Explicit ruin formulas for models with dependence among risks

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Mixing $\rightarrow$ Pareto distribution

$$X \sim Exp(\Theta) \quad \text{with} \quad \Theta \sim Gamma(\alpha, \beta)$$

Resulting mixing distribution for X

$$1 - F_X(x) = \int_0^\infty e^{-\theta x} f_\Theta(\theta) d\theta = \left(1 + \frac{x}{\beta}\right)^{-\alpha}, \quad x \geq 0$$

Mixing $\rightarrow$ Weibull distribution

If we mix an exponential with a stable distribution of order $1/2$, we obtain a Weibull distribution with fixed shape parameter $p = 1/2$.

$$X \sim Exp(\Theta) \quad \text{with} \quad f_\Theta(\theta) = \frac{\alpha}{2\sqrt{\pi\theta^3}} e^{-\alpha^2/4\theta}$$

Resulting mixing distribution for X

$$1 - F_X(x) = exp\{-\alpha x^{1/2}\}, \quad x \geq 0.$$

Outline

- Simple idea of mixing in ruin theory
- Examples
- Dependence described by copulas
- Alternative stochastic representation
- Mixing results in the Bulletin de l'Association des Actuaires suisses (1972, 1977)

Classical ruin model

$$R(t) = u + ct - \sum_{k=1}^{N(t)} X_k,$$

- $u \geq 0$: initial surplus
- $N(t)$ stochastic process: number of claims up to time t, $E\{N(t)\} = \lambda t$
- X_k iid random variables: claim amounts, $E\{X\} < \infty$
- $c > 0$: constant premium intensity
- $ct > E\{N(t)\}E\{X\}$: net profit condition

Ruin

Time of ruin

$$T_u = \inf_{t \geq 0} \left(R(t) < 0 \mid R(0) = u \right)$$

Probability of ruin

$$\psi(u) = P\left(T_u < \infty \right)$$

Classical result (Cramer, 1930) for compound Poisson model with $X \sim Exp(\theta)$ claim amounts

$$\psi(u) = \min \left\{ \frac{\lambda}{\theta c} e^{-(\theta - \frac{\lambda}{c})u}, 1 \right\}, \quad u \geq 0.$$

Mixing idea

Denote by

$$\psi_\theta(u) = P\left(T_u < \infty \mid \Theta = \theta\right)$$

Then, the ruin probability is given by

$$\psi(u) = \int_0^\infty \psi_\theta(u) dF_\Theta(\theta).$$

Claims: dependent Pareto

- Compound Poisson risk model ($\tau \sim Exp(\lambda)$)
- Claims $X \sim Exp(\theta)$, where $\theta \sim \Gamma(\alpha, \beta)$

Ruin probability is

$$\begin{aligned}\Psi(u) &= \int_0^{\lambda/c} 1 \cdot \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\beta\theta} d\theta \\ &+ \int_{\lambda/c}^{\infty} \underbrace{\frac{\lambda}{\theta c} e^{-\theta u} e^{\frac{\lambda}{c} u}}_{\Psi_\theta(u)} \cdot \underbrace{\frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\beta\theta}}_{f_\theta(\theta)=\Gamma(\alpha,\beta)} d\theta\end{aligned}$$

Claims: dependent Pareto

$$\begin{aligned}\Psi(u) &= 1 - \frac{\Gamma(\alpha, \beta\theta_0)}{\Gamma(\alpha)} + \frac{\beta}{\Gamma(\alpha)} \\ &\times (\beta\theta_0)^{\alpha-1} e^{-\beta\theta_0} \underbrace{(u+\beta)^{-1}}_{\to_{u\to\infty} 0} \underbrace{\frac{\Gamma(\alpha-1, (\beta+u)\theta_0)}{((\beta+u)\theta_0)^{\alpha-2} e^{-(\beta+u)\theta_0}}}_{\to_{u\to\infty} 1}\end{aligned}$$

Probability of ruin decays to a constant

$$\lim_{u\to\infty} \Psi(u) = 1 - \frac{\Gamma(\alpha, \frac{\beta\lambda}{c})}{\Gamma(\alpha)} > 0$$

as fast as u^{-1}!

Claims: dependent Pareto

$$\Psi(0) = 1 - \frac{\Gamma(\alpha, \frac{\beta\lambda}{c})}{\Gamma(\alpha)} + \frac{\beta\lambda}{c}\frac{\Gamma(\alpha-1, \frac{\beta\lambda}{c})}{\Gamma(\alpha)},$$

where

$$\Gamma(\alpha, x) = \int_x^\infty w^{\alpha-1}e^{-w}dw$$

is the incomplete Gamma function.

Mixing dependence structure

For $X \sim Exp(\Theta)$, with $\Theta \sim F_\Theta$, for each n,

$$P\left(X_1 > x_1, \ldots, X_n > x_n \mid \Theta = \theta\right) = \prod_{k=1}^{n} e^{-\theta x_k}. \tag{1}$$

That is, given $\Theta = \theta$, the X_k $(k \geq 1)$ are conditionally independent and distributed as $Exp(\theta)$.

Net profit condition

Since for $\theta \leq \theta_0 = \lambda/c$ the net profit condition is violated,

$$\psi(u) = \int_0^\infty \psi_\theta(u) dF_\Theta(\theta) = F_\Theta(\theta_0) + \int_{\theta_0}^\infty \psi_\theta(u) dF_\Theta(\theta).$$

An immediate consequence is that in this dependence model

$$\lim_{u \to \infty} \psi(u) = F_\Theta(\theta_0),$$

which is positive whenever the random variable Θ has probability mass at or below $\theta_0 = \lambda/c$ (probability of net profit condition not being fulfilled).

Claims: dependent Weibull

For the classical Cramér Lundberg risk model with claims

$$X \sim Exp(\Theta), \quad \text{with} \quad f_\Theta(\theta) = \frac{\alpha}{2\sqrt{\pi\theta^3}} e^{-\alpha^2/4\theta},$$

the probability of ruin is

$$\begin{aligned}\Psi(u) &= \int_0^{\theta_0} 1 \cdot \frac{\alpha}{2\sqrt{\pi\theta^3}} e^{-\alpha^2/4\theta} d\theta \\ &+ \int_{\theta_0}^{\infty} \underbrace{\frac{\lambda}{\theta c} e^{-\theta u} e^{\frac{\lambda}{c} u}}_{\Psi_\theta(u)} \cdot \underbrace{\frac{\alpha}{2\sqrt{\pi\theta^3}} e^{-\alpha^2/4\theta}}_{f_\theta(\theta) = dH} d\theta\end{aligned}$$

Here the net profit condition holds for any $\theta > \theta_0 = \frac{\lambda}{c}$.

Claims: dependent Weibull

$$
\begin{aligned}
\Psi(u) &= \mathrm{Erfc}\left(\frac{\alpha}{2\sqrt{\theta_0}}\right) \\
&+ \theta_0 \frac{1}{\sqrt{\pi}\alpha^2} e^{\theta_0 u} \{ e^{-\alpha\sqrt{u}} \sqrt{\pi} \left(1 + \alpha\sqrt{u} + (1 + \alpha\sqrt{u}) \mathrm{Erf}\left(\frac{\alpha}{2\sqrt{\theta_0}} - \sqrt{u\theta_0}\right)\right) \\
&- e^{\alpha\sqrt{u}} \sqrt{\pi} \left(1 - \alpha\sqrt{u} + (-1 + \alpha\sqrt{u}) \mathrm{Erf}\left(\frac{\alpha}{2\sqrt{\theta_0}} + \sqrt{u\theta_0}\right)\right) - \frac{2\alpha}{\sqrt{\theta_0}} e^{-\theta_0 u - \frac{\alpha^2}{4\theta_0}} \}
\end{aligned}
$$

Since

$$
\mathrm{Erf}(x) = 1 - \frac{\Gamma\left(\frac{1}{2}, x^2\right)}{\sqrt{\pi}}
$$

one can write the result in terms of incomplete Gamma functions.

Interarrival times: dependent Pareto

In a classical Cramér Lundberg risk model with exponential claims,

$$\Lambda \sim \Gamma(\alpha, \beta)$$

one obtains Pareto inter-arrival times.

$$\begin{aligned}
\Psi(u) &= \int_0^{\lambda_0} \underbrace{\frac{\lambda}{\theta c} e^{-\theta u} e^{\frac{\lambda}{c} u}}_{\Psi_\lambda(u)} \underbrace{\frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\beta\lambda}}_{f_\lambda(\lambda)=\Gamma(\alpha,\beta)} d\lambda \\
&+ \int_{\lambda_0}^{\infty} 1 \cdot \frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\beta\lambda} d\lambda
\end{aligned}$$

Interarrival times: dependent Pareto

$$\Psi(u) = e^{-u\theta}\beta^{\alpha}(\beta - \frac{u}{c})^{-\alpha-1}\frac{\Gamma(\alpha+1) - \Gamma(\alpha+1, c\beta\theta - u\theta)}{c\theta\Gamma(\alpha)} + \frac{\Gamma(\alpha, \beta c\theta)}{\Gamma(\alpha)}$$

Thus, the probability of ruin decays again as fast as u^{-1} to

$$\lim_{u\to\infty}\Psi(u) = \frac{\Gamma(\alpha, \beta c\theta)}{\Gamma(\alpha)}$$

the probability of net profit condition not being fulfilled.

$$\Psi(0) = \frac{\Gamma(\alpha+1) - \Gamma(\alpha+1, c\beta\theta)}{\beta c\theta\Gamma(\alpha)} + \frac{\Gamma(\alpha, \beta c\theta)}{\Gamma(\alpha)}$$

Interarrival times: dependent Weibull

Using Mathematica,

$$\begin{aligned}\Psi(u) &= \int_0^{\lambda_0} \frac{\alpha}{2\sqrt{\pi\lambda^3}} e^{-\alpha^2/4\lambda} d\lambda \\ &+ \frac{\alpha}{4\sqrt{cu}\theta} i e^{-i\sqrt{u/c}\alpha - u\theta} \left(-\mathrm{Erfc}\left(\frac{\alpha}{2\sqrt{c\theta}} - i\sqrt{u\theta} \right) \right. \\ &+ \left. e^{2i\sqrt{u/c}\alpha} \mathrm{Erfc}\left(\frac{\alpha}{2\sqrt{c\theta}} + i\sqrt{u\theta} \right) \right)\end{aligned}$$

Proposition

The dependence model characterized by

$$P\left(X_1 > x_1, \ldots, X_n > x_n \mid \Theta = \theta\right) = \prod_{k=1}^{n} e^{-\theta x_k}$$

can equivalently be described by having marginal claim sizes $X_1, X_2, \ldots$ that are completely monotone, with a dependence structure due to an Archimedean survival copula with generator $\varphi = \left(\widetilde{F}_\Theta\right)^{-1}$ for each subset $(X_{j_1}, \ldots, X_{j_n})$ (for $j_1, \ldots, j_n$ pairwise different), where $\widetilde{F}_\Theta$ denotes the Laplace-Stieltjes transform of F_Θ.

Archimedean generator

$$\varphi(t) = \left(\widetilde{F}_\Theta\right)^{-1}(t)$$

As the inverse of a Laplace-Stieltjes transform of a cdf,

- $\varphi : [0,1] \to [0,\infty]$ is a continuous strictly decreasing function
- $\varphi(0) = \infty$ and $\varphi(1) = 0$
- φ^{-1} is completely monotone

Thus, the Archimedean copula is well-defined for all n (see e.g. Nelsen, 2006).

Other conditional tails

From the general construction of Archimedean copulas the conditional tail of the marginals can be also written in a more general form, i.e. the power form

$$P\left(X_1 > x_1, \ldots, X_n > x_n \mid \Theta = \theta\right) = \prod_{k=1}^{n} (\overline{G}(x_k))^{\theta}$$

for all n, i.e. θ is the common mixing parameter.
The dependence structure is again Archimedean with generator

$$\varphi(t) = (\widetilde{F}_{\Theta})^{-1}(t),$$

where now

$$\overline{F}_X(x_i) = \widetilde{F}_{\Theta}(-\log \overline{G}(x_i))$$

Example

- If in the independent risk model, the claim sizes are $X \sim \textit{Pareto}(\alpha, \beta)$ distributed, where α would now be the mixing parameter θ (so that $\overline{G}$ would be the tail of a Pareto$(1, \beta)$ random variable)

- However, for Pareto distributed claims there is no fully explicit formula for $\psi_\theta(u)$ available (but see e.g. Albrecher and Kortschak 2009 for an integral representations of $\psi_\theta(u)$).

Claims: dependent Gamma

Gamma(α, β) distribution with shape parameter $\alpha \leq 1$ is completely monotone.

$$\overline{F}_X(x_i) = \int_0^\infty e^{-\theta x_i} dF_\Theta(\theta) = \widetilde{F}_\Theta(x_i), \quad i = 1, \ldots, n,$$

one identifies the mixing distribution by inverse Laplace transformation. Using complex analysis techniques

$$f_\Theta(\theta) = \frac{\sin(\alpha\pi)}{\pi\theta} \left(\frac{\theta}{\beta} - 1 \right)^{-\alpha}, \quad \theta > \beta,$$

i.e. the mixing distribution has a Pareto-type tail.

Stochastic representation

$$P\left(X_1 > x_1, \ldots, X_n > x_n \mid \Theta = \theta\right) = \prod_{k=1}^{n} e^{-\theta x_k}$$

has also the stochastic representation

$$X_i \stackrel{d}{=} \frac{1}{\Theta} Y_i, i \geq 1,$$

where $Y_i \sim Exp(1)$, $i \geq 1$.

Ruin probability

Mixing argument

$$\psi(u) = \int_0^\infty \psi_\theta(u) dF_\Theta(\theta)$$

Stochastic representation argument

$$\psi(u) = \int_0^\infty \tilde{\psi}(u/s) dF_S(s)$$

where $\tilde{\psi}$ denotes the probability of ruin when the claims are unit exponentials.

Dependent Gamma claims revisited

$X_i \sim Gamma(\alpha, 1)$ dependent.

From beta-gamma algebra, one has that

$$X \overset{d}{=} SY$$

- $Y \sim Exp(1)$
- $S \sim Beta(\alpha, 1-\alpha)$ with density

$$f_S(s) = \frac{s^{\alpha-1}(1-s)^{-\alpha}}{\Gamma(\alpha)\Gamma(1-\alpha)}, \quad 0 < s < 1, \quad \alpha \in (0,1)$$

implying $f_\Theta(\theta) = \frac{\sin(\alpha\pi)}{\pi\theta}(\theta-1)^{-\alpha} \quad \theta > 1$

Bühlmann 1972

The probability of ruin

$$\psi_T(u) = P(R(t) < 0, \quad \text{for at least one} \quad t \in [0, T])$$

is discussed in the case of a fluctuating risk parameter λ continuoulsy estimated on the basis of past experience, by mixing

$$\psi_T(u) = \int \psi_T(u \mid \lambda) dU(\lambda),$$

$N(t)$ describes the number of claims in an Ove Lundberg model (random process for sickness and accident statistics), where

$$P(N(t) = k) = \int \frac{(\lambda t)^k}{k!} e^{-\lambda t} dU(\lambda)$$

There results a random walk problem which is considerably more complicated that in the classical case. However one can derive cautious bounds which are convenient for numerical calculations.

Dubey 1977

For $N(t) \sim Poisson(\Lambda)$

$$R(t) = c\int_0^t \hat{\lambda}(s)ds - \sum_{j=0}^{N(t)} Y_j, \quad t \geq 0$$

Estimates for λ

- $\hat{\lambda}(t) = E\{\Lambda \mid N(t)\} \rightarrow$ exact form for the ruin probability
- $\hat{\lambda}(t) = \frac{N(t)}{t} \rightarrow$ moments of ruin
- $\hat{\lambda}(t) = \frac{a+N(t)}{b+t} \rightarrow$ approximation of the probability of ruin, considering $\hat{\lambda}$ to be the "credibility" estimate.

Note: This model was theoretical basis for the Bonus-Malus system for the Swiss obligatory car insurance.

Thank you for your attention!

$$\psi(u) = \int_0^\infty \psi_\theta(u) dF_\Theta(\theta).$$

De Finetti's theorem

An exchangeable family of random variables can, in great generality, be generated as a mixture over a common parameter of an i.i.d. sequence.
(Bühlmann 1960, Feller 1970)

Exchangeability may be seen as a natural assumption in the risk model context.