

# Scenario Aggregation for Solvency Regulation

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# Outline

## Scenario Aggregation in the Swiss Solvency Test

### Minimum $\phi$ -Divergence Approach

Minimum  $\phi$ -Divergence Scenario Aggregation

Robustness of Capital Requirement

Solving the Optimization Problem

### Case Studies

Case Study 1

Case Study 2

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# Internal Models

- ▶ Probability space  $(\Omega, \mathcal{F}, \mathbb{P})$
  - ▶ Annual loss  $L$ : random variable assigning a loss  $L(\omega)$  to any possible state of the world  $\omega \in \Omega$
- $\hookrightarrow (\Omega, \mathcal{F})$  is universal
- $\hookrightarrow \mathbb{P}$  and  $L$  are insurer specific (internal model)

## Assumption:

Regulator trusts the mapping  $L : \Omega \rightarrow \mathbb{R}$  (no ambiguity)

## Fact:

Regulator wants to challenge  $\mathbb{P}$ , or distribution  $F_L(x) = \mathbb{P}[L \leq x]$

# Scenarios

A scenario is an event  $S \in \mathcal{F}$ : *a narrative description of a possible state of the world*

For factor models, this notion includes

- ▶ **point scenarios**  $S = \{\mathbf{x} \in \mathbb{R}^n \mid x_j = c_j \text{ for some } j\}$ , e.g.
  - ↪ 1-year loss of EUR 100 mio in Eurowind,
  - ↪ 1-year drop of  $-20\%$  for SMI,
- ▶ **quadrants**  $S = \{\mathbf{x} \in \mathbb{R}^n \mid x_j \geq c_j \text{ for some } j\}$ , e.g.
  - ↪ 1-year change in EURCHF  $\leq -20\%$ ,
  - ↪ 1-year change European Credit Spreads AAA  $\geq 50\%$ .

## Scenarios from a Regulatory point of view

“Mit den SST-Szenarien sollen die Mängel aus verteilungsbasierten Modellen korrigiert werden. So können analytische Modelle extreme Ereignisse oft nur unzureichend abbilden, sowohl in Bezug auf die Heavy-Tailedness der Randverteilungen als auch in Bezug auf die so genannte Tail-Dependency.”

— FINMA, *Wegleitung für die Erarbeitung des SST-Berichtes 2013*, 2012

# SST Scenario Aggregation

## Given by FINMA

- ▶ Scenarios  $S_1, \dots, S_d$  along with auxiliary probabilities  $\pi_i > 0$ 
  - ↪ In general,  $\pi_i \neq \mathbb{P}[S_i]$
  - ↪ It is natural to set  $S_0 = \Omega$  and  $\pi_0 = 1 - \sum_{i=1}^d \pi_i$

Scenario	Probability of occurrence
Industrial	0.5%
Pandemic	1%
Accident on a works outing	0.5%
Accident: Panic in a football stadium	Type 2: not relevant for target capital.
Hail scenario	Type 2: not relevant for target capital.
Disability	0.5%
Daily allowance for sickness	0.5%

Source: FOPI, *SST Technical Documents*, 2006

# SST Scenario Aggregation

## Ansatz

- ▶ Scenario  $S_i$  causes an extra-ordinary loss

$$\ell_i = \mathbb{E}[L \mid S_i] - \mathbb{E}[L]$$

to be determined by actuary, with  $\ell_0 = 0$

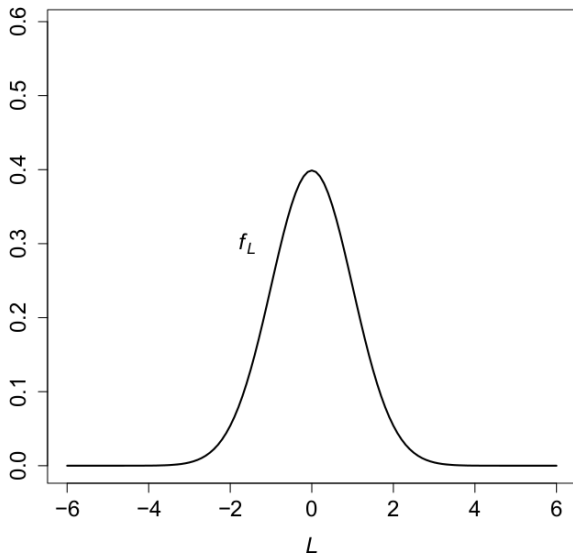
- ▶ Loss distribution conditional on scenario  $S_i$  is  $F_L(x - \ell_i)$

## Aggregation

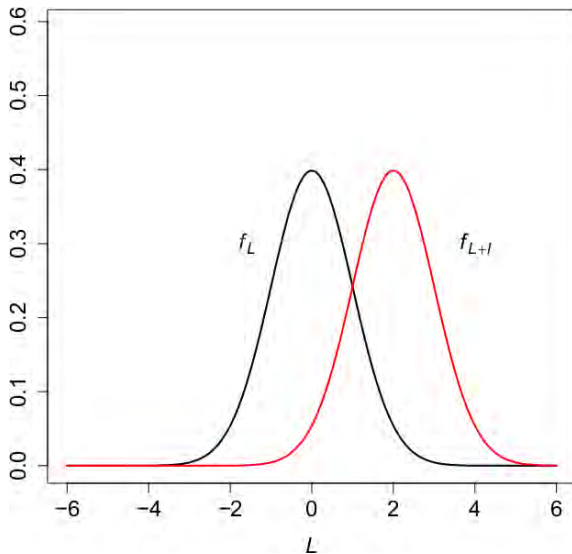
- ▶ Replace  $F_L(x)$  by aggregated loss distribution

$$F_L^{\text{aggr}}(x) = \sum_{i=0}^d \pi_i F_L(x - \ell_i)$$

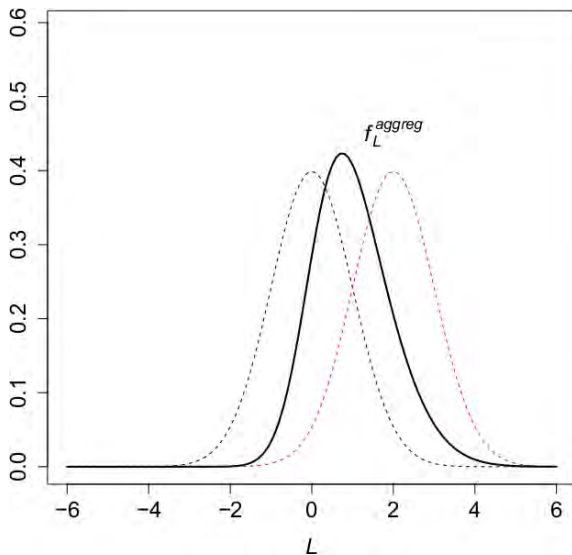
# SST Scenario Aggregation



# SST Scenario Aggregation



# SST Scenario Aggregation



## SST Method: Discussion

- ▶ Fact:  $F_L^{\text{aggr}}(x)$  is the cdf of  $L + Z$  for an independent extra-ordinary loss random variable  $Z$  with  $\mathbb{P}[Z = \ell_i] = \pi_i$
- ▶ Lemma:

$$\text{ES}^{\text{aggr}}[L] = \text{ES}[L + Z] \geq \text{ES}[L] + \mathbb{E}[Z]$$

- ▶ Consequence: if  $\mathbb{E}[Z] > 0$  then

$$\text{ES}^{\text{aggr}}[L] > \text{ES}[L]$$

no matter how conservative the internal model for  $L$  is

⇒ scenario aggregation penalizes conservative internal models

## SST Method: Discussion

- ▶ No control on how far  $F_L^{\text{aggr}}(x)$  from  $F_L(x)$  is
- ▶ No control on how far  $\text{ES}^{\text{aggr}}(L)$  from  $\text{ES}(L)$  is
- ▶ Confusion among stakeholders about “double-counting”
- ▶ High degree of subjectivity about auxiliary weights  $\pi_i$
- ▶ Focus is on the mean but not on the tail of the loss distribution
- ▶ Aggregation is on the level of  $F_L(x)$ , not  $\mathbb{P}$

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# A Moment's Reflection on Stress Tests

- ▶ Stress test: selected states of the world  $\omega_j \in \Omega$
- ▶ Leads to a maximal insurer specific loss  $\ell = \max_j L(\omega_j)$
- ▶ Internal model (null hypothesis)  $\mathbb{P}$  passes the stress test if not rejected on significance level  $1 - \alpha = 1\%$ . That is, if

$$\ell \leq \text{VaR}_\alpha(L)$$

- ▶ Equivalently,

$$\mathbb{P}[S] \geq 1 - \alpha$$

for the scenario  $S = \{L \geq \ell\}$

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# Definition of Views

- ▶ Given a collection of scenarios  $S_1, \dots, S_d \in \mathcal{F}$
- ▶ Given a vector of target probabilities  $\mathbf{c} = (c_1, \dots, c_d)^\top$
- ▶ Define  $S_0 = \Omega \setminus \cup_{i=1}^d S_i$
- ▶ Denote  $\mathcal{M} =$  set of probability measures on  $(\Omega, \mathcal{F})$
- ▶ **Views** on  $\mathbb{Q} \in \mathcal{M}$ :

$$\mathbb{Q}[S_i] \geq c_i, \quad i = 1, \dots, d \quad (\text{views})$$

## Views in Terms of Atoms

- ▶ Let  $U_0, \dots, U_n$  be the atoms of  $\mathcal{S} = \sigma(S_1, \dots, S_d)$ :

$$S_0 = U_0, \quad S_i = \cup_{j \in J(i)} U_j, \quad i = 1, \dots, d$$

- ▶ **Views** on  $\mathbb{Q} \in \mathcal{M}$ :

$$\sum_{j \in J(i)} q_j \geq c_i, \quad i = 1, \dots, d \quad (\text{views})$$

for vector  $q_j = \mathbb{Q}[U_j]$

- ▶ **Views** in matrix form:

$$A \mathbf{q} \geq \mathbf{c} \quad (\text{views})$$

for matrix  $A_{ij} = 1_{J(i)}(j)$

# Scenario Aggregation

- ▶ Modification of internal model: find minimizer for

$$\begin{array}{ll} \text{minimize} & d(Q, \mathbb{P}) \\ \text{subject to} & \text{(views)} \end{array}$$

with domain  $\mathcal{M}$

- ▶  $d(\cdot, \mathbb{P})$  measures the difference from  $\mathbb{P}$  on  $\mathcal{M}$

# $\phi$ -Divergence

- ▶  $\phi$ -divergence

$$d(\mathbb{Q}, \mathbb{P}) = \begin{cases} \mathbb{E}[\phi(d\mathbb{Q}/d\mathbb{P})], & \text{if } \mathbb{Q} \ll \mathbb{P} \\ +\infty, & \text{otherwise} \end{cases}$$

where  $\phi$  is convex, and strictly convex at 1 with  $\phi(1) = 0$

- ▶ Standard measure for difference of  $\mathbb{Q}$  from  $\mathbb{P}$  in statistics, Csiszar (1963)
- ▶ Fact:  $d(\mathbb{Q}, \mathbb{P})$  is not a metric, but convex in  $\mathbb{Q}$

# Examples and Facts

- ▶ Examples:

$$\phi(t) = \begin{cases} t \log t, & \text{relative entropy} \\ (\sqrt{t} - 1)^2, & \text{Hellinger distance} \\ |t - 1|^p, & L^p\text{-distance, } p \geq 1 \end{cases}$$

- ▶ Facts:

$$\begin{aligned} \|d\mathbb{Q}/d\mathbb{P} - 1\|_1 &\leq \sqrt{2d_E(\mathbb{Q}, \mathbb{P})} \\ d_H(\mathbb{Q}, \mathbb{P}) &\leq \|d\mathbb{Q}/d\mathbb{P} - 1\|_1 \leq \sqrt{2d_H(\mathbb{Q}, \mathbb{P})} \\ \|d\mathbb{Q}/d\mathbb{P} - 1\|_1 &= d_{TV}(\mathbb{Q}, \mathbb{P}) \quad \text{total variation} \end{aligned}$$

- ▶ Fact: all but the  $L^1$ -distance are strictly convex in  $\mathbb{Q}$

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# Robustness Check

- ▶ Question: is the capital requirement robust under minimum  $\phi$ -divergence scenario aggregation?
- ▶ In other words: is VaR and ES continuous with respect to  $d(\mathbb{Q}, \mathbb{P})$  ?

## Recall Definitions of VaR and ES

- ▶ Value at risk

$$\text{VaR}(X) = q_{\alpha}^{-}(X), \quad \text{left } \alpha\text{-quantile}$$

- ▶ Expected shortfall

$$\begin{aligned} \text{ES}(X) &= \frac{1}{1-\alpha} \mathbb{E} [(X - q)^+] + q \\ &= \frac{1}{1-\alpha} (\mathbb{E}[X \mathbf{1}_{\{X > q\}}] + q (\mathbb{P}[X \leq q] - \alpha)) \end{aligned}$$

for any  $\alpha$ -quantile  $q \in [q_{\alpha}^{-}(X), q_{\alpha}^{+}(X)]$

- ▶ Folk theorem: VaR is more robust than ES ...

# Lemma for Value at Risk

## Lemma 2.1.

If  $d\mathbb{P}_n/d\mathbb{P} \rightarrow 1$  in  $L^1$  then

$$\sup_x |\mathbb{P}_n[X \leq x] - \mathbb{P}[X \leq x]| \rightarrow 0$$

and

$$q_{\alpha}^{-}(X) \leq \liminf_n q_n \leq \limsup_n q_n \leq q_{\alpha}^{+}(X)$$

for any sequence  $(q_n)$  of  $\alpha$ -quantiles of  $X$ , for any  $X \in L^0$ .

# Non-Robustness of Value at Risk: Example

- ▶ Define  $X = 0$  or  $1$  with  $\mathbb{P}[X = 0] = \alpha$
- ▶ Define

$$d\mathbb{P}_n/d\mathbb{P} = \begin{cases} 1 + (1 - \alpha)(-1)^n/(\alpha n), & \text{on } \{X = 0\} \\ 1 + (-1)^{n+1}/n, & \text{on } \{X = 1\} \end{cases}$$

- ▶ VaR does not converge:

$$\text{VaR}_n(X) = \begin{cases} 0 = q_\alpha^-(X), & \text{for } n \text{ even} \\ 1 = q_\alpha^+(X), & \text{for } n \text{ odd} \end{cases}$$

# Robustness of Expected Shortfall

## Theorem 2.2.

Let  $p \in [1, \infty]$ . If  $d\mathbb{P}_n/d\mathbb{P} \rightarrow 1$  in  $L^p$  then

$$\text{ES}_n(X) \rightarrow \text{ES}(X)$$

for all  $X \in L^r$ , where  $p^{-1} + r^{-1} = 1$ .

Proof.

Using previous lemma and

$$(1 - \alpha) |\text{ES}_n(X) - \text{ES}(X)| \leq \mathbb{E} [|d\mathbb{P}_n/d\mathbb{P} - 1|(X - q_n)^+] \\ + \mathbb{E} [|(X - q_n)^+ - (X - q)^+|] + (1 - \alpha) |q_n - q|$$

for any converging (sub-)sequence of  $\alpha$ -quantiles  $q_n \rightarrow q$ . □

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## Back to the Optimization Problem

- Modification of internal model: find minimizer for

$$\begin{array}{ll} \text{minimize} & d(\mathbb{Q}, \mathbb{P}) \\ \text{subject to} & (\text{views}) \end{array} \quad (\text{P})$$

with domain  $\mathcal{M}$

### Lemma 2.3.

For every  $\mathbb{R} \ll \mathbb{P}$  satisfying the views there exists a

$$\mathbb{R}' \in \mathcal{Q} := \{\mathbb{Q} \ll \mathbb{P} \mid d\mathbb{Q}/d\mathbb{P} \text{ is } \mathcal{S}\text{-measurable}\}$$

satisfying the views and  $d(\mathbb{R}', \mathbb{P}) \leq d(\mathbb{R}, \mathbb{P})$ .

### Proof.

Set  $d\mathbb{R}'/d\mathbb{P} = \mathbb{E}[d\mathbb{R}/d\mathbb{P} \mid \mathcal{S}]$  and use Jensen's inequality. □

# Existence and Uniqueness

- ▶ Note  $\dim \mathcal{Q} = n + 1$ : identify  $\mathbb{Q} \in \mathcal{Q}$  with  $\mathbf{q}$  by

$$q_j = \mathbb{Q}[U_j], \quad j = 0, \dots, n$$

## Theorem 2.4.

*There exists a solution of (P) in  $\mathcal{Q}$ . Moreover, if  $\phi$  is strictly convex then the solution is unique.*

# Solution of Optimization Problem

- ▶ Define  $\mathbf{p} \in (0, 1)^{n+1}$  by  $p_j = \mathbb{P}[U_j]$
- ▶ The optimization problem (P) reduces to

$$\begin{aligned} & \text{minimize} && \sum_{j=0}^n p_j \phi(\mathbf{q}_j/p_j) \\ & \text{subject to} && A\mathbf{q} \geq \mathbf{c} \\ & && \mathbf{1}^\top \mathbf{q} = \mathbf{1} \end{aligned} \tag{P}$$

with domain  $(0, 1)^{n+1}$

- ▶ Solution via dual problem or Kuhn–Tucker (FOC) conditions
- ▶ Reference e.g. Boyd and Vandenberghe (2004)

## Special Case: $d = 1$ Scenario

### Corollary 2.5.

For  $d = 1$  scenario  $S_1$  a (the) solution to (P) is given by

$$\frac{d\mathbb{Q}^*}{d\mathbb{P}} = \frac{1 - \max\{c_1, p_1\}}{p_0} 1_{S_0} + \frac{\max\{c_1, p_1\}}{p_1} 1_{S_1}$$

independently of the choice of the (strictly) convex divergence function  $\phi$ .

Proof.

Convexity of  $\phi$  implies that  $q_1 \mapsto d(\mathbb{Q}(q_1), \mathbb{P})$  is non-decreasing in  $q_1$  for  $q_1 > p_1$ . □

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## Special Case: Stress Testing ( $d = 1$ Scenario)

- ▶ Recall stress testing is equivalent to  $d = 1$  views

$$\mathbb{Q}[S_1] \geq 1 - \alpha$$

on the scenario  $S_1 = \{L \geq \ell\}$  with  $\ell = \max_i L(\omega_i)$

- ▶ In this case we obtain a closed form expression for ES:

### Corollary 2.6.

*The expected shortfall under  $\mathbb{Q}^*$  given in Corollary 2.5 satisfies*

$$\text{ES}_{\mathbb{Q}^*, \alpha}(L) = \text{ES}_{\mathbb{P}, \max\{\mathbb{P}[L < \ell], \alpha\}}(L).$$

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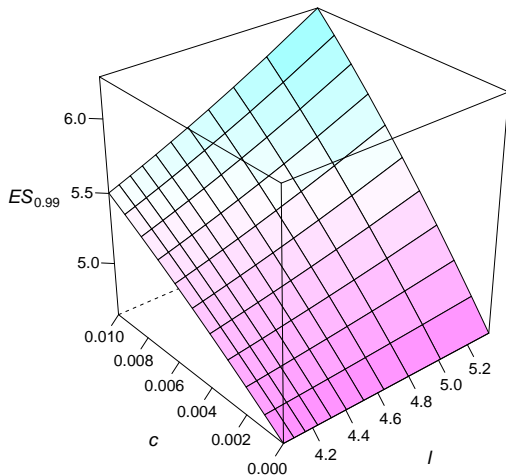
Case Study 2

## Case Study 1: Setup

- ▶ Loss  $L \sim \mathcal{N}(0, 10.2)$
- ▶ Compare scenario aggregation using SST and minimum entropy method

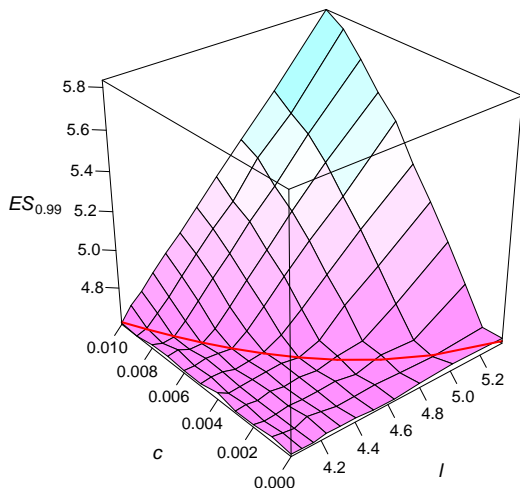
## Case Study 1: SST Method

- ▶  $d = 1$  scenario with probability  $c$  and extra-ordinary loss  $\ell$
- ▶ Recall:  $F_L^{\text{aggr}}(x) = (1 - c) F_L(x) + c F_L(x - \ell)$



## Case Study 1: Minimum Entropy Method

- ▶  $d = 1$  scenario  $S_1 = \{L \geq \ell\}$
- ▶ View:  $\mathbb{Q}[L \geq \ell] \geq c$  for some auxiliary level  $c$



# Case Study 1: Difference SST - Minimum Entropy Method

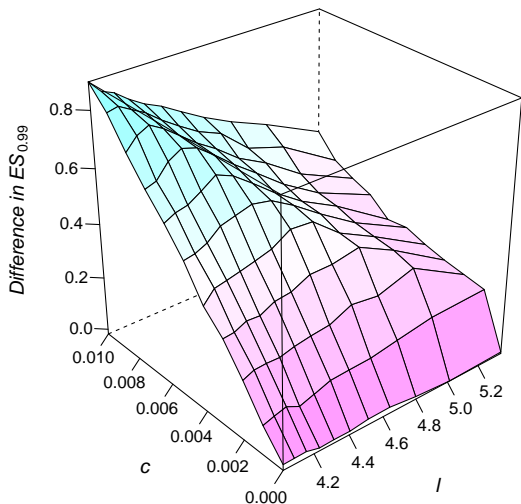
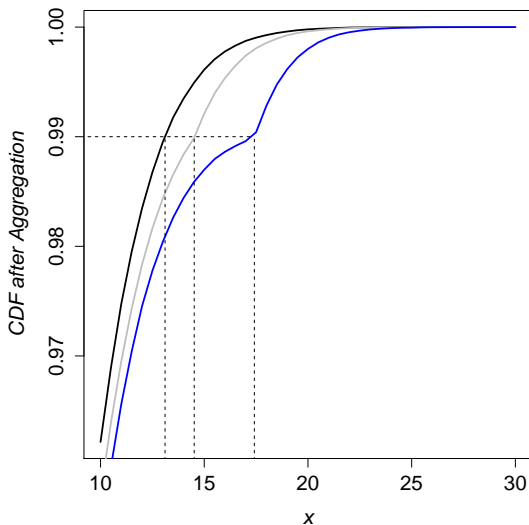


Figure :  $ES^{\text{aggr}}[L] - ES_{Q^*}[L]$  as function of  $c$  and  $l$

## Case Study 1 (Stress Test): Minimum Entropy Method



**Figure :** Impact on cumulative distribution function for  $\ell = \text{VaR}_{\alpha}(L)$  with  $\alpha = 0.99$  (black),  $\alpha = 0.995$  (grey),  $\alpha = 0.999$  (blue)

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## Case Study 2: Setup

- ▶ Two risk factors ( $X_1, X_2$ ) normal with mean  $\mathbf{0}$ , and  $\text{var}(X_1) = 1$ ,  $\text{var}(X_2) = 4$ , and  $\text{corr}(X_1, X_2) = -0.5$
- ▶  $X_1$ : change in interest rates
- ▶  $X_2$ : risk factor related to CAT events with reinsurance
- ▶ Loss

$$L = \max\{X_1, -1\} + \max\{\min\{X_2, 5\}, -1\}$$

is capped in  $X_2$  (reinsurance), and gains are capped at 1

- ▶  $d = 2$  scenarios

$$S_1 = \{X_1 \geq 1, X_2 \geq 1\} \quad \text{and} \quad S_2 = \{X_1 < -2\}$$

## Case Study 2: Shortfall Region

- ▶ Shortfall region  $W = \{L > \text{VaR}_{0.99}(L)\}$  overlaps with  $S_1$ ,

$$W \cap S_1 \neq \emptyset$$

but not with  $S_2$ ,

$$W \cap S_2 = \emptyset$$

- ▶ Conditional expected losses given  $S_1$  and  $S_2$  are positive

$$\ell_1 = \mathbb{E}[L \mid S_1] = 3.2, \quad \ell_2 = \mathbb{E}[L \mid S_2] = 1.32$$

- ▶ SST aggregation of  $S_2$  leads to a capital increase even though  $S_2$  does not intersect with the shortfall region  $W$

## Case Study 2: Scenarios and Shortfall Region

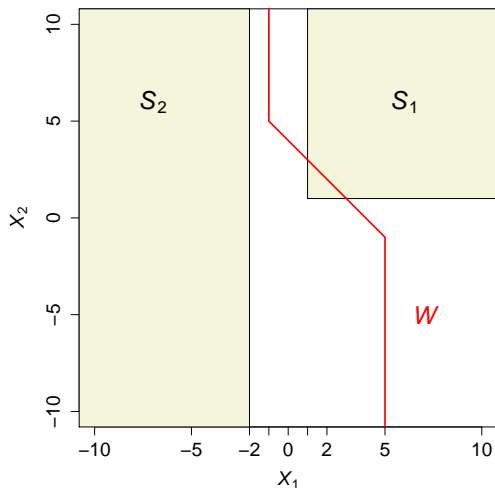


Figure : Scenarios  $S_1$ ,  $S_2$ , shortfall region  $W$

## Case Study 2: SST Method

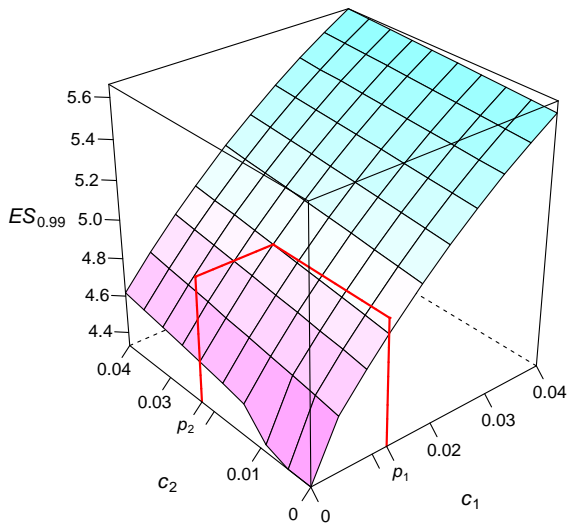


Figure : Sensitivity of  $ES^{\text{aggr}}[L]$  with respect to  $c_1$  and  $c_2$

## Case Study 2: Minimum Entropy Method

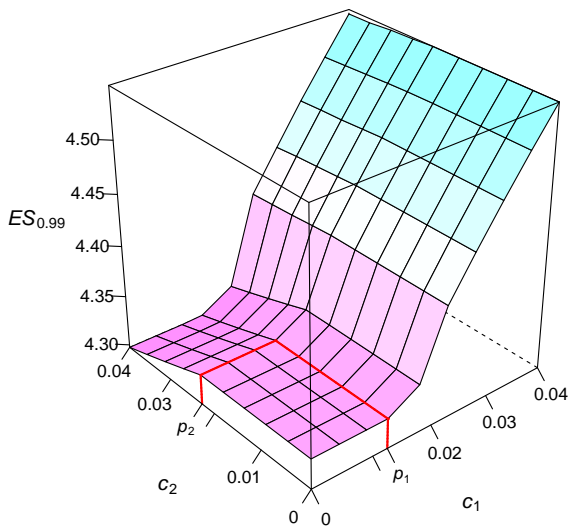


Figure : Sensitivity of  $ES_{Q^*} [L]$  with respect to  $c_1$  and  $c_2$

## Case Study 2: Difference SST - Minimum Entropy Method

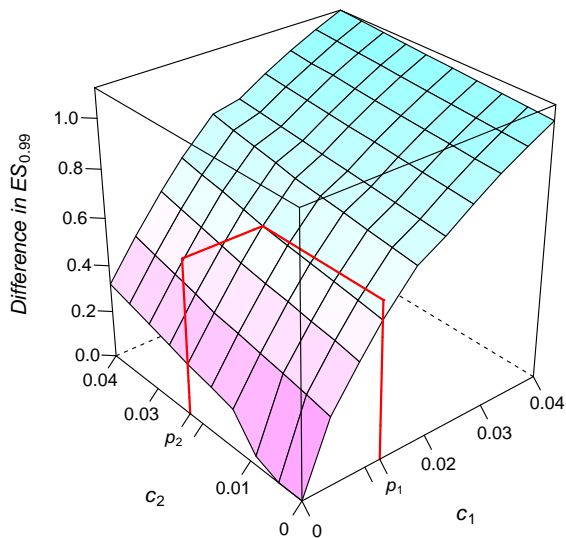


Figure :  $ES^{\text{aggr}}[L] - ES_{Q^*}[L]$  as function of  $c_1$  and  $c_2$

# Conclusion

- ▶ Scenario aggregation is vital part of
  - ▶ risk-based solvency regulation
  - ▶ internal risk modelling
- ▶ Current (SST) methods subject to critical review
- ▶ Minimum  $\phi$ -divergence approach is a coherent scenario aggregation method:
  - ▶ No penalty for conservative internal models
  - ▶ Focus on tail events
  - ▶ Control over distance from internal model
  - ▶ Robustness of capital requirement
  - ▶ Highly tractable (closed form solutions sometimes)