shift of survivor curve, not rectangular, the shape of survivor curve gets closer to a rectangle by which the rate of survivor in young age is increasing and that in old age is decreasing rapidly. From the result, Ishii shows limiting age has no boundary or has the boundary at very old age.

But as there is not enough data for estimation of the oldest-old mortality in Japan, we can not draw the conclusion either limiting age has boundary or no.

3 The Brownian Bridge

If we assume that the residue of mortality model is the Wiener process,the tail of distribution is extended as time passes. This allows that life span is infinite. Where we assume that life span is finite,that we adapt Wiener process in residue of mortality model is not consistent. Then we construct the mortality model that mortality converges in 1 at limiting age by using the Brownian bridge instead of Wiener prcess.

The Brownian bridge is the stochastic process that attains some point at some non-negative time T almost surely. On the probability space $(\Omega, \mathcal{F}, P)$, let $W(t)$ denote Wiener process.If $T \geq 0$ is constant, the Brownian bridge from 0 to 0 on $[0, T]$ is defined by

$$X(t) = W(t) - \frac{t}{T} W(T) \quad (0 \leq t < \leq T) \tag{1}$$

$0 \leq t \leq T$, $X(t)$ is not adapted to the filtration $\mathcal{F}(t)$ generated by $W(t)$ because $W(T)$ included in $X(t)$is not $\mathcal{F}(t)$-measurable. Then we define the stochastic process $Y(t)$ by

$$Y(t) = \begin{cases} (T-t) \displaystyle\int_0^t \frac{1}{T-u} dW(u) & (0 \leq t < T) \\ 0 \end{cases} \tag{2}$$

$Y(t)$ is adapted to the filtration $\mathcal{F}(t)$ and have the distribution same as the Brownian bridge from 0 to 0 on $[0, T]$. (e.g.See [7](Steven E.Shreve(2004))). Then the stochastic differential equation $Y(t)$ is denoted by

$$dY(t) = -\frac{Y(t)}{T-t} dt + dW(t) = -(\int_0^t \frac{1}{T-u} dW(u)) dt + dW(t) \tag{3}$$

4 The modelling

4.1 Modelling the mortality model by the Brownian bridge

We construct the mortality model by using the Brownian bridge.

On the probability space$(\Omega, \mathcal{F}, P)$, Let $\mathcal{F}(t)$ the filtration generated by Wiener process $W(t)$, where x is entry age, t is calendar year. And let $L(x,t)$ stochastic differential equation for the number of survivors as follows;

$$dL_t(x,s) = -L_t(x,s)\mu_{x,t+s}ds + \sigma(x,s)L_t(x,s)dY(s) \tag{4}$$

$\mu_{x,t}$:the force of mortality at entry age x,calendar year t
$\sigma(x,t)$:the $\mathcal{F}(t)$-adapted
$Y(s)$:the Brownian bridge
Then $L_t(x,s)$ is denoted by the closed solution (by Ito formula)

$$L_t(x,s) = L_t(x,0)\exp(\int_0^s(\mu_{x,t+u} - \frac{1}{2}\sigma(x,u)^2)du + \int_0^s \sigma(x,u)dY(u)) \tag{5}$$

In general,If $\frac{1}{2}\sigma(x,u)^2$ is a negligible amount we can approximate $L_t(x,s)$ as follow;

$$L_t(x,s) = L_t(x,0)\exp(\int_0^s \mu_{x,t+u}du + \int_0^s \sigma(x,u)dY(u)) \tag{6}$$

In general, $\frac{1}{2}\sigma(x,u)^2$ in mortailty model is a negligible amount. But we would pay attention to the level of $\sigma(x,u)$.
Since then we call the mortality and survival age models defined by $L_t(x,s)$ Brownian-Bridge Mortality Model.

4.2 The merit of Brownian bridge model

When we construct Brownian bridge model,we should pay attention that Brownian bridge model is the model to consider characteristic of diffusion coefficient. Because it is important to understand the change of mortality for considering the risk such as longevity risk. This model has two merits. First, we can specify the effect of longevity risk by the changing limiting age. Second, we can apply various mortality model by parametarizing drift coefficient. And if T is infinity, the Brownian bridge converge Wiener process(see Appendix A and B). So it is equal to no boundary of limiting age that we choose the Wiener process for diffusion coefficient.

4.3 Lee-Carter model

Let $m_{x,t}$ be the central death rate for age x in calendar year t
R.D.Lee,and L.R.Carter(1992) modelled $m_{x,t}$ as follows;

$$\log(m_{x,t}) = \alpha_x + \beta_x \kappa_t + \epsilon_{xt} \tag{7}$$

Where α_x, β_x and ϵ_{xt} are parameters. We take limitation between parameters as follows to identify;

$$\sum_{x=0}^{\omega} \beta_x = 1, \sum_{t=1}^{T} \kappa_t = 0 \tag{8}$$

Every parameters means;
α_x:log mortality or the force of mortality of age effect
κ_t:parameter of calendar year effect
β_x:parameter of age-specific responses

And ϵ_{xt} is the residue satisfied $E[\epsilon_{xt}] = 0$. By using various stochastic model as the residue, we can extend various mortality model from Lee-Carter model.

4.4 Lee-Carter Brownian Bridge Model

The characteristic of Japanese mortality can be expressed by Lee-Carter model pretty accurately. The Lee-Carter model is utilized for a large number of studies in Japanese mortality, and was used for projection of population in Japan on January 2002. We construct the mortality model in Japan by using the Brownian bridge for the residue of Lee-Carter model. We call this model Lee-Carter Brownian Bridge Model.

Specifically, we fit the drift coefficient of the Brownian bridge model for that of Lee-Carter model. We denote the survival rate in Brownian bridge model as follows;

$$\int_0^1 \mu_{x,t+s} ds = \alpha_x + \beta_x \kappa_t \tag{9}$$

And the parameters is defined by

α_x:log survivor rate of age x effect
κ_t:parameter of calendar year t
β_x:parameter of age-specific responses

In general, we construct the Lee-Carter model by parameterizing log mortality or rate of change of death, now we construct the model with the parameterizing the survivor rate instead of mortality.

$$L_{t+n}(x,n) = L_t(x,0) \exp(\sum_{i=0}^{n} (\alpha_{x+i} + \beta_{x+i}\kappa_{t+i}) + \int_0^n \sigma(x,u) dY(u)) \tag{10}$$

5 Application of the Japanse mortality

In this section, we fit Lee-Carter Brownian bridge model for Japanese mortality, and project the mortality.

5.1 Data

We used mortality data of the Japanese population downloaded by Human Mortality Database, relative to the age range 0-109 and 1995-2009 period for male and female total.

5.2 Assumption of parameter

We assume the parameters by using general least squares method. We get the estimation of parameter by minimizing the following formula;

$$\sum_{x=0}^{\omega} \sum_{t=1}^{T} (\log(\frac{L_{t+1}(x,1)}{L_t(x,0)} - \alpha_x - \beta_x \kappa_t))^2 \tag{11}$$

Where ω is limiting age, and T is measure period

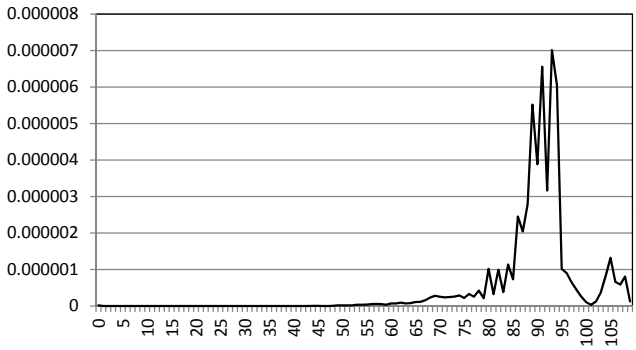

Figure 2: The variance of mortality by age

We assume α_x by differentiating (12) α_x, β_x and κ_t by using the singular value decomposition method. The variance of $\sigma(x,s)$ is uneven by age, so we assume $\sigma(x,s)(=\sigma(x))$ every age.

We calculate the variance of (10).

$$\log(\frac{L_{t+1}(x,1)}{L_t(x,0)} - \alpha_x - \beta_x\kappa_t) = \int_0^1 \sigma(x)dY(u)$$

By (16)(See Appendix.B), we can calculate the variance of the left side of (10) as follows;

$$Var(\log(\frac{L_{t+1}(x,1)}{L_0(x,0)} - \alpha_x - \beta_x\kappa_t)) = Var(\int_0^1 \sigma(x)dY(u))$$

$$= \sigma(x)^2 Var(Y(1)) = \sigma(x)^2\frac{(T-1)}{T}$$

Then $\sigma(x,s)$ is denoted by

$$\sigma(x)^2 = \frac{1}{\nu_x}\sum_t \frac{(\log(\frac{L_{t+1}(x,1)}{L_t(x,0)}) - \alpha_x - \beta_x\kappa_t)^2}{\frac{(T-1)}{T}} \tag{12}$$

where ν_x is the number of data age x.

5.3 Fitting the model

Now by comparing with the mortality calculated by Lee-Carter model and the total(male and female) mortality in 2009, we verify the mortality by Lee-Carter Brownian bridge model. Specifically, we compare Akaike Information Criterion (AIC) by Lee-Carter Brownian bridge model with Lee-Carter and mortality. AIC is defined by AIC = - 2log L + 2p where log L is maximum log likelihood and p is the number of parameters. And we assume the distribution of the

mortality model	LC-BBM	LC
maximum log likelihood	-13.39	-13.16
AIC	-1,467	-1,442

Table 1: The Lee-Carter Brownian bridge model Lee-Carter model

residue of mortality is a normal distribution easily. Table1 shows the result of AIC by each model and mortality. AIC by Lee-Carter Brownian bridge model is approximately the same as that by Lee-Carter model. Table2 shows the mortality by Lee-Carter Brownian bridge model is apploximately the same as by Lee-Carter model and that in 2009.

age/model	Lee-Carter Brownian bridge	Lee-Carter	mortality in 2009
0	0.00243	0.00241	0.00233
10	0.00007	0.00007	0.00008
50	0.00260	0.00256	0.00254
60	0.00578	0.00573	0.00572
70	0.01281	0.01263	0.01177
80	0.03637	0.03610	0.03652
90	0.10964	0.10922	0.10709
100	0.27997	0.27919	0.27980

Table 2: The mortality calculated by The Lee-Carter Brownian bridge model and Lee-Carter model,and in 2009

6 Effect that uncertainty of limiting age gives the longevity risk

After we make Lee-Carter Brownian bridge model fit the mortality in Japan, we consider the effect that the uncertainty of limiting age gives the lonvevity risk for the insurance company with a portfolio with single premium immediate annuity for life.

6.1 Definition of longevity risk

The capital requirement for longevity risk is defined as a result of longevity scenario as follows;

$$Life_{long} = (\Delta NAV | longevityshock) \tag{13}$$

where

ΔNAV:The change in the net value of assets minus liabilities
longevityshock: a (permanent) 20 percentage decrease in mortality rates for each age and each policy where the payment of benefits (either lump sum or

multiple payments) is contingent on longevity risk

Although, in the definition, proposed in SolvencyⅡ longevityshock is defined as a parmament 20 percentage decrease in mortality, we consider the capital required with 99.5 percentage probability, the unexpected losses on a 1-year time horizon as a SolvencyⅡ risk.

6.2 Result

The longevity risk for the insurance company with a portfolio of 100 thousand single premium immediate annuity for life every entry age and limiting age, and assume no other risk exist.

Table3 shows the value of longevity risk evaluated by Lee-Carter Brownian bridge model.

entry age	limiting age	120	130	140	∞	LC	Liability
50	SolvencyⅡ	10,093	10,093	10,093	10,093	10,060	265,473
	Var99.5	8,204	8,362	8,483	9,652	21,127	
60	SolvencyⅡ	11,465	11,465	11,465	11,465	11,439	217,790
	Var99.5	7,473	7,623	7,734	8,913	21,810	
70	SolvencyⅡ	12,183	12,183	12,183	12,183	12,167	162,917
	Var99.5	5,482	5,591	5,667	6,705	19,077	
80	SolvencyⅡ	11,634	11,634	11,634	11,634	11,640	106,153
	Var99.5	3,416	3,484	3,528	4,459	13,544	

Table 3: The value of longevity risk every entry age(million yen)

Table3 shows the longevity risk evaluated by Lee-Carter Brownian bridge model is less than that by SolvencyⅡ in every entry age and limiting age.

Figure3, Figure4 and Figure5 show mortality ratio for SolvencyⅡ and that for Lee-Carter Brownian bridge model at entry age 50, 60, 70 and limiting age 120.

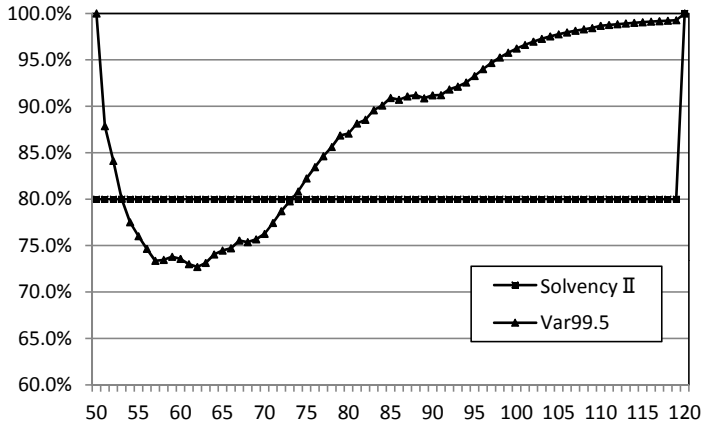

Figure 3: At entry age 50

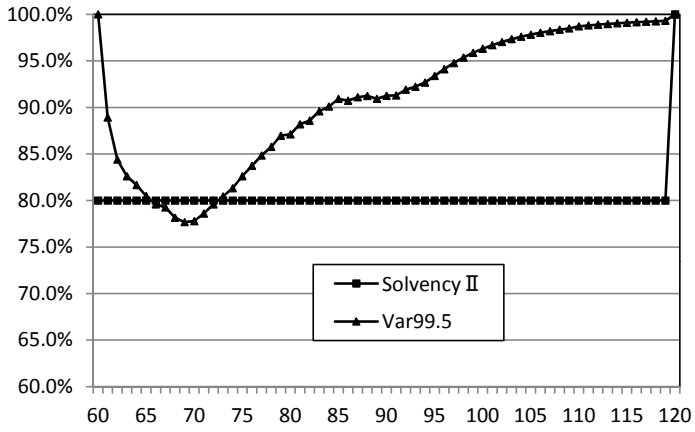

Figure 4: At entry age 60

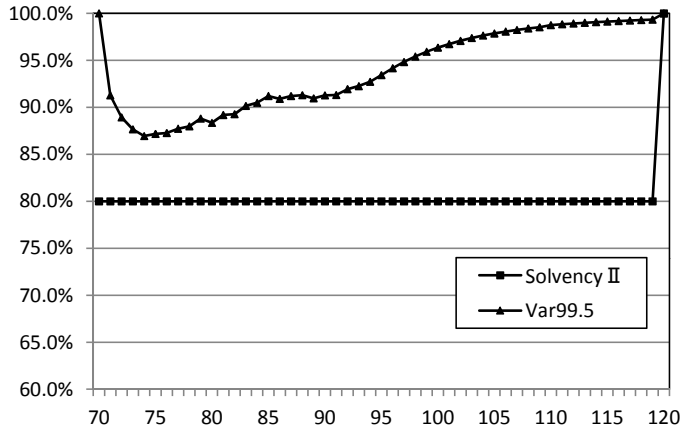

Figure 5: At entry age 70

After age 75, the mortality of the stress scenario by Lee-Carter Brownian bridge model is less than that of SolvencyⅡ.

Around 75 years old, since the absolute numbers of death is small, one death has a big influence on the mortality. Therefore a ratio of standard deviation for the death rate is big. On the other hand, after age 75 a ratio of standard deviation for the death rate is small.

After age 75, the stress of mortality evaluated by SolvencyⅡ is more than that by Lee-Carter Brownian bridge model, so it is the reason why the longevity risk by Lee-Carter Brownian bridge model is less than that of SolvencyⅡ.

6.3 The change of longevity risk from the change of limiting age

Table4 shows the change of longevity risk from change of limiting age.

The change of longevity risk of every limiting age is caused by the change

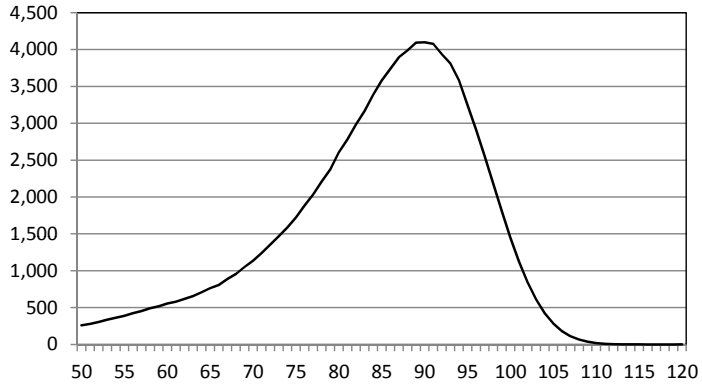

Figure 6: The number of death in the 100 thousand policyholders entry age 50

of survivors of age 70-90. The mortality ratio calculated by SolvencyⅡ does not depend on limiting age as longevity risk is not changed by limiting age.

entry age	limiting age	110→ 120	120 →130	130→140	140 → ∞
50	SolvencyⅡ	6	0	0	0
	Var99.5	216	158	121	1,169
60	SolvencyⅡ	9	0	0	0
	Var99.5	216	151	111	1,179
70	SolvencyⅡ	13	0	0	0
	Var99.5	169	109	77	1,038
80	SolvencyⅡ	21	0	0	0
	Var99.5	119	68	45	931

Table 4: The amount of changes of longevity risk(million yen)

age	50	60	70	80	90	100	110
limiting age							
110	100,000	96,989	91,205	76,409	42,488	6,715	62
120	100,000	96,994	91,247	76,544	42,684	6,784	64
130	100,000	96,998	91,278	76,643	42,828	6,835	64

Table 5: The number of survivors

7 Conclusions

We get the result that the longevity risk evaluated by LC-BBM model for the insurance company with a portfolio with single premium immediate annuity for life is less than that by the definition proposed in SolvencyⅡ.

From the view point of prudence, it is nessesary to evaluate the longevity consavertively by the definition proposed in SolvencyⅡ. This is because SolvencyⅡ treats the risk over 75 age too high. But excessive prudence limits an opportunity of the creation of the future value. Although there are many challenges(for example, elaboration of estimation of parameters and so on), it is relatively easy to implement the model. We hope that this leads to elaboration of the longevity risk evaluation and the development of ERM.

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Appendix.A The Ito isometry

Theorem.1. Let $W(t)$ Wiener process on probability space$(\Omega, \mathcal{F}, P)$.Then

$$E[(\int_S^T f(t,\omega)dW(t))^2] = E[\int_S^T f^2(t,\omega)dt] \quad \forall f \in \mathcal{V}(S,T) \tag{14}$$

Proof. See [7](Steven E.Shreve(2004))

Appendix.B proof of Proposition

Proposition.1. Let $W(t)$ Wiener process on probability space $(\Omega, \mathcal{F}, P)$ and $Y(t)$ the Brownian bridge.
Then on Hilbert spece it is satisfied that

$$\lim_{T\to\infty} Y(t) = W(t) \tag{15}$$

Proof.

$$\| \lim_{T\to\infty} Y(t) - W(t)\| = \| \int_0^t (\lim_{T\to\infty} \frac{T-t}{T-u} - 1)dW(u))\| =$$

$$(\int_\Omega | \int_0^t (\lim_{T\to\infty} \frac{T-t}{T-u} - 1)dW(u) |^2 \, dP)^{\frac{1}{2}} = (E[| \int_0^t (\lim_{T\to\infty} \frac{T-t}{T-u} - 1)dW(u) |^2])^{\frac{1}{2}}$$

$$= (E[| \int_0^t (\lim_{T\to\infty} \frac{T-t}{T-u} - 1)^2 du |)^{\frac{1}{2}} \text{ (by The Ito's isometry)}$$

Hence

$$\int_0^t (\lim_{T\to\infty} \frac{T-t}{T-u} - 1)^2 du = [\lim_{T\to\infty} (\frac{(T-t)^2}{T-u} + 2(T-t)\log(T-u) + u)]_0^t$$

$$= \lim_{T\to\infty} (\frac{(T-t)t}{T} - 2t\frac{\log(1+\frac{t}{T-t})}{\frac{t}{T-t}} + t) \leq \lim_{T\to\infty} (\frac{(T-t)t}{T} - 2t(1-\frac{\frac{t}{T-t}}{2}) + t) \searrow 0$$

For

$$\log(1+\frac{t}{T-t}) = \frac{t}{T-t} - \frac{1}{2}(\frac{t}{T-t})^2 + \frac{1}{3}(\frac{t}{T-t})^3 - \cdots \geq \frac{t}{T-t} - \frac{1}{2}(\frac{t}{T-t})^2$$

Therefore

$$\| \lim_{T\to\infty} Y(t) - W(t)\| = 0$$

Proposition.2. Let $Y(t)$ the Brownian bridge on probability space $(\Omega, \mathcal{F}, P)$.
Then

$$Var(Y(t)) = \frac{(T-t)t}{T} \tag{16}$$

Proof.

$$Var(Y(t)) = E[Y(t)^2] - (E[Y(t)])^2$$

$$= (T-t)^2 E[(\int_0^t \frac{1}{T-u} dW(u))^2] - (T-t)^2 E[(\int_0^t \frac{1}{T-u} dW(u))]^2$$

$$= (T-t)^2 E[(\int_0^t (\frac{1}{T-u})^2 du))] = (T-t)^2 [\frac{1}{T-u}]_0^t = \frac{(T-t)t}{T}$$

(by Ito's isometry)

Appendix.C Ito-integral of the Brownian bridge

Proposition.3. We approximate (3) by

$$Y(k) - Y(k-1) \equiv -\sum_{i=1}^{k} \frac{W(i) - W(i-1)}{T-x-i} + W(k) - W(k-1)$$

$$W(k) - W(k-1) \sim N_k(0,1)$$

(Between $N_i(0,1)$ and $N_j(0,1)$ are independent if $i \neq j$) then

$$\int_0^n \sigma(x,u) dY(u) \sim N(0, \sum_{i=1}^{n} (\sigma(x,i) - \frac{\sum_{j=i+1}^{n} \sigma(x,j)}{T-x-i})^2)$$

Proof.

$$\int_0^n \sigma(x,u) dY(u) = \sum_{i=1}^{n} \sigma(x,i)(Y(i) - Y(i-1)) = \sum_{i=1}^{n} \sigma(x,i)(-\sum_{j=1}^{i} \frac{N_j(0,1)}{T-x-j} + N_i(0,1))$$

$$= \sum_{i=1}^{n} (\sigma(x,i) - \frac{\sum_{j=i+1}^{n} \sigma(x,j)}{T-x-i}) N_i(0,1) \sim N(0, \sum_{i=1}^{n} (\sigma(x,i) - \frac{\sum_{j=i+1}^{n} \sigma(x,j)}{T-x-i})^2)$$

(by reproductive property of normal distribution)