ASTIN, AFIR/ERM and IACA Colloquia
Innovation & Invention

23-27 August 2015 | Amora Hotel Jamison Sydney

Actuaries Institute

Immunization and Hedging of Longevity Risk

Changyu Estelle Liu and Michael Sherris

Outline

1. Motivation and Key Insights
2. Interest Rate and Longevity Risk
3. Optimal Bond Portfolios
4. Portfolio Surplus Distribution and Hedge Effectiveness
5. Summary and Wrap Up

Outline

1. **Motivation and Key Insights**
2. Interest Rate and Longevity Risk
3. Optimal Bond Portfolios
4. Portfolio Surplus Distribution and Hedge Effectiveness
5. Summary and Wrap Up

Introduction

Motivation

- Pension funds and life insurers offering annuities hold long term liabilities linked to longevity
- Risk management of life annuity portfolios should immunize or hedge both interest rate and mortality risks
- Standard fixed interest duration-convexity hedging must be adapted to allow for both interest rate and longevity risk

Our aims

- To develop an immunization approach along with a delta-gamma based approach allowing for both interest rate and mortality risk.
- To compare and assess immunization and hedge effectiveness of fixed-income coupon bonds, annuity bonds, and longevity bonds, using simulations of portfolio surplus outcomes for an annuity portfolio.

Key insights

- Fixed-income annuity bonds can more effectively match cash flows but provide limited additional hedge effectiveness over coupon bonds.
- Longevity bonds, including deferred longevity bonds, reduce risk significantly compared to coupon and annuity bonds, reflecting the long duration of the typical life annuity and the exposure to longevity risk.
- Longevity bonds are shown to be effective in immunizing surplus over short and long horizons based on Fisher-Weil duration and convexity of cash flows.
- Delta gamma hedging is only effective over short horizons, requiring more re-balancing than immunization with longevity bonds.
- Delta gamma hedging is model dependent introducing increased model risk compared to Fisher-Weil immunization.

Approach

1. **Stochastic Mortality Risk:** Schrager (2006) Two-factor Model
2. **Stochastic Interest Rate Risk:** CIR Model
3. **Liabilities:** Whole-life annuities (Monthly) for Australian Male current age 65
4. **Assets:**
   - Currently Issued Fixed-income Securities (monthly, quarterly, semi-annually): 39 Coupon Bonds and 24 Annuity Bonds (ASX and FIIG)
   - 10 Assumed Longevity Bonds (Yearly) for Australian Male current age 65: 5,10,..., 50 years
5. **Optimal Asset Allocation:** Extend Shiu's Linear Programming
   - Duration-convexity immunization: minimizing assets convexity, matching surplus and duration of assets and liabilities
   - Delta-gamma hedging: minimizing assets gamma, matching surplus and delta of assets and liabilities
6. **Hedge Performance:** Compare distribution of P&L at future time points (1, 10, and 50 years)
   - Compare risk reduction measures

Outline

1. Motivation and Key Insights
2. **Interest Rate and Longevity Risk**
3. Optimal Bond Portfolios
4. Portfolio Surplus Distribution and Hedge Effectiveness
5. Summary and Wrap Up

Risk Models and Measures for Life Annuity and Longevity Bonds

Mortality model

$$\mu_x(t) = Y_1(t) + Y_2(t)c^x \tag{1}$$

$Y_1(t)$ and $Y_2(t)$ are base and age-dependent mortality risk factors. Stochastic differential equations for the mortality risk factors:

$$dY_i(t) = -a_i Y_i(t)dt + \sigma_i dW_i^{\mathbb{Q}}(t) \text{ , for } i = 1, 2 \tag{2}$$

Interest rate model

$$dr(t) = \kappa_r(\theta_r - r(t))dt + \sigma_r\sqrt{r(t)}dW_r^{\mathbb{Q}}(t) \tag{3}$$

Fisher-Weil Duration and Convexity, Delta and Gamma

$\widetilde{\mathcal{D}} = \frac{\mathcal{D}}{P}$	$\widetilde{\mathcal{C}} = \frac{\mathcal{C}}{P}$	
$\widetilde{\Delta}_{Y_1(t)} = \frac{\Delta_{Y_1(t)}}{P}$ $\widetilde{\Gamma}_{Y_1(t)} = \frac{\Gamma_{Y_1(t)}}{P}$	$\widetilde{\Delta}_{Y_2(t)} = \frac{\Delta_{Y_2(t)}}{P}$ $\widetilde{\Gamma}_{Y_2(t)} = \frac{\Gamma_{Y_2(t)}}{P}$	$\widetilde{\Delta}_{r(t)} = \frac{\Delta_{r(t)}}{P}$ $\widetilde{\Gamma}_{r(t)} = \frac{\Gamma_{r(t)}}{P}$

Risk Measures for Life Annuity and Longevity Bonds

Life Annuity:

Code	Maturity	TTM	Freq	Price	$\widetilde{\Delta}_{Y_1(t)}$	$\widetilde{\Delta}_{Y_2(t)}$	$\widetilde{\Delta}_{r(t)}$	$\widetilde{\Gamma}_{Y_1(t)}$	$\widetilde{\Gamma}_{Y_2(t)}$	$\widetilde{\Gamma}_{r(t)}$	$\widetilde{\mathcal{D}}$	$\widetilde{\mathcal{C}}$
IA-WL	∞	∞	12	127.67	-7.79	-5.08E+03	-2.27	98.20	7.85E+07	5.78	8.12	109.23

Assumed Longevity Bonds:

Code	Maturity	TTM	Freq	Price	$\widetilde{\Delta}_{Y_1(t)}$	$\widetilde{\Delta}_{Y_2(t)}$	$\widetilde{\Delta}_{r(t)}$	$\widetilde{\Gamma}_{Y_1(t)}$	$\widetilde{\Gamma}_{Y_2(t)}$	$\widetilde{\Gamma}_{r(t)}$	$\widetilde{\mathcal{D}}$	$\widetilde{\mathcal{C}}$
LB65-19	30/06/2019	5	1	420.94	-2.83	-1.00E+03	-1.71	9.92	1.31E+06	3.23	2.86	10.17
LB65-24	30/06/2024	10	1	699.86	-4.74	-1.96E+03	-2.11	29.81	5.66E+06	4.89	4.83	31.23
LB65-29	30/06/2029	15	1	866.81	-6.19	-2.94E+03	-2.26	53.41	1.45E+07	5.58	6.36	57.03
LB65-34	30/06/2034	20	1	956.71	-7.18	-3.86E+03	-2.33	74.98	2.88E+07	5.87	7.43	81.38
LB65-39	30/06/2039	25	1	998.30	-7.77	-4.59E+03	-2.35	90.50	4.70E+07	5.98	8.07	99.45
LB65-44	30/06/2044	30	1	1,013.60	-8.03	-5.05E+03	-2.36	98.84	6.47E+07	6.03	8.36	109.46
LB65-49	30/06/2049	35	1	1,017.70	-8.12	-5.24E+03	-2.36	101.87	7.64E+07	6.04	8.46	113.21
LB65-54	30/06/2054	40	1	1,018.40	-8.13	-5.30E+03	-2.36	102.52	8.10E+07	6.04	8.48	114.03
LB65-59	30/06/2059	45	1	1,018.40	-8.13	-5.30E+03	-2.36	102.58	8.19E+07	6.04	8.48	114.11
LB65-64	30/06/2064	50	1	1,018.40	-8.13	-5.30E+03	-2.36	102.59	8.20E+07	6.04	8.48	114.12

Note: Interest rate delta compared to duration. Delta $Y_1(t)$ compared to duration. Gamma $Y_1(t)$ compared to convexity.

Risk Measures for Life Annuity and Longevity Bonds

Life Annuity:

Code	Maturity	TTM	Freq	Price	$\widetilde{\Delta}_{Y_1(t)}$	$\widetilde{\Delta}_{Y_2(t)}$	$\widetilde{\Delta}_{r(t)}$	$\widetilde{\Gamma}_{Y_1(t)}$	$\widetilde{\Gamma}_{Y_2(t)}$	$\widetilde{\Gamma}_{r(t)}$	$\widetilde{\mathcal{D}}$	$\widetilde{\mathcal{C}}$
IA-WL	∞	∞	12	127.67	-7.79	-5.08E+03	-2.27	98.20	7.85E+07	5.78	8.12	109.23

Assumed Longevity Bonds:

Code	Maturity	TTM	Freq	Price	$\widetilde{\Delta}_{Y_1(t)}$	$\widetilde{\Delta}_{Y_2(t)}$	$\widetilde{\Delta}_{r(t)}$	$\widetilde{\Gamma}_{Y_1(t)}$	$\widetilde{\Gamma}_{Y_2(t)}$	$\widetilde{\Gamma}_{r(t)}$	$\widetilde{\mathcal{D}}$	$\widetilde{\mathcal{C}}$
LB65-19	30/06/2019	5	1	420.94	-2.83	-1.00E+03	-1.71	9.92	1.31E+06	3.23	2.86	10.17
LB65-24	30/06/2024	10	1	699.86	-4.74	-1.96E+03	-2.11	29.81	5.66E+06	4.89	4.83	31.23
LB65-29	30/06/2029	15	1	866.81	-6.19	-2.94E+03	-2.26	53.41	1.45E+07	5.58	6.36	57.03
LB65-34	30/06/2034	20	1	956.71	-7.18	-3.86E+03	-2.33	74.98	2.88E+07	5.87	7.43	81.38
LB65-39	30/06/2039	25	1	998.30	-7.77	-4.59E+03	-2.35	90.50	4.70E+07	5.98	8.07	99.45
LB65-44	30/06/2044	30	1	1,013.60	-8.03	-5.05E+03	-2.36	98.84	6.47E+07	6.03	8.36	109.46
LB65-49	30/06/2049	35	1	1,017.70	-8.12	-5.24E+03	-2.36	101.87	7.64E+07	6.04	8.46	113.21
LB65-54	30/06/2054	40	1	1,018.40	-8.13	-5.30E+03	-2.36	102.52	8.10E+07	6.04	8.48	114.03
LB65-59	30/06/2059	45	1	1,018.40	-8.13	-5.30E+03	-2.36	102.58	8.19E+07	6.04	8.48	114.11
LB65-64	30/06/2064	50	1	1,018.40	-8.13	-5.30E+03	-2.36	102.59	8.20E+07	6.04	8.48	114.12

Note: Interest rate delta compared to duration. Delta $Y_1(t)$ compared to duration. Gamma $Y_1(t)$ compared to convexity.

Outline

1. Motivation and Key Insights
2. Interest Rate and Longevity Risk
3. **Optimal Bond Portfolios**
4. Portfolio Surplus Distribution and Hedge Effectiveness
5. Summary and Wrap Up

Linear Programming: Duration-Convexity Immunization

Matches the Fisher-Weil dollar durations and minimizes portfolio risk arising from convexity:

$$\max_{w_k, w_j} \left\{ \sum_k w_k \cdot C[a_k] + \sum_j w_j \cdot C[a_j] \right\}$$

subject to:

- $\sum_{t>0} n_t (t-h)^+ \leq 0,$ for all positive h
- Match Surplus at t=0
- Match Duration at t=0

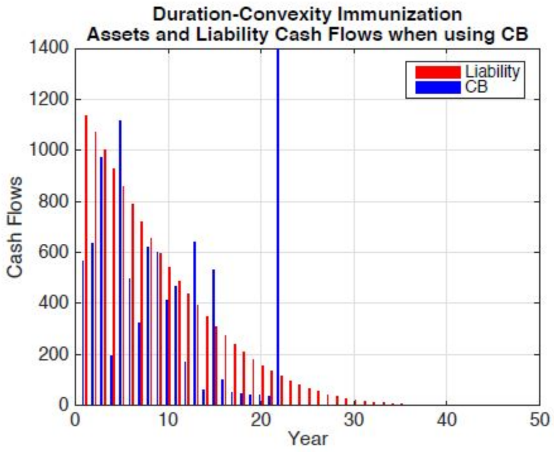
where w_k is the allocation to each fixed-income asset (Coupon and Annuity Bond), and w_j is the allocation to each Longevity Bond.

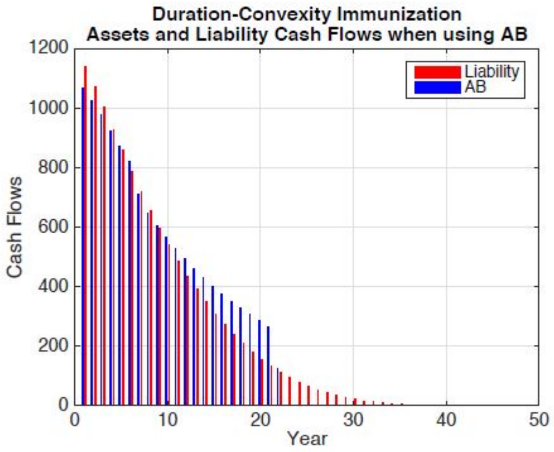
Optimal Bond Portfolio to Immunize a Life Annuity Issued to 65 year olds

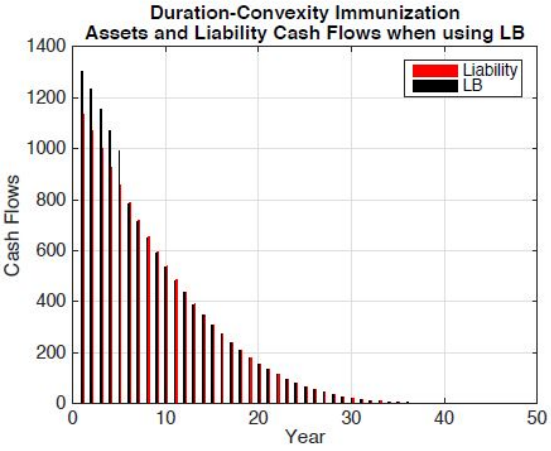
Bond	Weight	Bond	Weight
	Only Coupon Bonds		
GSBS-CB-14	0.03	GSBK-CB-16	0.01
GSBS-CB-15	0.04	GSBC-CB-17	0.08
GSBS-CB-18	0.09	GSBE-CB-19	0.01
GSBG-CB-23	0.06	GSBG-CB-20	0.05
GSBG-CB-24	0.04	GSBI-CB-21	0.03
GSBG-CB-25	0.05	ACG-CB-22	0.05
GSBG-CB-26	0.01	ACT-CB-30	0.01
GSBG-CB-27	0.09	NSWTC-CB-25	0.01
GSBG-CB-29	0.07	NSWTC-CB-35	0.24
GSBG-CB-15	0.02	SYDAIR-CB-20	0.02
	Only Annuity Bonds		
Praeco-AB-20	0.05	JEM(NSWSch)-AB-35	0.94
JEM-AB-35	0.01	-	-
	Only Longevity Bonds		
LB65-19	0.06	LB65-64	0.08
LB65-59	0.86	-	-
	Coupon Bonds and Longevity Bonds		
GSBS-CB-14	0.04	LB65-64	0.08
LB65-59	0.87	-	-
	Annuity Bonds and Longevity Bonds		
LB65-19	0.06	LB65-64	0.08
LB65-59	0.86	-	-

Optimal Duration-Convexity Bond Portfolio – Cash Flows

Annualized Asset and Liability Cash Flows - Different Bond Portfolios







Linear Programming: Delta-Gamma Hedging

Minimize the weighted gamma of assets:

$$\min_{w_k, w_j} \left| \sum_k w_k \cdot \left\{ \sigma_r \cdot \Gamma_{r(t)}[a_k] \right\} + \sum_j w_j \cdot \left\{ \sigma_r \cdot \Gamma_{r(t)}[a_j] + \sigma_1 \cdot \Gamma_{Y_1(t)}[a_j] + \sigma_2 \cdot \Gamma_{Y_2(t)}[a_j] \right\} \right| \quad (4)$$

subject to conditions:

- Match Surplus at t=0
- Match Delta at t=0

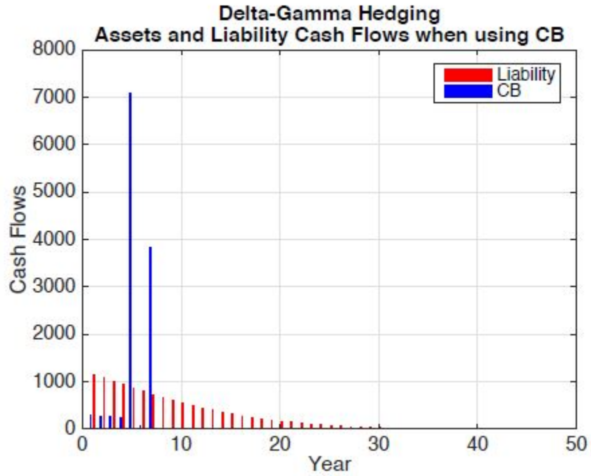
where w_k is the allocation to each fixed-income asset, and w_j is the allocation to each longevity-linked asset.

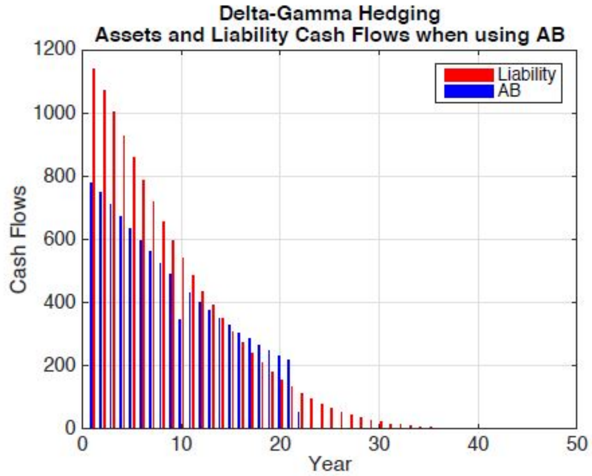
Optimal Bond Portfolio to Delta-Gamma Hedge a Life Annuity Issued to 65 year olds

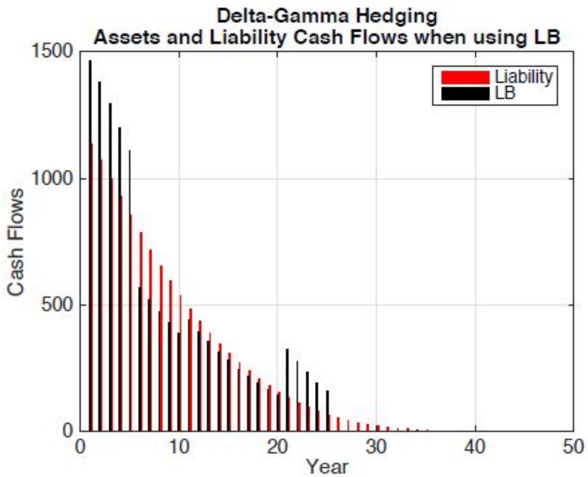
Bond	Weight	Bond	Weight
	Only Coupon Bonds		
GSBS-CB-18	0.65	RABO-CB-20	0.35
	Only Annuity Bonds		
NSWWAB10-AB-23	0.24	JEM-AB-35	0.76
	Only Longevity Bonds		
LB65-19	0.22	LB65-34	-1.34
LB65-24	-0.12	LB65-39	2.23
	Coupon Bonds and Longevity Bonds		
GSBG-CB-15	0.02	LB65-34	-1.44
LB65-19	0.13	LB65-39	2.28
	Annuity Bonds and Longevity Bonds		
LB65-19	0.22	LB65-34	-1.34
LB65-24	-0.12	LB65-39	2.23

Note: short and long positions in longevity bonds - a deferred longevity bond.

Optimal Delta-Gamma Hedge Bond Portfolio – Cash Flows



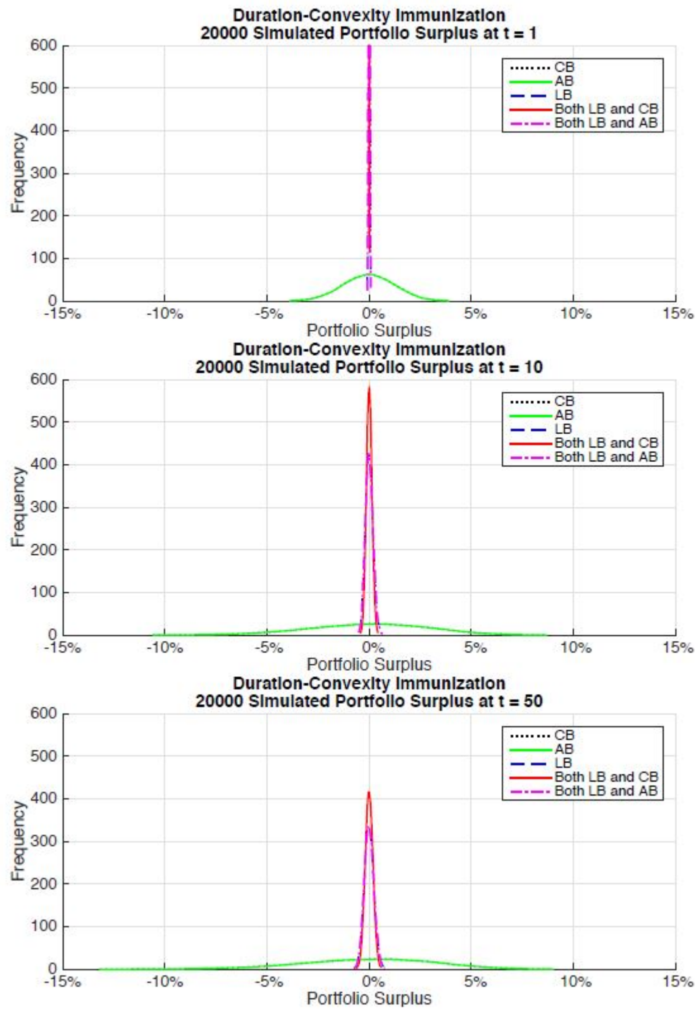




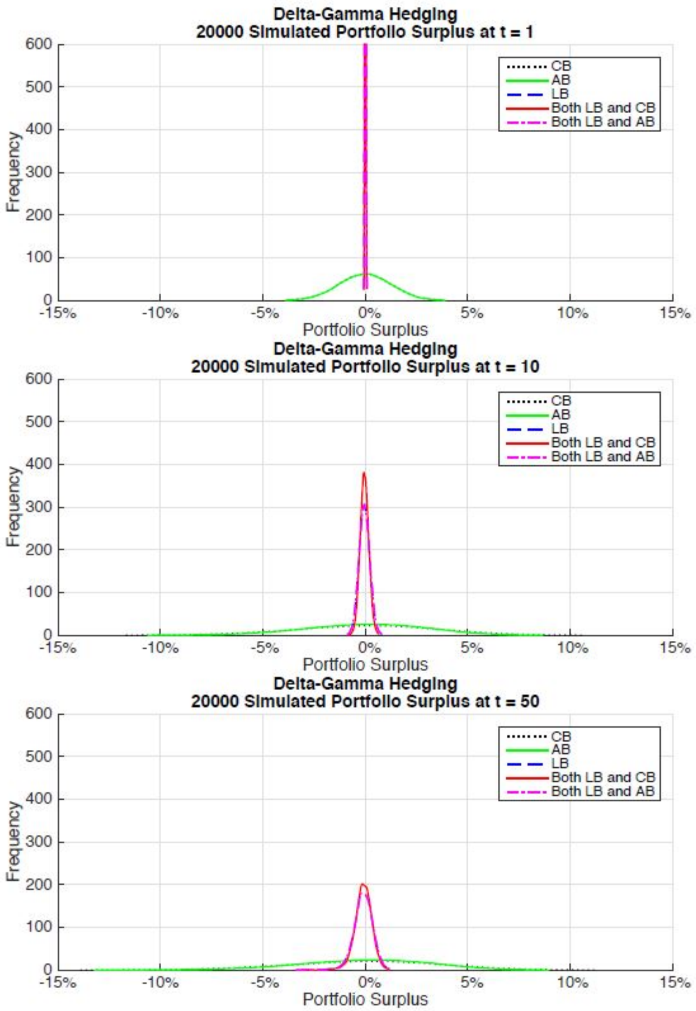
Outline

1. Motivation and Key Insights
2. Interest Rate and Longevity Risk
3. Optimal Bond Portfolios
4. **Portfolio Surplus Distribution and Hedge Effectiveness**
5. Summary and Wrap Up

Optimal Duration-Convexity Bond Portfolio – Surplus Distribution



Optimal Delta-Gamma Hedge Bond Portfolio – Surplus Distribution



Measuring Hedge Effectiveness

- Risk measures used for portfolio surplus - VaR, Volatility and Expected Shortfall
- Measures risk reduction as a percentage of the risk

$HE(VaR_\alpha)$	$HE(\sigma)$	$HE(ES_\alpha)$
$HE(VaR_\alpha) = \frac{VaR_\alpha(S_t) - VaR_\alpha(S_t^*)}{VaR_\alpha(S_t)}$	$HE(\sigma) = \frac{\sigma(S_t) - \sigma(S_t^*)}{\sigma(S_t)}$	$HE(ES_\alpha) = \frac{ES_\alpha(S_t) - ES_\alpha(S_t^*)}{ES_\alpha(S_t)}$

Hedge effectiveness-Annuity Bonds and Longevity Bonds

Measure	t=1	t=10	t=50
	Delta-Gamma Hedging: AB compared to CB		
$HE(\sigma)$	0 %	-13 %	-13 %
$HE(VaR_{0.5\%})$	-0 %	-8 %	-8 %
$HE(ES_{0.5\%})$	-0 %	-8 %	-6 %
	Immunization: AB compared to CB		
$HE(\sigma)$	2 %	1 %	-7 %
$HE(VaR_{0.5\%})$	0 %	0 %	-0 %
$HE(ES_{0.5\%})$	-0 %	0 %	-1 %
	Delta-Gamma Hedging: LB compared to CB		
$HE(\sigma)$	-98 %	-93 %	-87 %
$HE(VaR_{0.5\%})$	-98 %	-92 %	-83 %
$HE(ES_{0.5\%})$	-98 %	-92 %	-75 %
	Immunization: LB compared to CB		
$HE(\sigma)$	-98 %	-94 %	-93 %
$HE(VaR_{0.5\%})$	-98 %	-95 %	-94 %
$HE(ES_{0.5\%})$	-98 %	-95 %	-93 %

Note: Annuity bonds are little different compared to coupon bonds (no longevity risk hedged). Longevity bonds practically eliminate all risk.

Hedge effectiveness-Delta-Gama Hedging compared to Immunization

Measure	t=1	t=10	t=50
Using CB: Delta-Gamma Hedging compared to Immunization			
$HE(\sigma)$	- 0 %	15 %	14 %
$HE(VaR_{0.5\%})$	0 %	8 %	8 %
$HE(ES_{0.5\%})$	0 %	9 %	6 %
Using AB: Delta-Gamma Hedging compared to Immunization			
$HE(\sigma)$	-0 %	0 %	0 %
$HE(VaR_{0.5\%})$	0 %	-0 %	-0 %
$HE(ES_{0.5\%})$	0 %	0 %	0 %
Using LB: Delta-Gamma Hedging compared to Immunization			
$HE(\sigma)$	-13 %	41 %	113 %
$HE(VaR_{0.5\%})$	-20 %	62 %	228 %
$HE(ES_{0.5\%})$	-20 %	77 %	272 %

Note: Annuity bonds are effective over all horizons. Coupon bonds are more effective for long horizons with immunization. For delta-gamma hedging longevity bonds need frequent re-balancing.

Outline

1. Motivation and Key Insights
2. Interest Rate and Longevity Risk
3. Optimal Bond Portfolios
4. Portfolio Surplus Distribution and Hedge Effectiveness
5. **Summary and Wrap Up**

Summary

- Fisher-Weil immunization can effectively immunize both interest rate and mortality (longevity) risk.
- Annuity bonds provide better cash flow matching than coupon bonds, however the hedge effectiveness are similar.
- Delta-gamma hedging with longevity bonds include both deferred (negative short maturity bond and positive longer maturity bond) and immediate longevity bonds.
- Longevity bonds alone provide very effective hedging of longevity risk in life annuities - no surprise here but it is surprising that these are not available in government or private bond markets.
- Raises policy issues as to why the market and government should consider the design of (deferred) longevity bonds to support life insurers offering annuities.

Research Contributions

Extensions on the current mortality immunization literature Tsai and Chung (2013), Lin and Tsai (2013), and Luciano et al. (2012):

- Incorporated fixed-income securities and longevity-linked securities to manage both interest rate risk and mortality risk.
- Developed linear programming methods to solve the optimal asset allocation problems considering any number of assets in the portfolio (not limited by the number of equality conditions).
- Compared delta-gamma hedging with duration-convexity hedging and used stochastic models to assess both interest rate and mortality risks.

Thank you – Questions or Discussion

Thank you for your attention!

Contact: email m.sherris@unsw.edu.au or estelle.liu@mine.com.au

Find the paper on SSRN

http://papers.ssrn.com/sol3/papers.cfm?abstract_id=2596989

Reference

Cox, J. C., Ingersoll Jr, J. E. & Ross, S. A. (1985), 'A theory of the term structure of interest rates', *Econometrica* **53**(2), 385–408.

Lin, T. & Tsai, C. C.-L. (2013), 'On the mortality/longevity risk hedging with mortality immunization', *Insurance: Mathematics and Economics* **53**(3), 580–596.

Luciano, E., Regis, L. & Vigna, E. (2012), 'Delta–gamma hedging of mortality and interest rate risk', *Insurance: Mathematics and Economics* **50**(3), 402–412.

Schrager, D. F. (2006), 'Affine stochastic mortality', *Insurance: mathematics and economics* **38**(1), 81–97.

Tsai, C. C.-L. & Chung, S.-L. (2013), 'Actuarial applications of the linear hazard transform in mortality immunization', *Insurance: Mathematics and Economics* **53**(1), 48–63.

Ziveyi, J., Blackburn, C. & Sherris, M. (2013), 'Pricing European options on deferred annuities', *Insurance: Mathematics and Economics* **52**(2), 300–311.