Scenario Aggregation
(working paper)

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This paper elaborates on scenario aggregation for regulatory purposes. The existing approach proposed with the *Swiss Solvency Test* (SST) is presented and discussed. We then propose a more general and coherent framework for scenario aggregation based on divergence from the reference probability measure subject to scenario constraints. The continuity of standard risk measures with respect to changes in the reference probability measure is discussed. This new scenario aggregation approach is the illustrated with examples.

1 Background

The last decades have seen strong developments in the statistical measurement of risk. The quantitative methods used by banks and insurances for risk management serve many purposes such as capital allocation or reporting to regulators. The latter have required regulated institutions to implement and document internal models that, once approved, should be used to report their amount of capital which is bearing the risk and to show that they would remain solvent in case of extreme scenarios.

Although the risk modeling methodology of an insurance company's internal model is reported and subject to approval, model risk remains inherent and should therefore be challenged. The risk of inappropriate modeling can be raised at many levels. One could question a specific choice of risk factors, the marginal distribution of these risk factors or the dependence structure between them (see, e.g., [Embrechts et al., 2012]). Model risk assessment is therefore of crucial importance. In consequence, both the regulators and the regulated entities themselves have understood the necessity to complement statistical risk assessment with scenarios and stress tests.

Each regulator has its own view on how stress tests and scenarios should be defined and implemented. On the banking side, the Basel Committee on Banking Supervision, see e.g, [BCBS, 2005], requires banks to perform stress tests which can then be used to set associated capital charges, see e.g., [BCBS, 2008]. On the insurance side, although Solvency II does not require inclusion of results of scenarios in the capital calculation, it is a predominant point of the Swiss Solvency Test (SST) implementation, see [FOPI, 2006].

2 Internal Models

An internal model is formalized as random annual loss variable L defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, which assigns to any possible state of the world $\omega \in \Omega$ a loss $L(\omega)$. While $(\Omega, \mathcal{F})$ can be viewed as universal object, the probability measure $\mathbb{P}$ and the random variable L are insurer specific. Stress testing and scenario aggregation will be done via modifying the measure $\mathbb{P}$, or alternatively, the loss distribution function $F_L(x) = \mathbb{P}[L \leq x]$.

The insurance company would then be required by regulators to maintain an amount of capital, that serves as a buffer for these possible losses. This amount of capital can, for example, be computed via risk measures

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such as the Value-at-Risk (VaR) at level α, $\mathrm{VaR}_\alpha(L)$, or the expected shortfall at level α, $\mathrm{ES}_\alpha(L)$. Details on these risk measures can be found in, e.g. [McNeil et al., 2005] Section 2.2.4 or [Föllmer and Schied, 2011] Section 4.4.

The *required capital* (RC) required to be maintained by the Swiss Solvency Test is given by $\mathrm{ES}_\alpha(L)$ with $\alpha = 99\%$, see [FOPI, 2006].

3 SST Scenario Aggregation Method

One SST requirement is to evaluate a given list of d scenarios that have a small probability of occurrence, and that would have a negative effect on the annual loss L. Each scenario is a narrative description of a potential, and typically extremal, event. This includes, a market crash, or a pandemic or natural catastrophe, etc. Scenario i comes along with an auxiliary probability weight c_i set subjectively by the regulator. It causes an extra-ordinary loss $\ell_i \geq 0$ to be determined by the company's actuary. The stressed loss distribution function conditional on scenario i is then set to be $F_L(x - \ell_i)$. Scenario aggregation is then done via mixing, which leads to the following modified loss distribution

$$F_L^{\mathrm{aggr}}(x) = c_0\, F_L(x) + \sum_{i=1}^{d} c_i\, F_L(x - \ell_i)$$

where $c_0 = 1 - \sum_{i=1}^{d} c_i$ is the implied probability weight for a "normal unstressed year.

Note that $F_L^{\mathrm{aggr}}(x)$ is the distribution function of $L+S$, where S is a discrete random variable, independent of L, that takes values $\ell_0 = 0, \ell_1, \ldots, \ell_d$ with probabilities c_i, $i = 0, \ldots, d$, respectively. We note $q_\alpha(X)$ and $\mathrm{ES}_\alpha(X)$ the quantile and expected shortfall of the random variable X, see Section 5 for more details and definitions. Assuming that L has a continuous distribution, $L + S$ will also be continuous, and using the Definition 5.2, we note

$$\begin{aligned}
(1-\alpha)\left[\mathrm{ES}_\alpha(L+S) - \mathrm{ES}_\alpha(L)\right] &= \sum_{i=0}^{d} c_i \mathbb{E}\left((L+s_i)\,\mathbf{1}_{\{L+s_i \geq q_\alpha(L+S)s_i\}}\right) - \mathbb{E}\left(L\,\mathbf{1}_{\{L \geq q_\alpha(L)\}}\right) \\
&= \mathbb{E}\left(\sum_{i=0}^{d} c_i(L+s_i)\left(\mathbf{1}_{\{L+s_i \geq q_\alpha(L+S)\}} - \mathbf{1}_{\{L \geq q_\alpha(L)\}}\right)\right) + \sum_{i=0}^{d} c_i s_i (1-\alpha) \\
&\geq q_\alpha \left(\sum_{i=0}^{d} c_i \mathbb{E}\left(\mathbf{1}_{\{L+s_i \geq q_\alpha(L+S)\}}\right) - \mathbb{E}\left(\mathbf{1}_{\{L \geq q_\alpha(L)\}}\right)\right) + \mathbb{E}(S)(1-\alpha) \\
&= \mathbb{E}(S)(1-\alpha).
\end{aligned}$$

Combining this statement with the subadditivity property of the expected shortfall we get

$$\mathrm{ES}_\alpha(L) + \mathbb{E}(S) \leq \mathrm{ES}_\alpha(L+S) \leq \mathrm{ES}_\alpha(L) + \mathrm{ES}_\alpha(S). \tag{1}$$

3.1 Discussion

The approach described above on the incorporation of scenarios in the required capital calculation is perfectible and can be criticized on several points.

3.1.1 Scenarios and probability definition

The definition of scenarios in the SST methodology is entirely prescribed by the regulators and is therefore strongly subjective. In particular,

- scenario probabilities or the scenarios themselves can be unrealistic or unjustified;

- if the regulator decides to change the scenarios definition and their probabilities from a year to another, this might lead to substantial moves of capital over the entire insurance industry;
- scenarios should be adapted in periods of distress;
- scenarios impacts and their probabilities should be reasonably easy to track and report for insurance companies, probabilities of point mass scenarios are generally difficult to determine for an insurer (if possible at all).

Despite these pitfalls it should be acknowledged that, because they are purely subjective, these scenarios avoid the model risk that would be involved in the choice of a risk factor distribution. In any case, the idea of choosing scenarios and their probabilities in a subjective way has been criticised from both the academic research and industry. Recent works focus on a more systematic approach in choosing scenarios for stress-testing, see, e.g., [Breuer and Csiszár, 2010] or [McNeil and Smith, 2012].

From a regulator point of view, it is important to treat equally all regulated institutions so it is reasonable to require that the scenarios to evaluate should be the same for all. In addition, this would allow any external third party to have access to a full definition of such scenarios.

3.1.2 Scenario aggregation

The theoretical approach of scenarios aggregation as proposed by SST should also be criticised.

- the fact that scenarios impacts are modeled as shift in the one-year loss distribution is too simplistic and does not fulfill any specific requirement;
- the scenarios impact should focus on the tail of the one-year loss distribution, not on its mean;
- there is no control on how far $F_L^{\text{aggr}}(x)$ from $F_L(x)$ is;
- the scenarios aggregation approach will penalize *any* model to which it is applied, including the most conservative ones. Indeed, we have that $\mathrm{ES}_\alpha(L+S) > \mathrm{ES}_\alpha(L)$ as long as $\mathbb{E}(S) > 0$ from Equation (1).

Such flaws in the scenarios aggregation methodology proposed by SST are due to a non-appropriate mathematical modeling. Scenario aggregation for scenarios defined on risk-factors has been studied in, e.g. [Meucci, 2008], [Meucci, 2010] or [McNeil and Smith, 2012].

In order to define a coherent mathematical approach for scenarios aggregation, we however need to clearly define the regulatory requirement that the scenario aggregation serves. It should indeed be stated by the regulator what condition should an internal model satisfy once evaluated with the scenarios. In the case where the condition is not fulfilled, an appropriate aggregation method would then correct the internal model. From this starting point, we propose a new approach to scenario aggregation in the next section.

4 A New Framework

Point of departure is a moment's reflection on stress tests. A stress test is based on a selected universal state of the world (point-scenario) $\omega \in \Omega$, which leads to an insurance specific loss of $\ell = L(\omega)$. The internal model (null hypotheses $\mathbb{P}$) is said to pass the stress test if it is not rejected based on the resulting loss ℓ for a pre-specified significance level of, say, $1-\alpha = 1\%$. This means $\ell \leq \mathrm{VaR}_\alpha(L)$, or equivalently, $\mathbb{P}(S) \geq 1-\alpha$, for the event (scenario) $S = \{L \geq \ell\}$. If $\mathbb{P}$ does not satisfy this constraint, it needs to be replaced by some modified measure $\mathbb{Q}$ which satisfies the constraint and is within minimal divergence from $\mathbb{P}$.

This leads to the following formalization. Given are d possibly overlapping scenarios $S_1, \ldots, S_d \in \mathcal{F}$, $S_0 := \Omega \setminus \cup_{i=1}^d S_i$. Denote $\mathcal{S} := \sigma(S_1, \ldots, S_d)$ the σ-field generated by the scenarios. Let $U_0, \ldots, U_n$ be the unique (disjoint) atoms of $\mathcal{S}$. That is, for every i there exists an index set $J(i) \subset \{1, \ldots, n\}$ such that $S_i = \cup_{j \in J(i)} U_j$, and $U_0 = S_0$, and we have $\mathcal{S} := \sigma(U_0, \ldots, U_n)$. Denote by $\mathbf{p} \in \mathbb{R}_+^{n+1}$ the vector of probabilities $p_j = \mathbb{P}[U_j]$.

Remark 4.1. *If the scenarios S_i are mutually disjoint then $n = d$ and $S_i = U_i$, for all $i = 1, \ldots, d$.*

The views on a probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ are defined by an auxiliary vector of minimal (or target) probabilities $\mathbf{c} \in \mathbb{R}_+^d$ with $\mathbf{1}^\top \mathbf{c} \leq \mathbf{1}$ as

$$\mathbb{Q}[S_i] \geq c_i, \quad i = 1, \ldots, d. \tag{2}$$

In compact notation this reads $\sum_{j \in J(i)} q_j \geq c_i$, or in matrix form

$$A\mathbf{q} \geq \mathbf{c}$$

for the vector $\mathbf{q} \in \mathbb{R}_+^{n+1}$ of probabilities $q_j = \mathbb{Q}[U_j]$, and for the $d \times (n+1)$-matrix A defined as $A_{ij} = 1$ if $j \in J(i)$ and 0 otherwise, $i = 1, \ldots, d$, $j = 0, \ldots, n$. Note that the first column of A consists of zeros only: $A_{i0} = 0$ for all i.

For a given divergence function d, the general problem of scenario aggregation can now be defined as finding a minimizer of the problem

$$\begin{aligned} \text{minimize} \quad & d(\mathbb{Q}, \mathbb{P}) \\ \text{subject to} \quad & \text{views (2)} \end{aligned} \tag{3}$$

with domain $\mathcal{M}$, the set of all probability measures on $(\Omega, \mathcal{F})$.

As a divergence function, we shall use $d(\mathbb{Q}, \mathbb{P}) = \mathbb{E}\left(\varphi\left(\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P}\right)\right)$, where φ is convex, with $\varphi(1) = 0$. In particular, these cover the following well-known divergences

$$\varphi(t) = \begin{cases} t \ln t & \text{for the relative entropy;} \\ (\sqrt{t} - 1)^2 & \text{for the Hellinger divergence;} \\ |t - 1|^p & \text{for the } L^p \text{ distance}, p \geq 1. \end{cases} \tag{4}$$

Whenever $\mathbb{Q} \ll \mathbb{P}$ is not verified, we set $d(\mathbb{Q}, \mathbb{P}) := +\infty$ so that the application $\mathbb{Q} \mapsto d(\mathbb{Q}, \mathbb{P})$ is defined on $\mathcal{M}$.

Remark 4.2. *For $\mathbb{Q} \ll \mathbb{P}$, we identify $\mathbb{Q}$ with $Z := d\mathbb{Q}/d\mathbb{P}$. We can therefore define $D(Z) := \mathbb{E}\left(\varphi\left(Z\right)\right)$. In the sequel, we will interchangeably use the notations $d(\mathbb{Q}, \mathbb{P})$ and $D(Z)$. In addition, for all $\mathbb{Q} \ll \mathbb{P}$, we have that $\mathbb{Q}$ satisfies the views* (2) *if and only if*

$$\mathbb{E}(Z\mathbf{1}_{S_i}) \geq c_i, \quad i = 1, \ldots, d.$$

We know from [Csiszár, 1967] that $d(\mathbb{Q}, \mathbb{P}) \geq 0$. In addition, for functions φ that are strictly convex at 1, which is the case for all the divergences in (4), equality holds if and only if $\mathbb{P} = \mathbb{Q}$. In general, strict convexity of φ on $(0, \infty)$ holds only for the relative entropy, the Hellinger divergence and the L^p-distance, whenever $p > 1$.

The relative entropy, $d_E(\mathbb{Q}, \mathbb{P}) = D_E(Z) = \mathbb{E}\left(Z \ln\left(Z\right)\right)$ is sometimes also called Kullback-Leibler divergence. Note that d_E is not a metric on $\mathcal{M}$ since it does not fulfills the symmetry property but it is strictly convex. It is often used in information theory, see [Kullback, 1968], or as a smoothing device for inverse problems. Using the following Lemma, we note that the convergence in the relative entropy implies the convergence in the L_1-distance.

Lemma 4.3. *For $\mathbb{Q} \ll \mathbb{P}$, let $Z = d\mathbb{Q}/d\mathbb{P}$. Then*

$$\|Z - 1\|_1 \leq \left(2 d_E(\mathbb{Q}, \mathbb{P})\right)^{1/2}. \tag{5}$$

Proof. See Theorem 6.1 in [Kemperman, 1969].

□

The Hellinger distance, $d_H(\mathbb{Q}, \mathbb{P}) = D_H(Z) = \mathbb{E}((\sqrt{Z} - 1)^2)$ share many properties with the L_1-distance, see, e.g. Section 1 in [Devroye, 1987]. Note that $(\sqrt{t} - 1)^2 \leq |\sqrt{t} - 1||\sqrt{t} + 1|$, so that $d_H(\mathbb{Q}, \mathbb{P}) \leq \|Z - 1\|_1$. In fact, the convergence in the Hellinger distance is equivalent to the convergence in the L_1-distance as the following Lemma shows.

Lemma 4.4. *For $\mathbb{Q} \ll \mathbb{P}$, let $Z = d\mathbb{Q}/d\mathbb{P}$. Then*

$$d_H(\mathbb{Q},\mathbb{P}) \leq \|Z-1\|_1 \leq (2d_H(\mathbb{Q},\mathbb{P}))^{1/2}. \tag{6}$$

Proof. See Theorem 1.3 in [Devroye, 1987].

□

The L^1 distance

$$\left\|\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}-1\right\|_1 = \mathbb{E}\left(\left|\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}-1\right|\right), \tag{7}$$

can in fact be rewritten as a distance between probability measures, namely the total variation defined by

$$d_{TV}(\mathbb{Q},\mathbb{P}) := \sup_{A\in\mathcal{F}} |\mathbb{Q}(A)-\mathbb{P}(A)|.$$

Indeed,

$$\begin{aligned}\left\|\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}-1\right\|_1 &= 2\cdot\mathbb{E}\left(\left(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}-1\right)\mathbf{1}_{\{\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\geq 1\}}\right)\\ &= 2\cdot\sup_{A\in\mathcal{F}}\left|\mathbb{E}\left(\left(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}-1\right)\mathbf{1}_{\{A\}}\right)\right| = 2\cdot d_{TV}(\mathbb{Q},\mathbb{P}).\end{aligned} \tag{8}$$

An interesting property of the L_1-distance is that it remains unchanged if we transform the space Ω by any one-to-one transformation. In addition, the link with the total variation in (8) makes the interpretation easy as it is twice the maximum amount by which two probability measures differ for any event $A \in \mathcal{F}$.

We note from (8) and Lemma 4.4 that convergence with respect to the relative entropy convergence, or the L_1-distance implies convergence in total variation, which is a very strong notion of convergence. In particular, it implies the uniform convergence of the distribution functions.

Proposition 4.5. *Suppose that the distance function is given by $D(Z) = \mathbb{E}(\varphi(Z))$, for some convex function φ. There exists a solution to* (3) *in $\mathcal{M}$, if and only if there exists a solution to* (3) *in the subspace*

$$\mathcal{Q} = \left\{\mathbb{Q} \ll \mathbb{P} : \frac{d\mathbb{Q}}{d\mathbb{P}} = \frac{\mathbb{Q}(S_i)}{\mathbb{P}(S_i)} \text{ on } S_i, i=1,\ldots,n\right\} \subseteq \mathcal{M}. \tag{9}$$

Proof. Note that one implication of the assertion is trivial from the fact that $\mathcal{Q} \subseteq \mathcal{M}$.

Let us show the other implication. Let $\mathbb{Q} \in \mathcal{M}$ be a solution to (3) and note $Z := \mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P}$. We show that $\mathbb{E}(Z|\mathcal{S})$ is a solution to (3). Note first $\mathbb{E}(Z|\mathcal{S})$ satisfies the views (2) whenever Z does. Indeed, using the law of total expectation, $c_1 \leq \mathbb{Q}(S_i) = \mathbb{E}\left(Z\mathbf{1}_{\{S_i\}}\right) = \mathbb{E}\left(\mathbb{E}(Z|\mathcal{S})\mathbf{1}_{\{S_i\}}\right)$. In addition, we have by Jensen's inequality that

$$D(Z) \geq D(\mathbb{E}(Z|\mathcal{S})),$$

so that a minimum of D will also be attained at $\mathbb{E}(Z|\mathcal{S})$, which proves the assertion.

□

We also obtain unicity of the solution if we require the function φ to be strictly convex on $(0,\infty)$.

Proposition 4.6. *Suppose that the distance function is given by $D(Z) = \mathbb{E}(\varphi(Z))$, for some strictly convex function φ on $(0,\infty)$. The solution to* (3) *is unique in $\mathcal{M}$.*

Proof. First note that the views (2) form a convex subset of $\mathcal{M}$. Unicity of the solution to (3) then directly follows from strict convexity of φ on $(0,\infty)$.

□

In virtue of Propositions 4.5 and 4.6, we conclude that whenever the distance function of the minimization problem (3) is such that φ is strictly convex on $(0,\infty)$, then its unique solution is an absolutely continuous probability measure $\mathbb{Q}^*$ that is a scaling of $\mathbb{P}$ on each scenario set. More precisely, the solution is such that

$$\mathbb{Q}^*(A) = \sum_{j=0}^{n} \frac{q_j^*}{p_j} \mathbb{P}(A \cap U_j), \quad A \in \mathcal{F}. \tag{10}$$

In particular, this is true whenever the distance function d is chosen to be the relative entropy, the Hellinger distance or the L^p- distance for $p > 1$.

Remark 4.7. *Note that no discontinuities are induced on the distribution of X under $\mathbb{Q}^*$. Indeed, for any absolutely continuous scaling α, we have, for all $A \in \mathcal{F}, \mathbb{Q}^*(A) = \alpha\,\mathbb{P}(A) = 0$ if and only if $\mathbb{P}(A) = 0$.*

Remark 4.8. *Put here a counter-example of non-unicity of the solution in the L^1-case.*

Note that this approach overcomes some of the weaknesses of the SST scenario aggregation method. First and foremost, an internal model which is conservative enough so that $p_i \geq c_i$ for some view i is not penalized by aggregating scenario i anymore. Second, the modified model is an interpolation between the internal model and the regulator's views within minimal distance from the internal model. The regulator has remote control on the capital requirements as he can tune the trade off between idiosyncracy (internal model) and standardization (regulator's views) via increasing the number of scenarios d. Intuitively, the more scenarios are prescribed the less weight the internal model has on the capital requirement.

4.1 Relative entropy

It is remarkable that this infinite-dimensional problem can be reduced to a finite-dimensional problem which has a unique and sometimes explicit solution

$$\begin{array}{rl} \text{minimize} & \sum_{j=0}^{n} q_j \log \frac{q_j}{p_j} \\ \text{subject to} & A\mathbf{q} \geq \mathbf{c} \\ & \mathbf{1}^\top \mathbf{q} = \mathbf{1} \end{array} \tag{11}$$

with domain $\mathbb{R}_+^{n+1}$. The solution to this problem will be noted $\mathbf{q}^*$.

The Lagrangian function is

$$L(\mathbf{q}, \boldsymbol{\lambda}, \nu) = \sum_{j=0}^{n} q_j \log \frac{q_j}{p_j} + \boldsymbol{\lambda}^\top (\mathbf{c} - A\mathbf{q}) + \nu \left(\mathbf{1}^\top \mathbf{q} - 1 \right).$$

The dual problem is

$$\begin{array}{rl} \text{minimize} & \sum_{j=0}^{n} p_j \mathrm{e}^{(A^\top \boldsymbol{\lambda})_j - \nu - 1} - \mathbf{c}^\top \boldsymbol{\lambda} + \nu \\ \text{subject to} & \boldsymbol{\lambda} \geq 0 \end{array} \tag{12}$$

with domain $\boldsymbol{\lambda} \in \mathbb{R}^d$ and $\nu \in \mathbb{R}$. Suppose that we have found the solution $(\boldsymbol{\lambda}^*, \nu^*)$ of the dual problem (12), then the unique minimizer of the Lagrangian as a function of $(\boldsymbol{\lambda}^*, \nu^*)$ is given by

$$\widetilde{q}_j = p_j e^{(A^\top \boldsymbol{\lambda}^*)_j - \nu^* - 1} \quad j = 0, \ldots, d.$$

The Slater's condition for our convex optimization problem is that there exists a $\mathbf{q} > 0$ such that $A\mathbf{q} \geq \mathbf{c}$ and $\mathbf{1}^\top \mathbf{q} = 1$. This condition is sufficient for the existence of $(\boldsymbol{\lambda}^*, \nu^*)$ and to guarantee strong duality, i.e $\mathbf{q}^* = \widetilde{\mathbf{q}}$, see [Boyd and Vandenberghe, 2004].

Given that Slater's condition is fulfilled, the KKT conditions provide a necessary and sufficient conditions for optimality. These conditions read

$$\lambda \geq 0, \quad Aq \geq c, \quad \lambda^\top (Aq - c) = 0 \tag{13}$$

$$\mathbf{1}^\top q = 1 \tag{14}$$

$$\log q - \log p + \mathbf{1} - A^\top \lambda + \nu \mathbf{1} = \mathbf{0} \tag{15}$$

It is understood that (15) implies that $\mathbf{q} > 0$.

In general, finding the solution $(\boldsymbol{\lambda}^*, \nu^*)$ of the dual problem (12) or the solving the KKT system (13)–(15) need to be done numerically, see e.g. [Boyd and Vandenberghe, 2004] or [Fang et al., 1997]. In what follows, we present two special cases where the solution $\mathbf{q}^*$ can be calculated explicitly, without numerical procedures. This first special case covers independent scenarios and the second correspond to the general case with $d = 2$ dimensions.

4.1.1 Disjoint scenarios

Assume the scenarios S_i are mutually disjoint and $S_i = U_i$. Then $A = (\mathbf{0}_{d\times 1}\, \mathrm{Id}_{d\times d})$. The solution of the KKT conditions (13)–(15), and thus of problem (11), is then given explicitly as follows.

The KKT conditions read

$$\begin{aligned} q_i^* &\geq c_i, \quad i = 0, \ldots, d \\ \mathbf{1}^\top \mathbf{q} &= 1 \\ \lambda_i^* &\geq 0, \quad i = 0, \ldots, d \\ \ln\left(\frac{q_i^*}{p_i}\right) + 1 + \nu^* - \lambda_i^* &= 0, \quad i = 0, \ldots, d \\ \lambda_i^*(c_i - q_i^*) &= 0, \quad i = 0, \ldots, d. \end{aligned}$$

Note that the inequality constraint $\lambda_i^* \geq 0$ and the second last constraint can be reduced to one inequality constraint. It is therefore said that λ_i^* is a slack variable. The KKT conditions become

$$q_i^* \geq c_i, \quad i = 0, \ldots, d \tag{16}$$

$$\mathbf{1}^\top \mathbf{q}^* = 1 \tag{17}$$

$$\ln\left(\frac{q_i^*}{p_i}\right) + 1 + \nu^* \geq 0, \quad i = 0, \ldots, d \tag{18}$$

$$\left(\ln\left(\frac{q_i^*}{p_i}\right) + 1 + \nu^*\right)(c_i - q_i^*) = 0, \quad i = 0, \ldots, d. \tag{19}$$

We are now going to solve for ν^*.

(i) Suppose $\nu^* \geq -\left(1 + \ln(c_i/p_i)\right)$. This forces $q_i^* = c_i$, otherwise, if $q_i^* > c_i$, then $\nu^* \geq -1\left(1 + \ln(c_i/p_i)\right) > -1\left(1 + \ln(q_i^*/p_i)\right)$, which is in contradiction with condition (19).

(ii) Suppose $\nu^* < -\left(1 + \ln(c_i/p_i)\right)$. This forces $q_i^* > c_i$, otherwise, if $q_i^* = c_i$, condition (18) does not hold.

Therefore, we have

$$q_i^* = \max\left\{c_i, p_i e^{-\nu^* - 1}\right\}, \quad i = 0, \ldots, d. \tag{20}$$

It is easy to check that $\mathbf{q}^*$ as defined in (20) will verify condition (16). In order to fulfill condition (17), we need to solve for ν^* such that $\sum_{i=0}^d \max\left\{0, p_i e^{-\nu^* - 1} - c_i\right\} = 1 - \sum_{i=0}^d c_i$.

Denote

$$f(\nu) = \sum_{i=0}^{d} \max\left\{0, p_i e^{-\nu^*-1} - c_i\right\},$$

which means that we need to solve $f(\nu^*) = 1 - \sum_{i=0}^{d} c_i$. This is a decreasing function of ν that is stepwise exponential with steps at

$$-\left(1 + \ln\left(\frac{c_1}{p_1}\right)\right) \geq \cdots \geq -\left(1 + \ln\left(\frac{c_d}{p_d}\right)\right),$$

where we supposed w.l.o.g that the c_i's and p_i's are ordered so. Note also that $\lim_{\nu\to\infty} f(\nu) = 0$. Take k^* such that

$$f\left(-\left(1 + \ln\left(\frac{c_{k^*+1}}{p_{k^*+1}}\right)\right)\right) > 1 - \sum_{i=0}^{d} c_i \geq f\left(-\left(1 + \ln\left(\frac{c_{k^*}}{p_{k^*}}\right)\right)\right),$$

and set

$$\nu^* = -\ln\left(\frac{1 - \sum_{i=k^*+1}^{d} c_i}{p_0 + \sum_{i=1}^{k^*} p_i}\right) - 1.$$

Therefore,

$$q_j^* = \max\left\{c_j, p_j \frac{1 - \sum_{i=k^*+1}^{d} c_i}{p_0 + \sum_{i=1}^{k^*} p_i}\right\}.$$

We note from (20), together with the constraint $\mathbf{1}^\top \mathbf{q}^*$, that $\nu \geq -1$. In consequence, the case $\mathbf{p} \leq \mathbf{c}$ will immediately lead to the solution $\mathbf{q}^* = \mathbf{c}$. An illustration is given in Figure 1 of the two cases $\mathbf{p} \leq \mathbf{c}$ and $p_1 > c_1, p_2 \leq c_2$ with $d = 2$ scenarios.

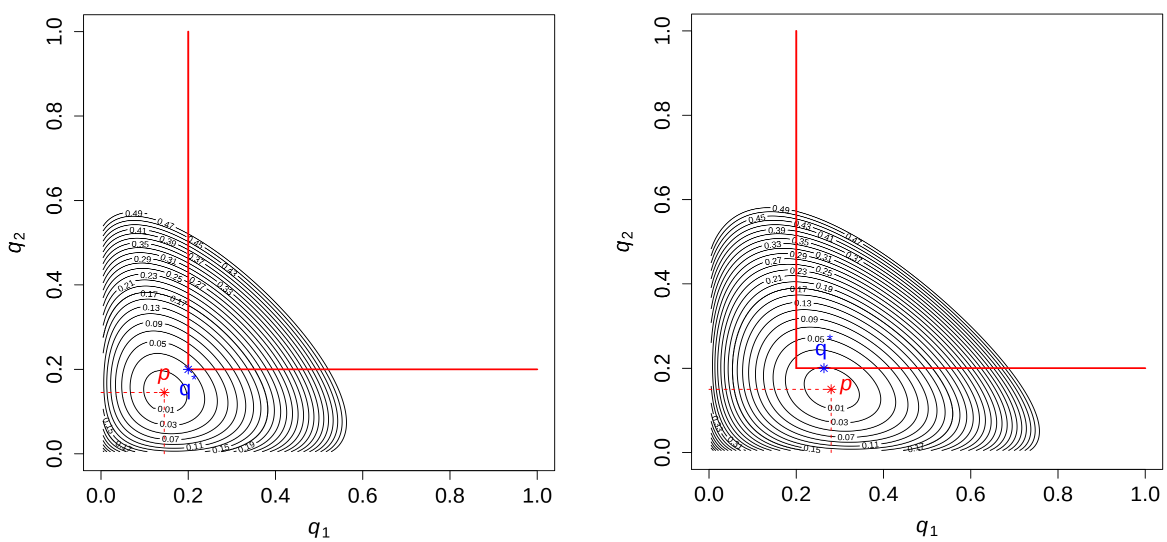

Figure 1: Optimal solution $\mathbf{q}^*$ of the optimization problem (3) with $\mathbf{c} = (2,2)^\top$, $d = 2$ scenarios and using the relative entropy. Left picture illustrates the case $\mathbf{p} \leq \mathbf{c}$. Right picture illustrates the case $p_1 > c_1, p_2 \leq c_2$.

4.1.2 General case, $d = 2$ dimensions

Let us consider $d = 2$ overlapping scenarios, inducing $n = 4$ atoms given by $U_0 = S_0$, $U_1 = S_1 \cap S_2$, $U_2 = S_1 \setminus S_2$, $U_3 = S_2 \setminus S_1$. Note from the KKT conditions (13) and (15) that we can use λ_1 and λ_2 as slack variables in two of the three dimensions, we obtain that

$$q_2 = \max\{c_1 - q_1, p_2 e^{-\nu-1}\}, \quad q_3 = \max\{c_2 - q_1, p_3 e^{-\nu-1}\}. \tag{21}$$

Using the KKT condition (15) on q_1, we get

$$q_1 = e^{\nu+1} p_1 \max\left\{\frac{c_1 - q_1}{p_2}, e^{-\nu-1}\right\} \max\left\{\frac{c_2 - q_1}{p_3}, e^{-\nu-1}\right\}. \tag{22}$$

The values of q_1 and ν can the be solved to fulfil the KKT condition (14), namely $\sum_{i=0}^{3} q_i = 1$. In order to solve for q_1, we are going to distinct cases

Case 1: $(c_1 - q_1)/p_2 \le e^{-\nu-1}, (c_2 - q_1)/p_3 \le e^{-\nu-1}$

We deduce that $\nu = -1$ and $q_i = p_i, i = 0, \ldots, 3$. Note that this corresponds to the case $\mathbf{p} \in I_1$,

$$I_1 := \left\{\mathbf{p} \in \mathbb{R}^3_+ : c_1 \le p_1 + p_2, c_2 \le p_1 + p_3\right\}.$$

Case 2: $(c_1 - q_1)/p_2 > e^{-\nu-1}, (c_2 - q_1)/p_3 \le e^{-\nu-1}$

We deduce that $\nu = -\left(1 + \log\left((1 - c_1)/(p_0 + p_3)\right)\right)$ and

$$q_0 = \frac{(1 - c_1)p_0}{p_0 + p_3}, \quad q_2 = \frac{p_2 c_1}{p_1 + p_2}$$
$$q_1 = \frac{p_1 c_1}{p_1 + p_2}, \quad q_3 = \frac{(1 - c_1)p_3}{p_0 + p_3}.$$

Note that this corresponds to the case $\mathbf{p} \in I_2$,

$$I_2 := \left\{\mathbf{p} \in \mathbb{R}^3_+ : c_1 > p_1 + p_2, c_2 \le p_3/(p_0 + p_3) + c_1(p_1/(p_1 + p_2) - p_3/(p_0 + p_3))\right\}.$$

Case 3: $(c_1 - q_1)/p_2 \le e^{-\nu-1}, (c_2 - q_1)/p_3 > e^{-\nu-1}$

We deduce that $\nu = -\left(1 + \log\left((1 - c_2)/(p_0 + p_2)\right)\right)$ and

$$q_0 = \frac{(1 - c_2)p_0}{p_0 + p_2}, \quad q_2 = \frac{(1 - c_2)p_2}{p_0 + p_2}$$
$$q_1 = \frac{p_1 c_2}{p_1 + p_3}, \quad q_3 = \frac{p_3 c_2}{p_1 + p_3}.$$

Note that this corresponds to the case $\mathbf{p} \in I_3$,

$$I_3 := \left\{\mathbf{p} \in \mathbb{R}^3_+ : c_1 \le p_2/(p_0 + p_2) + c_2(p_1/(p_1 + p_3) - p_2/(p_0 + p_2)), c_2 > p_1 + p_3\right\}.$$

Case 4: $(c_1 - q_1)/p_2 > e^{-\nu-1}, (c_2 - q_1)/p_3 > e^{-\nu-1}$

Denote $\Delta = p_0 p_1 - p_2 p_3$. We find that

$$q_1 = \frac{(c_1 + c_2)\Delta + p_2 p_3 + \sqrt{-4 c_1 c_2 p_0 p_1 \Delta + (-c_1 \Delta - c_2 \Delta - p_2 p_3)^2}}{2\Delta},$$

and $q_0 = 1 - c_1 - c_2 + q_1, q_2 = c_1 - q_1, q_3 = c_2 - q_1$.

Note that this corresponds to the case $\mathbf{p} \in I_4$,

$$I_4 := \left\{\mathbf{p} \in \mathbb{R}^3_+ : \mathbf{1}^\top \mathbf{q} = 1\right\} \setminus (I_1 \cup I_2 \cup I_3).$$

4.2 L^2 distance

We now consider solved the general problem (3) with the distance function $D(Z) = \mathbb{E}\left((Z-1)^2\right)$. The finite dimensional optimization problem that needs to be solved is therefore

$$\begin{aligned} \text{minimize} \quad & \textstyle\sum_{j=0}^{n} p_j \left(\frac{q_j}{p_j} - 1\right)^2 \\ \text{subject to} \quad & A\mathbf{q} \geq \mathbf{c} \\ & \mathbf{1}^\top \mathbf{q} = \mathbf{1} \end{aligned} \tag{23}$$

with domain $\mathbb{R}_+^{n+1}$. The solution to this problem will be noted $\boldsymbol{\lambda}^*$.

In this case, the dual problem is

$$\begin{aligned} \text{minimize} \quad & \textstyle\sum_{j=0}^{n} p_j \frac{(A^\top \boldsymbol{\lambda})_j - \nu}{2} \left((A^\top \boldsymbol{\lambda})_j - \nu + 1\right) - \mathbf{c}^\top \boldsymbol{\lambda} + \nu \\ \text{subject to} \quad & \boldsymbol{\lambda} \geq 0 \end{aligned} \tag{24}$$

with domain $\boldsymbol{\lambda} \in \mathbb{R}^d$ and $\nu \in \mathbb{R}$. Given the solution $(\boldsymbol{\lambda}^*, \nu^*)$ of the dual problem (24), the unique minimizer of the Lagrangian associated to the primal problem (23) as a function of $(\boldsymbol{\lambda}^*, \nu^*)$ is given by

$$\widetilde{q}_j = p_j \left((A^\top \boldsymbol{\lambda})_j - \nu + 1\right) \quad j = 0, \ldots, d.$$

The explicit solving of the special cases of disjoint scenarios and of $d = 2$ scenarios is in progress and will be included in the final version of the paper.

5 Economic Capital

We should now discuss the impact of the scenario aggregation approach proposed in Section 4 on an economic capital calculation.

Let us first introduce the risk measures used for the economic capital calculation. For more details on the risk measures, see, for example, Chapter 4 in [Föllmer and Schied, 2011] or Sections 2.2 and 6.1 in [McNeil et al., 2005].

Definition 5.1. *We define the Value-at-Risk at level $\alpha \in (0,1)$, $\mathrm{VaR}_\alpha(X)$, as the lower quantile at level α of X, that is*

$$\mathrm{VaR}_\alpha(X) := q_\alpha(X) = \inf\left\{x : \mathbb{P}(X \leq x) \geq \alpha\right\}.$$

Definition 5.2. *We define the expected shortfall at level $\alpha \in (0,1)$, $\mathrm{ES}_\alpha(X)$, as*

$$\begin{aligned} \mathrm{ES}_\alpha(X) &:= \frac{1}{1-\alpha}\mathbb{E}\left((X - q_\alpha(X))^+\right) + q_\alpha(X) \\ &= \frac{1}{1-\alpha}\mathbb{E}\left(X\mathbf{1}_{\{X > q_\alpha(X)\}}\right) + \frac{q_\alpha(X)}{1-\alpha}\left(\mathbb{P}(X \leq q_\alpha(X)) - \alpha\right). \end{aligned}$$

Remark 5.3. *If the cdf of X satisfies $\mathbb{P}(X \leq q_\alpha(X)) = \alpha$, in particular, if q_α is a point of continuity of $x \mapsto \mathbb{P}(X \leq x)$, then $\mathrm{ES}_\alpha(X)$ simply writes*

$$\mathrm{ES}_\alpha(X) = \frac{1}{1-\alpha}\mathbb{E}\left(X\mathbf{1}_{\{X > q_\alpha(X)\}}\right) = \mathbb{E}\left(X | X > q_\alpha(X)\right).$$

We will now enunciate a result that guarantees the continuity of the risk measure $\mathrm{ES}_\alpha(X)$, when changing the reference measure $\mathbb{P}$. For a family of probability measures $(\mathbb{P}_n)_{n\in\mathbb{N}} \in \mathcal{M}$, we shall denote $q_{n,\alpha}(X)$ and $\mathrm{ES}_{n,\alpha}(X)$ the quantile and expected shortfall of X under $\mathbb{P}_n$.

Theorem 5.4. *Let $p \in [1, \infty]$ and $r \in [1, \infty]$ such that $p^{-1} + r^{-1} = 1$. Let us assume that $d\mathbb{P}_n/d\mathbb{P} =: Z_n$ converges to 1 in L^p for $n \to \infty$. Then*

$$\mathrm{ES}_{n,\alpha}(X) \to \mathrm{ES}_\alpha(X), \quad \textit{for all } X \in L^q.$$

Proof. Let us note F_n the distribution function of X under $\mathbb{P}_n, n \in \mathbb{N}$. For simplicity, assume that quantiles are taken at level α throughout this proof, hence omitting the labelling. In addition, there should be no confusion between $F(q-)$, the left limit of F at q and $F(q^-)$, the evaluation of F at the left quantile q^-.

First of all, let us show that

$$q^- \leq \liminf_{n\to\infty} q_n \leq \limsup_{n\to\infty} q_n \leq q^+. \tag{25}$$

Indeed, assume that for some $\epsilon > 0$ there exists a subsequence $(q_{n_k})_{k\in\mathbb{N}}$ such that $q_{n_k} \leq q^- - \epsilon$ for all $k \in \mathbb{N}$. Therefore,

$$F_{n_k}(q_{n_k}) - F(q_{n_k}) \geq \alpha - F(q^- - \epsilon) > 0 \quad \text{for all } k \in \mathbb{N},$$

which is in contradiction with the fact that $\|Z_{n_k} - 1\|_p \to 0$. Similarly, assume that form some $\epsilon > 0$, there exists a subsequence $(q_{n_k})_{k\in\mathbb{N}}$ such that $q_{n_k} \geq q^+ + \epsilon$ for all $k \in \mathbb{N}$. Therefore,

$$F(q_{n_k}-) - F_{n_k}(q_{n_k}-) \geq F((q^+ + \epsilon)-) - \alpha > 0 \quad \text{for all } k \in \mathbb{N},$$

which is again in contradiction with $\|Z_{n_k} - 1\|_p \to 0$, hence allowing us to conclude (25).

Let us now assume that for some $\epsilon > 0$, there exists a subsequence $(n_k)_k \in \mathbb{N}$ such that

$$|\mathrm{ES}_{n_k}(X) - ES(X)| \geq \epsilon \quad \text{for all } k \in \mathbb{N}. \tag{26}$$

From (25), we know that there exists a subsequence of $(n_k)_k \in \mathbb{N}$ (still denoted $(n_k)_k \in \mathbb{N}$ for simplicity) such that

$$\lim_{k\to\infty} q_{n_k} = q,$$

for $q \in [q^-, q^+]$, some α-quantile of F. In consequence we obtain

$$\begin{aligned}
(1-\alpha)\left|\mathrm{ES}_{n_k}(X) - \mathrm{ES}(X)\right| &= \left|\mathbb{E}\left(Z_{n_k}(X - q_{n_k})^+\right) - \mathbb{E}\left((X-q)^+\right) + (1-\alpha)(q_{n_k} - q)\right| \\
&\leq \left|\mathbb{E}\left((Z_{n_k} - 1)(X - q_{n_k})^+\right) + \mathbb{E}(\underbrace{(X - q_{n_k})^+ - (X-q)^+}_{\leq |(q_{n_k} - q)|})\right| + (1-\alpha)\left|q_{n_k} - q\right| \\
&\leq \|Z_{n_k} - 1\|_p \|X\|_r + |q_{n_k} - q| + (1-\alpha)\left|q_{n_k} - q\right| \\
&\longrightarrow 0.
\end{aligned}$$

This is in contradiction with (26), hence proving the proposition.

□

We now show continuity of the expected shortfall with respect to the relative entropy.

Corollary 5.5. *Let $(\mathbb{P}_n)_{n\in\mathbb{N}}$ such that $d_E(\mathbb{P}_n, \mathbb{P}) \to 0$, then*

$$\mathrm{ES}_{n,\alpha}(X) \to \mathrm{ES}_\alpha(X), \quad \textit{for all } X \in L^\infty.$$

Proof. This follows from Lemma 4.4 and Theorem 5.4.

□

6 Case Studies

We present here two case studies. For each of them, a scenario aggregation problem is formulated and an impact of the aggregation on the required capital is studied. We shall assume thorough this section that the loss random variable L is given by a (not necessarily linear) function of the risk factors $(X_1, \ldots, X_m)$.

6.1 Case Study 1

We assume that we have a 2-dimensional vector of risk factors $\mathbf{X} = (X_1, X_2)^\top$, with mean vector $\mathbf{0}$ and covariance matrix $\mathbf{\Sigma}$, such that the correlation term is 0.6 are respective variances are 8 and 12. In this example, the loss function is given by $L = X_1 + X_2$.

Assume that a series of losses has been observed and denote l the largest loss. Let us consider the scenario $S_1 = \{L \geq l\}$, for a loss level $l \geq 0$, and $S_0 = S_1^c$. This is a typical stress-testing example for internal models, using hypothesis testing to test whether or not the null-hypothesis that our internal model $\mathbb{P}$ is the true model must be rejected. For more details on hypothesis testing, see, e.g., Section 7.3 in [Davison, 2003]. In this example, the rejection set at a significance level c is $\{l \geq \mathrm{VaR}_{1-c}(L)\}$. The internal model $\mathbb{P}$ is therefore not rejected at a significance level c if $l \leq \mathrm{VaR}_{1-c}(L)$, or equivalently, $\mathbb{P}(S_1) \geq c$. Typically, $c = 1\%$. Note that this test is equivalent to checking whether or not the model fulfils a view of type (2). In the case where the model is rejected, and therefore does not fulfil the view $\mathbb{P}(S_1) \geq c$, we would aggregate this loss scenario. The new model $\mathbb{Q}^*$ obtained from the aggregation would then used to compute the relevant risk measures, namely $\mathrm{VaR}_{0.99}^{\mathbb{Q}^*}(L)$ and $\mathrm{ES}_{0.99}^{\mathbb{Q}^*}(L)$.

In this example, we perform the stress test for different largest loss levels l, and different significance levels c. We plot in Figure 2 the values of $\mathrm{VaR}_{0.99}^{\mathbb{Q}^*}(L)$ and $\mathrm{ES}_{0.99}^{\mathbb{Q}^*}(L)$ for c ranging from 0.001 to 0.01 and l artificially equal to $\mathrm{VaR}_\alpha(L)$, with α ranging from 0.99 to 0.999. For a fixed $l = \mathrm{VaR}_\alpha(L)$, it is clear that in the proposed aggregation method, none of the two risk measures under interest will deviate from $\mathrm{VaR}_{0.99}(L)$ and $\mathrm{ES}_{0.99}(L)$ as long as $c \leq \alpha$, as the internal model $\mathbb{P}$ is not rejected and no aggregation is needed. When $\alpha < c \leq 0.01$, the internal model does not satisfy the view anymore, i.e it is rejected. In that case, an aggregation is performed, solving the problem (3) with the relative entropy distance. As there is only one scenario, the solution will be given by

$$\mathbb{Q}^*(A) = \frac{1-c}{p_0}\mathbb{P}(A \cap S_0) + \frac{c}{p_1}\mathbb{P}(A \cap S_1), \quad A \in \mathcal{F}.$$

In addition, note that in the case $c = 0.01$ and a given l, the required capital under $\mathbb{Q}^*$ can be computed directly as

$$\begin{aligned}
\mathrm{ES}_{0.99}^{\mathbb{Q}^*}(L) &= \mathbb{E}_{\mathbb{Q}^*}\left(L | L \geq \mathrm{VaR}_{0.99}^{\mathbb{Q}^*}(L)\right) \\
&= \mathbb{E}_{\mathbb{Q}^*}\left(L | L \geq l\right) = \frac{1}{c}\mathbb{E}_{\mathbb{Q}^*}\left(L\mathbf{1}_{\{L \geq l\}}\right) \\
&= \frac{1}{c}\frac{c}{p_1}\mathbb{E}\left(L\mathbf{1}_{\{L \geq l\}}\right) = \mathbb{E}_{\mathbb{P}}\left(L | L \geq l\right) \\
&= \mathrm{ES}_{\alpha = \mathbb{P}(L < l)}(L).
\end{aligned}$$

As a result of the scenario aggregation, the distribution function of L under $\mathbb{Q}^*$ will have an heavier right tail than under $\mathbb{P}$. For example, for $c = 0.01$ and $l \geq \mathrm{VaR}_{0.99}(L)$, the distribution under $\mathbb{Q}^*$ after aggregating the scenario $S_1 = \{L \geq l\}$, is an absolutely continuous modification of $\mathbb{P}$ such that its 99% quantile is equal to l. The distribution functions of L under $\mathbb{Q}^*$ for $c = 0.01$ and different values of $l \geq \mathrm{VaR}_{0.99}(L)$ is shown in Figure 3.

Although this example is of particular interest, it does not fit within a pure regulatory framework as the scenario S_1 is formulated in terms of the loss L, which is insurer specific, and not in terms of the risk factors variable $\mathbf{X}$ that is universal.

6.2 Case Study 2

Let us consider an insurer having the following loss function in terms of two risk factors X_1, X_2,

$$L(X_1, X_2) = \max(X_1, -1) + \max\left(\min(X_2, 5), -1\right).$$

The insurer would then incur losses (gains) in case of positive (negative) changes in X_1 and X_2. In addition, the insurer has capped his in X_2 losses at 5 and floored his gains in X_1 and X_2 at -1. The two risk factors are modelled with bi-dimensional vector $\mathbf{X} = (X_1, X_2)^\top$, with mean vector $\mathbf{0}$ and covariance matrix $\mathbf{\Sigma}$, such that the correlation term is -0.5 are respective variances are 1 and 4. Typically, X_1 could be an interest rate, say the Euro Zeros 1 year rates and X_2 could be a risk factor related to CAT events, for which the insurer is re-insured.

We consider two scenarios $S_1 = \{X_1 \geq 1, X_2 \geq 1\}$ and $S_2 = \{X_1 < -2\}$. Note that $\mathrm{VaR}_{0.99}(L) \approx 4.1$. Since the insurer has uncapped losses in X_1, the scenario set S_1 clearly has an intersection with the shortfall region of L, $\{L \geq \mathrm{VaR}_{0.99}(L)\}$. However, note that $L = X_1 - 1$ on S_2 and therefore $\{L \geq \mathrm{VaR}_{0.99}(L)\} \cap S_2 = \{\max\left(\min(X_2, 5), -1\right) - 1 \geq \mathrm{VaR}_{0.99}(L)\} = \emptyset$, as $\mathrm{VaR}_{0.99}(L) > 4$. Although the scenario set S_2 has an empty intersection with the shortfall region, we obtain a positive expected loss when conditioning on this scenario set, $\mathbb{E}(L|S_2) \approx 1.32$. In the case of the SST aggregation method, the aggregation of this scenario will therefore increase the required capital.

This case study is a toy example of a situation where, although it would lead to a positive expected loss - and hence an increase in required capital - in the sense of the SST aggregation, a scenario is not in the shortfall region of the insurer. Aggregation of such a scenario should therefore not lead to an increase of required capital, but rather a decrease as it is in the bulk of the loss distribution.

We consider the views $\mathbb{P}(S_1) \geq c_1$ and $\mathbb{P}(S_2) \geq c_2$. Impacts of the scenario aggregation on the Value-at-Risk and the Expected Shortfall at level $\alpha = 0.99$ are illustrated in Figure 4, for c_1 and c_2 ranging from 0 to 0.08.

As the expected losses $\mathbb{E}(L|S_1)$ and $\mathbb{E}(L|S_2)$ are strictly positive, we can observe from the top plots in Figure 4 that the SST aggregation method increase the required capital whenever c_1 or c_2 depart from 0. With the proposed scenario aggregation method, as the intersection between S_2 and the shortfall region is empty, increasing the probability c_2 will decrease the probability in the shortfall region, and hence induce a reduction in required capital. Conversely, as the intersection between S_1 and the shortfall region is not empty, increasing the probability c_1 will increase the probability in the shortfall region, and therefore increase the required capital.

Conclusion and perspectives

As the scenario aggregation is a vital part of risk-based solvency regulation, the problem must be embedded within a coherent mathematical framework. The minimum φ-divergence approach seem to be an appropriate tool for this as it would allow the regulator to focus on tail events, avoid to penalise the most conservative models and control the distance of our new model from the (original) internal model. In the case of a capital requirement based on the expected shortfall, it would be robust with such model changes. In the case of the Value-at-Risk, it would not. We can note in addition that such a scenario aggregation method can easily be implemented by the regulators and the regulated institutions as it is highly tractable.

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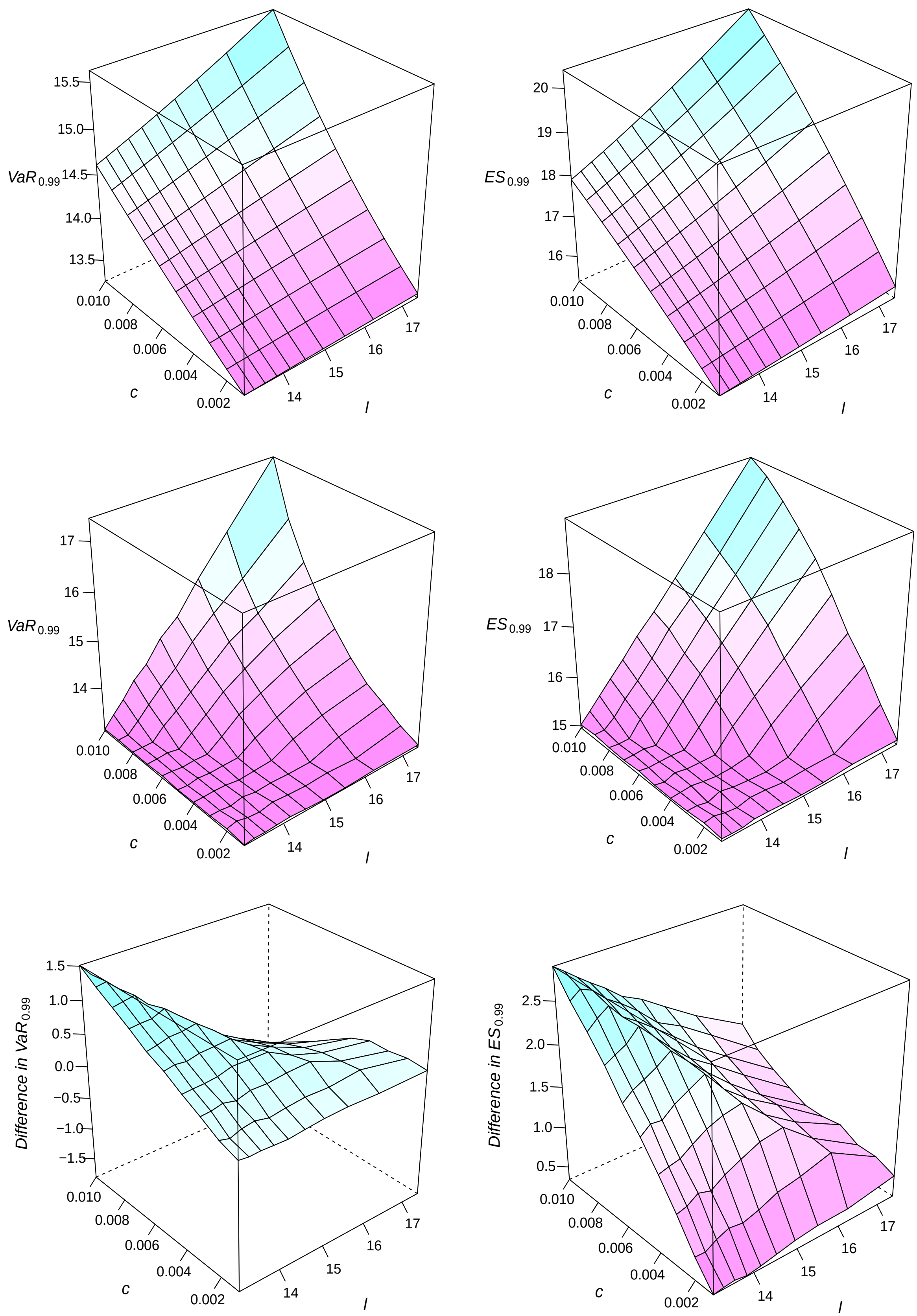

Figure 2: Impact on $\mathrm{VaR}_\alpha^{\mathbb{Q}^*}$ (left) and $\mathrm{ES}_\alpha^{\mathbb{Q}^*}$ (right), $\alpha = 99\%$, of the SST method (top) and the scenario aggregation method (middle). The difference between the risk measures computed after SST aggregation method and after the scenario aggregation method is given at the bottom.

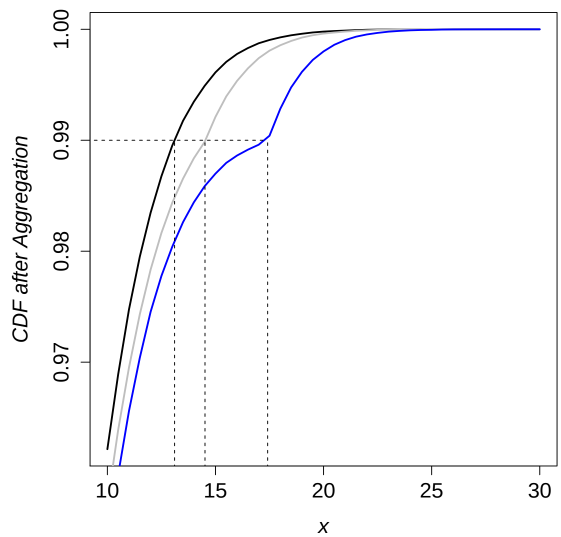

Figure 3: Distribution function of L under $\mathbb{Q}^*$, $x \mapsto \mathbb{Q}^*(L \leq x)$, resulting from scenario aggregation for $c = 0.01$ for different largest loss levels $l = \mathrm{VaR}_{0.99}(L)$ (black), $\mathrm{VaR}_{0.995}(L)$ (grey) and $\mathrm{VaR}_{0.999}(L)$ (blue). The vertical dashed lines draw the different levels of l, which must correspond to the 99% quantile of L under $\mathbb{Q}^*$.

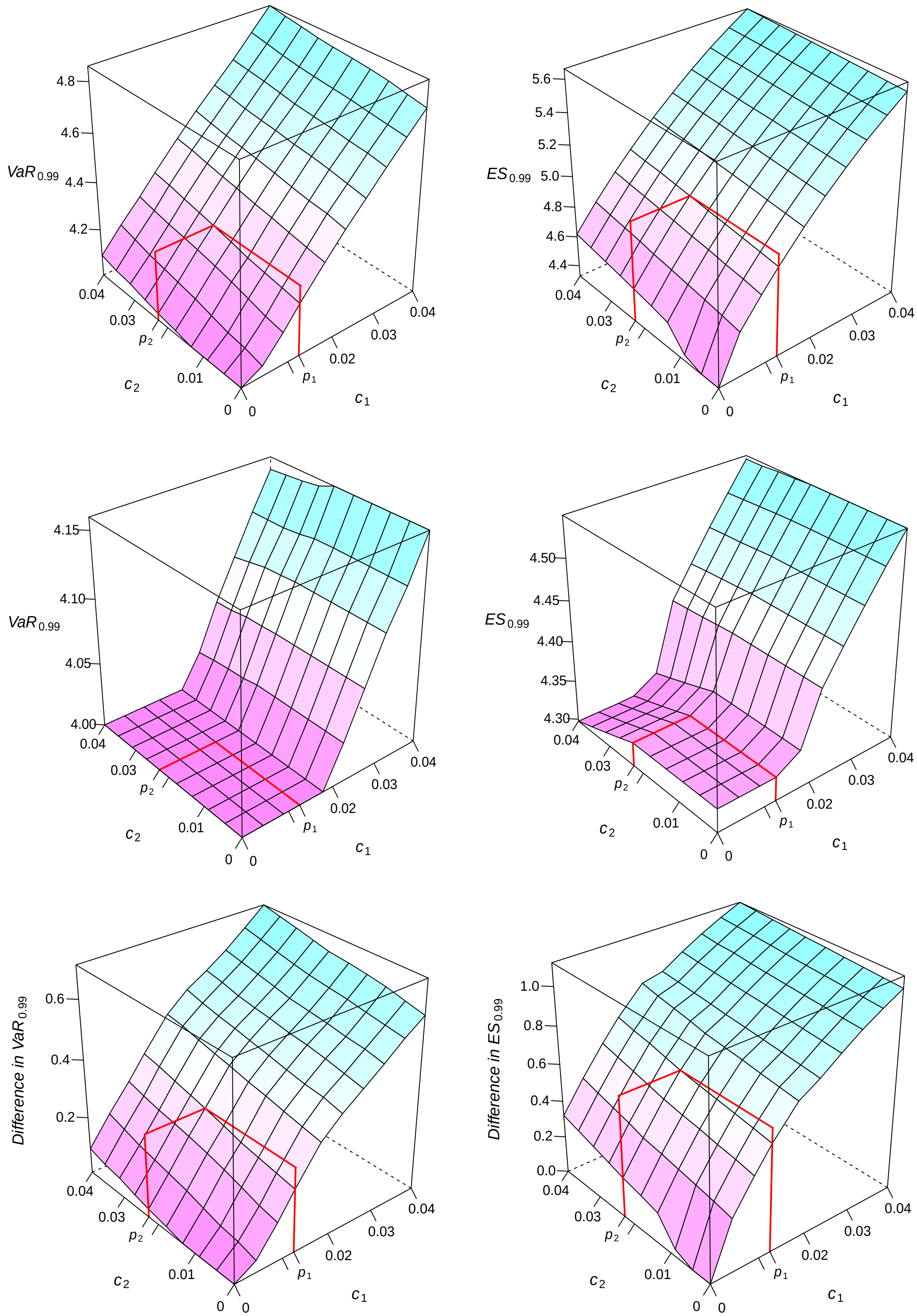

Figure 4: Impact on $\mathrm{VaR}_\alpha^{\mathbb{Q}^*}$ (left) and $\mathrm{ES}_\alpha^{\mathbb{Q}^*}$ (right), $\alpha = 99\%$, of the SST method (top) and the scenario aggregation method (middle). The difference between the risk measures computed after SST aggregation method and after the scenario aggregation method is given at the bottom.