Multivariate Tweedie Lifetimes: The Impact of Dependence

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Plan

- Introduction
- Dependent Lifetimes and a Multivariate Tweedie Distribution
- Parameter Estimation and Truncation
- Application to Specific Distributions
- Conclusion

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Introduction

- Motivation: Provide the means to assess the impact of dependent lifetimes on annuity valuation and risk management.
  - Underlying assumption is that systematic mortality improvements induce dependence.
    - ↳ Could reframe as cohort, or pool of similar-risks, analysis.

- Investigate a multivariate Tweedie distribution for two reasons.
  - It is a general family that includes many interesting distributions.
  - Induces dependence exactly in the manner we envision, namely, it categorizes mortality into systematic and idiosyncratic components.

- Resolve parameter estimation in the presence of truncation.
- Provide explicit solutions for an interesting subset of distributions.

The Exponential Dispersion Family

The Tweedie family of distributions is an important subclass of the exponential dispersion family (EDF).

Random variable $X \sim ED(\theta, \lambda)$ belongs to the EDF in the additive form if it has generalized density:

$$f(x; \theta, \lambda) = e^{[\theta x - \lambda \kappa(\theta)]} q(x; \lambda);$$

where

- θ is the canonical parameter belonging to the set

$$\Theta = \{\theta \in \mathbb{R} \,|\, \kappa(\theta) < \infty\},$$

- $\lambda \in \mathbb{R}_+$, the dispersion parameter,
- κ is twice differentiable,
- and q is a suitable normalizing constant.

Tweedie's Exponential Dispersion Family

The function κ determines the specific subclass of distributions.

↳ Alternatively, one may specify the structure of the underlying unit variance function, $V(\mu) = \kappa''((\kappa')^{-1}(\mu))$.

Definition (Tweedie Distribution, Tw_p)

The Tweedie class of the EDF is determined by a power variance structure,

$$V(\mu) = \mu^p,$$

where p is called the power parameter.

$\Rightarrow$ For the Tweedie family, $\kappa_p(\theta) = \kappa(\theta)$ has the form

$$\kappa(\theta) = \begin{cases} e^\theta, & p = 1, \\ -\log(-\theta), & p = 2, \\ \frac{\alpha-1}{\alpha}\left(\frac{\theta}{\alpha-1}\right)^\alpha, & p \neq 1, 2, \end{cases}$$

where $\alpha = (p-2)/(p-1)$.

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Modelling Dependent Lifetimes

Assume M pools of N lives.

↳ Suppose the lives within a pool are dependent.

→ Let $T_{i,j}$ be the lifetime of individual i in pool j.

We apply the following model for lifetimes:

$$T_{i,j} = Y_{0,j} + Y_{i,j},$$

$Y_{i,j} \sim Tw_p(\theta_j, \lambda_j)$ i.i.d. $(i \neq 0)$, $Y_{0,j} \sim Tw_p(\theta_j, \lambda_0)$ independent of $Y_{i,j}$.

- The intuition for this construction is that there is a common component $Y_{0,j}$, representative of systematic mortality.
  ↳ The value of $Y_{0,j}$ impacts each life in pool j.
- The individual component $Y_{i,j}$ is representative of idiosyncratic mortality.
  ↳ The parameters θ_j, λ_j that govern this component describe the general risk characteristics of the pool.

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Parameter Estimation

We estimate θ_j, λ_j and predict $Y_{0,j}$ using the method of moments.

↳ We condition on $Y_{0,j}$ since we consider one pool.

$$E[a_1(\mathbf{T}_j)|Y_{0,j}] = Y_{0,j} + E[a_1(\mathbf{Y}_j)] = Y_{0,j} + \lambda_j \kappa'(\theta_j), \quad (1)$$
$$E[\widetilde{m}_2(\mathbf{T}_j)|Y_{0,j}] = E[\widetilde{m}_2(\mathbf{Y}_j)] = \lambda_j \kappa''(\theta_j), \quad (2)$$
$$E[\widetilde{m}_3(\mathbf{T}_j)|Y_{0,j}] = E[\widetilde{m}_3(\mathbf{Y}_j)] = \lambda_j \kappa'''(\theta_j), \quad (3)$$

where a_k and $\widetilde{m}_k$ denote raw and unbiased central sample moments, respectively.

Note that the above central sample moments do not depend on $Y_{0,j}$.

↳ As a result, equations (2) and (3) can be used to estimate θ_j and λ_j.

$\rightarrow$ In order to estimate λ_0, multiple pools are required.

Truncated Observations

Our interest is on the effect of dependence on annuitants, where the focus is typically on retirement planning (e.g. ages 60+).

↳ We may consider translation or truncation.

→ Translation restricts the set of appropriate distributions.

↳ Translation results in residual lifetimes, which are conditional variables.

⇒ Truncation is more appropriate.

We develop parameter estimation that incorporates truncated variables founded on the method of moments.

↳ We require the ability to adjust moments to account for truncation.

Proposition (Additive Truncation Adjustments)

Consider $Y \sim ED(\theta, \lambda)$ *with survival function* $\overline{F}(y; \theta, \lambda)$. *Define the associated truncated random variable* ${}_\tau Y = Y | Y > \tau$, *where* $\tau \geq 0$. *The first raw moment and the second and third central moments of* ${}_\tau Y$ *are given by*

$$\begin{aligned} \alpha_1({}_\tau Y) &= \alpha_1(Y) + g_1(\tau), \\ \mu_2({}_\tau Y) &= \mu_2(Y) + g_2(\tau) - g_1(\tau)^2, \\ \mu_3({}_\tau Y) &= \mu_3(Y) + g_3(\tau) - 3g_2(\tau)g_1(\tau) + 2g_1(\tau)^3, \end{aligned}$$

where

$$g_k(\tau) = g_k(\tau; \theta, \lambda) = \frac{1}{\overline{F}(\tau; \theta, \lambda)} \frac{\partial^k \overline{F}(\tau; \theta, \lambda)}{\partial \theta^k}, \quad k = 1, 2, 3.$$

Sketch of Proof

We differentiate the survival function with respect to θ at point τ.

↳ Note that $\alpha_1(Y) = \lambda\kappa'(\theta)$.

$$
\begin{aligned}
\frac{\partial \overline{F}(\tau;\theta,\lambda)}{\partial\theta} &= \frac{\partial}{\partial\theta}\int_\tau^\infty e^{[\theta x - \lambda\kappa(\theta)]} dQ_\lambda(x) \\
&= \int_\tau^\infty (x - \lambda\kappa'(\theta)) e^{[\theta x - \lambda\kappa(\theta)]} dQ_\lambda(x) \\
&= \int_\tau^\infty x e^{[\theta x - \lambda\kappa(\theta)]} dQ_\lambda(x) - \lambda\kappa'(\theta)\overline{F}(\tau;\theta,\lambda) \\
&= (\alpha_1({}_\tau Y) - \alpha_1(Y))\overline{F}(\tau;\theta,\lambda).
\end{aligned}
$$

$$
\Rightarrow \alpha_1({}_\tau Y) = \alpha_1(Y) + \frac{1}{\overline{F}(\tau;\theta,\lambda)} \frac{\partial \overline{F}(\tau;\theta,\lambda)}{\partial\theta}.
$$

Introducing A Simplified Scenario

If we proceed as before, using modified versions of equations (1)–(3), the resulting system of non-linear equations is highly unstable.

↳ $Y_{0,j}$ is present in all equations, rendering the system irreducible.

We make some simplifying assumptions.

- Assume all pools share the same risk characteristics and truncation point.
    - $\theta_j \equiv \theta$, $\lambda_j \equiv \lambda$, and $\tau_j \equiv \tau$.

↳ Note that this does not imply that all pools have the same dependence.

The approach: one 3×3 system $\rightsquigarrow$ two 2×2 systems.

↳ In this way we require only g_1 and g_2.

Step 1: Consider the global sample, ${}_{\tau}\mathbf{T}$, to estimate θ.

→ This is a pragmatic way to postpone consideration of $Y_{0,j}$.

↳ We are left with two unknowns, θ, and $\tilde{\lambda} = \lambda_0 + \lambda$.

⇒ Using an iterative global algorithm, we obtain $\widehat{\theta}$.

Step 2: Consider pool j, ${}_{\tau}\mathbf{T}_j$, to estimate λ and predict $Y_{0,j}$.

→ We make use of estimate $\widehat{\theta}$.

⇒ Using an iterative local algorithm, we obtain $\widehat{\lambda}^{(j)}$ and $\widehat{Y}_{0,j}$.

Step 3: Obtain $\widehat{\lambda}$ by averaging $\widehat{\lambda}^{(j)}$, and $\widehat{\lambda}_0$ using predictions $\widehat{Y}_{0,j}$ and $\widehat{\theta}$.

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The Additive Adjustment Functions

Recall the additive adjustment functions g_k,

$$g_k(\tau) = g_k(\tau; \theta, \lambda) = \frac{1}{\overline{F}(\tau; \theta, \lambda)} \frac{\partial^k \overline{F}(\tau; \theta, \lambda)}{\partial \theta^k}.$$

→ Looks very nice using the EDF parametrization!

- ↳ Usefulness depends on the availability of a numerical Tweedie package.
- ↳ Generally, we require functions for:
  - the distribution,
  - the density, and
  - the first derivative of the density.

Alternatively, we may investigate g_k for some specific distributions:

1. the normal distribution,
2. the gamma distribution,
3. the inverse Gaussian distribution,
4. the compound Poisson distribution.

The Normal Distribution ($p = 0$)

Formulate the normal distribution using Tweedie parametrization:

$$Y \sim Tw_{p=0}(\theta, \lambda) \qquad \Leftrightarrow \qquad Y \sim N(\mu = \theta\lambda, \sigma^2 = \lambda).$$

Using this equivalence, we have that

$$\overline{F}_Y(\tau; \theta, \lambda) = \bar{\Phi}((\tau - \theta\lambda)/\sqrt{\lambda}),$$

where $\bar{\Phi}(x)$ is the standard normal survival function.

↳ It is straightforward to obtain g_1 and g_2.

$$\begin{aligned} g_1(\tau; \theta, \lambda) &= \sqrt{\lambda}\, \frac{\varphi((\tau - \lambda\theta)/\sqrt{\lambda})}{\bar{\Phi}((\tau - \lambda\theta)/\sqrt{\lambda})}, \\ g_2(\tau; \theta, \lambda) &= -\lambda\, \frac{\varphi'((\tau - \lambda\theta)/\sqrt{\lambda})}{\bar{\Phi}((\tau - \lambda\theta)/\sqrt{\lambda})}, \end{aligned}$$

where $\Phi(x)$ and $\varphi(x)$ are the standard normal distribution and density functions.

The Normal Distribution – Numerical Results

We simulate truncated multivariate normal lifetimes where

$$\begin{aligned} Y_{i,j} &\sim Tw_{p=0}(\theta = 0.2, \lambda = 375) \equiv N(\theta\lambda = 75, \lambda = 375), \\ Y_{0,j} &\sim Tw_{p=0}(\theta = 0.2, \lambda_0 = 25) \equiv N(\theta\lambda_0 = 5, \lambda_0 = 25). \end{aligned}$$

Consequently, we have that each individual lifetime is normally distributed with mean 80 and standard deviation 20,

$$T_{i,j} \sim Tw_{p=0}(\theta = 0.2, \tilde{\lambda} = \lambda + \lambda_0 = 400) \equiv N(\theta\tilde{\lambda} = 80, \tilde{\lambda} = 400).$$

The global algorithm, $\tau = 60$.

N	10K	1K	10K
M	50	1K	1K
$\widehat{\tilde{\lambda}}$	393	403	401
$\widehat{\theta}$	0.202	0.197	0.199

The local algorithm, $Y_0 = 5$.

N	100	1K	10K	1M
$\widehat{Y}_0$	3.5	10.4	6.6	5.0
$\widehat{\lambda}$	378	349	369	375

The Gamma Distribution ($p = 2$)

Formulate the gamma distribution using Tweedie parametrization:

$$Y \sim Tw_{p=2}(\theta, \lambda) \qquad \Leftrightarrow \qquad Y \sim \Gamma(\alpha = \lambda, \beta = -\theta).$$

Using this equivalence, we have that

$$\overline{F}_Y(\tau; \theta, \lambda) = \frac{(-\theta)^\lambda}{\Gamma(\lambda)} \int_\tau^\infty x^{\lambda-1} e^{\theta x} dx.$$

Differentiation with respect to θ yields,

$$\begin{aligned} g_1(\tau; \theta, \lambda) &= \frac{\lambda}{\theta}\Big(1 - K_1(\tau; \theta, \lambda)\Big), \\ g_2(\tau; \theta, \lambda) &= \frac{\lambda}{\theta^2}\Big((\lambda - 1) - 2\lambda K_1(\tau; \theta, \lambda) + (\lambda + 1) K_2(\tau; \theta, \lambda)\Big), \end{aligned}$$

where

$$K_k(\tau; \theta, \lambda) = \frac{\overline{F}_Y(\tau; \theta, \lambda + k)}{\overline{F}_Y(\tau; \theta, \lambda)}, \quad k = 1, 2.$$

$\Rightarrow$ Requires only the gamma distribution function!

The Gamma Distribution – Numerical Results

We simulate truncated multivariate gamma lifetimes where

$$\begin{aligned} Y_{i,j} &\sim Tw_{p=2}(\theta = -0.2, \lambda = 15) \equiv \Gamma(\lambda = 15, \beta = 0.2), \\ Y_{0,j} &\sim Tw_{p=2}(\theta = -0.2, \lambda_0 = 1) \equiv \Gamma(\lambda_0 = 1, \beta = 0.2). \end{aligned}$$

Consequently, we have that each individual lifetime is gamma distributed with mean 80 and standard deviation 20,

$$T_{i,j} \sim Tw_{p=2}(\theta = -0.2, \tilde{\lambda} = \lambda + \lambda_0 = 16) \equiv \Gamma(\tilde{\lambda} = 16, \beta = 0.2).$$

The global algorithm, $\tau = 60$.

N	10K	1K	10K
M	50	1K	1K
$\widehat{\tilde{\lambda}}$	16.16	15.97	15.97
$\widehat{\theta}$	-0.204	-0.200	- 0.201

The local algorithm, $Y_0 = 5$.

N	100	1K	10K	1M
$\widehat{Y}_0$	22.6	7.6	3.5	5.4
$\widehat{\lambda}$	12.0	14.6	15.3	14.9

The Inverse Gaussian Distribution ($p = 3$)

Formulate the inverse Gaussian using Tweedie parametrization:

$$Y \sim Tw_{p=3}(\theta, \lambda) \quad \Leftrightarrow \quad IG(\mu = \lambda/\sqrt{-2\theta}, \phi = \lambda^2),$$

where μ and ϕ are the mean and the shape parameter.

$$\Rightarrow \overline{F}_Y(\tau; \theta, \lambda) = 1 - \Phi(z_1(\tau)) - e^{2\lambda\sqrt{-2\theta}}\Phi(z_2(\tau)),$$

where

$$z_1(\tau) = z_1(\tau; \theta, \lambda) = \sqrt{-2\tau\theta} - \frac{\lambda}{\sqrt{\tau}}, \quad z_2(\tau) = z_2(\tau; \theta, \lambda) = -\sqrt{-2\tau\theta} - \frac{\lambda}{\sqrt{\tau}}.$$

Differentiation with respect to θ yields,

$$\begin{aligned} g_1(\tau; \theta, \lambda) &= \frac{\sqrt{\tau}\varphi(z_1(\tau)) - e^{2\lambda\sqrt{-2\theta}}\left(\sqrt{\tau}\varphi(z_2(\tau)) - 2\lambda\Phi(z_2(\tau))\right)}{\sqrt{-2\theta}(1 - \Phi(z_1(\tau)) - e^{2\lambda\sqrt{-2\theta}}\Phi(z_2(\tau)))}, \\ g_2(\tau; \theta, \lambda) &= \frac{\tau\varphi'(z_1(\tau)) - \sqrt{\frac{\tau}{-2\theta}}\varphi(z_1(\tau)) - e^{2\lambda\sqrt{-2\theta}}h(\tau; \theta, \lambda)}{2\theta(1 - \Phi(z_1(\tau)) - e^{2\lambda\sqrt{-2\theta}}\Phi(z_2(\tau)))}. \end{aligned}$$

The Inverse Gaussian Distribution – Numerical Results

We simulate truncated multivariate inverse Gaussian lifetimes where

$$\begin{aligned} Y_{i,j} &\sim Tw_{p=3}(\theta = -0.1, \lambda = 33.54) \equiv IG(\mu = 75, \phi = 1125), \\ Y_{0,j} &\sim Tw_{p=3}(\theta = -0.1, \lambda_0 = 2.24) \equiv IG(\mu = 5, \phi = 5). \end{aligned}$$

Consequently, we have that each individual lifetime is inverse Gaussian distributed with mean 80 and standard deviation 20,

$$T_{i,j} \sim Tw_{p=2}(\theta = -0.1, \tilde{\lambda} = \lambda + \lambda_0 = 35.78) \equiv IG(\mu = 80, \phi = 1280).$$

The global algorithm, $\tau = 60$.

N	10K	1K	10K
M	50	1K	1K
$\widehat{\tilde{\lambda}}$	36.06	35.65	35.47
$\widehat{\theta}$	-0.102	-0.1	-0.099

The local algorithm, $Y_0 = 5$.

N	100	1K	10K	1M
$\widehat{Y}_0$	17.0	16.5	1.5	4.8
$\widehat{\lambda}$	29.4	28.8	35.0	33.6

The Compound Poisson Distribution ($p \in (1, 2)$)

Formulate the compound Poisson using Tweedie parametrization:

$$Y \sim Tw_{p\in(1,2)}(\theta, \lambda) \quad \Leftrightarrow \quad CP(N \sim P(\lambda\kappa(\theta)), S_1 \sim \Gamma(-\alpha, -\theta)),$$

where $\lambda\kappa(\theta)$, $-\alpha$, and $-\theta$ are the mean of the Poisson distribution and the shape and rate parameters of the gamma distribution, respectively.
↳ Recall, $\alpha = (p - 2)/(p - 1)$, therefore, $\alpha \in (-\infty, 0)$.

→ Y may be written as a random sum of independent and identically distributed gamma random variables,

$$Y = S_1 + S_2 + \ldots + S_N.$$

Therefore, we have,

$$\overline{F}_Y(\tau; \theta, \lambda) = 1 - \sum_{n=0}^{\infty} P_\theta[S_n \leq \tau] P_\theta[N = n],$$

where $S_n \sim \Gamma(-\alpha n, -\theta)$.

The Compound Poisson – Numerical Results

We simulate truncated multivariate compound Poisson lifetimes where

$$
\begin{aligned}
Y_{i,j} &\sim Tw_{p=1.5}(\theta = -0.4, \lambda = 3) \equiv CP(N \sim P(30), S_1 \sim \Gamma(1, 0.4)), \\
Y_{0,j} &\sim Tw_{p=1.5}(\theta = -0.4, \lambda_0 = 0.2) \equiv CP(N \sim P(2), S_1 \sim \Gamma(1, 0.4)).
\end{aligned}
$$

Consequently, we have that each individual lifetime is compound Poisson distributed with mean 80 and standard deviation 20,

$$T_{i,j} \sim Tw_{p=1.5}(\theta = -0.4, \tilde{\lambda} = 3.2) \equiv CP(N \sim P(32), S_1 \sim \Gamma(1, 0.4)).$$

The global algorithm, $\tau = 60$.

N	10K	1K	10K
M	50	1K	1K
$\widehat{\tilde{\lambda}}$	3.00	3.18	3.15
$\widehat{\theta}$	-0.403	-0.399	-0.398

The local algorithm, $Y_0 = 5$.

N	100	1K	10K	1M
$\widehat{Y}_0$	37.6	9.1	5.9	4.8
$\widehat{\lambda}$	1.86	2.84	2.98	3.0

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Thank you!