Kurtosis and skewness estimation for non-life reserve risk distribution

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Abstract: In the daily tasks of a non-life actuary, the reserve risk distribution plays a central role. For example, the estimation of the cost of capital used in commutation pricing relies heavily on the assumption retained for the shape of the non-life reserve risk distribution.
Even though some distributions are widely used in the actuarial community (e.g. Lognormal distribution), it is interesting to note that very little is known on the determinants of the shape of the non-life reserve risk distribution. In general, the mean is usually defined as the Best Estimate and the standard deviation can be estimated using different methods (e.g. Mack 1993a). In terms of higher moments, Generalized Linear Models (GLM) and bootstrap technics offer different possibilities of quantifying moments and quantiles (see Wüthrich-Merz 2008 and England and Verrall 2002). However, these models require the specification of some explicit parametric distribution (e.g. for the residuals) in order to be applied.
Following a first introduction of skewness estimation of non-life reserve risk distribution (see Dal Moro 2012), this article investigates the possibility to estimate the kurtosis of the non-life reserve risk distribution. In addition, the robustness of the skewness and kurtosis estimation based on the proposed formulas is tested on eight different triangles.

Keywords. Skewness; Kurtosis; Platykurtic; Variance; Chain-Ladder; Reserve risk distribution; Correlation; Gaussian copula; Generalized Pareto Distribution; Johnson distribution.

1. INTRODUCTION

1.1 Background

Today, whenever a non-life actuary has the necessity to use and determine the full reserve risk distribution, he has several options:

- The most usual option consists in assuming that the full reserve risk distribution follows a lognormal distribution which parameters are fitted to the reserve Best Estimate and to the reserve standard deviation. The reserve standard deviation can be determined using the Mack method in the Chain-Ladder framework (see Mack 1993a) or in the Bornhuetter-Ferguson framework (see Mack 2008) or using the Hybrid Chain-Ladder method (see Arbenz 2010).
- Another option consists of using bootstrapping techniques to determine the full empirical distribution (see England and Verall 2006).
- A final option consists of using GLM techniques (see Merz and Wüthrich 2008).

The GLM and Lognormal assumptions essentially rely on the first and second moment of

the distribution and do not attempt to reflect any further knowledge of higher moments. In the same way, the bootstrapping techniques are based on resampling techniques which do not reflect any knowledge of higher moments.

This article tries, in a first step, to determine some knowledge on higher moments of the reserve risk distribution, in particular skewness and kurtosis. With this knowledge, it tries to draw general conclusions on the appropriateness of the Lognormal assumption on the basis of test cases.

In all the following, we will limit ourselves to the Chain-Ladder framework.

1.2 Outline

This paper is divided into the following sections:

- Section 2 provides basic definitions of chain-ladder coefficients
- Section 3 establishes the estimator of the kurtosis of the incurred claim amount for each development year of each accident year.
- Section 4 provides a description of the simulations done to estimate the skewness and the kurtosis for non-life reserves for each accident year of a triangle.
- Section 5 provides numerical examples.

2. DEFINITIONS

2.1 Definitions of Chain-Ladder elements

The following brief review of the Chain-Ladder method is based on Mack (1993a).

Let $C_{i,k}$ denote the cumulative incurred claims amount of accident year i after k years of development, 1 ≤ i, k ≤ I, where I denotes the most recent accident year. Then $C_{I-i+1,i}$ denotes the currently known claims amount of accident year I-i+1. The basic chain-ladder assumption is that there exists development factors $f_1, \ldots, f_{I-1}$ such that:

$$E\left(C_{i,k+1} \mid C_{i,1},\ldots,C_{i,k}\right) = f_k C_{i,k}, 1 \le i \le I, 1 \le k \le I-1 \quad (1)$$

where the link ratios (age-to-age factors) can be estimated as follows:

$$\hat{f}_k = \frac{\sum_{j=1}^{I-k} C_{j,k+1}}{\sum_{j=1}^{I-k} C_{j,k}}, 1 \le k \le I\text{-}1 \quad (2)$$

under the assumption that $\{C_{i,1},\ldots,C_{i,I}\}, \{C_{j,1},\ldots,C_{j,I}\}, i \neq j$ are independent.

In this paper, $\hat{f}$ will denote the estimator of the random variable f.

Note: The $\hat{f}_k$ are unbiased and uncorrelated (see Mack 1993a).

2.2 Variance of $C_{i,k}$

In the framework of distribution-free calculation of the standard error of the reserve estimates, several variance models exist. For the purpose of this article, we will focus on the Mack standard error.

As for the variance of $C_{i,k+1}$, T. Mack (1993a) induced that $Var\left(C_{i,k+1} \mid C_{i,1},...,C_{i,k}\right)$ (where $Var(A \mid B)$ denotes the conditional variance of A knowing B) should be proportional to $C_{i,k}$, i.e.:

$$Var\left(C_{i,k+1} \mid C_{i,1},...,C_{i,k}\right) = C_{i,k}\sigma_k^2 \text{ , } 1 \le i \le I, \; 1 \le k \le I\text{-}1 \tag{3}$$

where $\hat{\sigma}_k^2 = \frac{1}{I-k-1}\sum_{i=1}^{I-k} C_{i,k}\left(\frac{C_{i,k+1}}{C_{i,k}} - \hat{f}_k\right)^2 \quad for\ 1 \le k \le I\text{-}2$ (4)

It can be shown that the estimator $\hat{\sigma}_k^2$ is unbiased (see Mack 1993a).

2.3 Skewness of $C_{i,k}$

Let's denote $SK(A \mid B)$, the conditional skewness of A knowing B:

$$SK(A|B) = E\left(\left(A - E(A|B)\right)^3 | B\right)$$

Following the result in Dal Moro 2012, we assume that $SK\left(C_{i,k+1} \mid C_{i,1},...,C_{i,k}\right)$ is proportional to $C_{i,k}^{3/2}$, i.e. :

$$SK\left(C_{i,k+1} \mid C_{i,1},...,C_{i,k}\right) = C_{i,k}^{3/2} Sk_k^3 \text{ , } 1 \le i \le I, \; 1 \le k \le I\text{-}2 \tag{5}$$

with:

$$\hat{Sk}_k^3 = \frac{1}{\left(I-k-\frac{\left(\sum_{i=1}^{I-k} C_{i,k}^{3/2}\right)^2}{\left(\sum_{i=1}^{I-k} C_{i,k}\right)^3}\right)} \sum_{i=1}^{I-k} C_{i,k}^{3/2}\left(\frac{C_{i,k+1}}{C_{i,k}} - \hat{f}_k\right)^3 \quad \text{for } 1 \le k \le I\text{-}3 \tag{6}$$

For $k = I\text{-}2$ and $k = I\text{-}1$, as data is not available to estimate $\hat{Sk}_k^3$, it will be assumed that $\hat{Sk}_k^3 = 0$ for this article.

3. KURTOSIS ESTIMATOR

The aim of this section is to find an expression for the kurtosis of $C_{i,k}$ within the Chain-Ladder framework. Let's denote $KT(A|B)$, the conditional fourth centered moment of A knowing B:

$$KT(A|B) = E\left((A - E(A|B))^4 \mid B\right)$$

For the purpose of this article, it will be assumed that:

$\frac{KT\left(C_{i,k+1} \mid C_{i,1},...,C_{i,k}\right)}{\left[Var\left(C_{i,k+1} \mid C_{i,1},...,C_{i,k}\right)\right]^2}$ depends on k but not on i.

This assumption implies that, for one development year within a triangle, the kurtosis is the same for any accident year. This assumption can be applied in the context of a reserving portfolio for which risks would have the same characteristics for every accident year.

With such an assumption, we have:

$$\exists \gamma_k \text{ such that } \gamma_k = \frac{KT\left(C_{i,k+1} \mid C_{i,1},...,C_{i,k}\right)}{\left[Var\left(C_{i,k+1} \mid C_{i,1},...,C_{i,k}\right)\right]^2} = \frac{KT\left(C_{i,k+1} \mid C_{i,1},...,C_{i,k}\right)}{\left[\sigma_k^2 C_{i,k}\right]^2}$$

$$\Rightarrow KT\left(C_{i,k+1} \mid C_{i,1},...,C_{i,k}\right) = \gamma_k \left[\sigma_k^2 C_{i,k}\right]^2$$

Hence, we have the result that $KT\left(C_{i,k+1} \mid C_{i,1},...,C_{i,k}\right)$ is proportional to $C_{i,k}^2$, i.e. :

$$KT\left(C_{i,k+1} \mid C_{i,1},...,C_{i,k}\right) = C_{i,k}^2 Kt_k^4 \text{ , } 1 \le i \le I, \; 1 \le k \le I\text{-}3 \qquad (7)$$

Based on the definition above, the theorem below will give an unbiased estimator for $\hat{Kt}_k^4$.

Theorem:
Similar to the variance expression in equation (4), we show that the estimator below is unbiased.

$$\hat{Kt}_k^4 = \frac{\left[\sum_{i=1}^{I-k} C_{i,k}^2 \left(\frac{C_{i,k+1}}{C_{i,k}} - \hat{f}_k \right)^4 - 3\left(\hat{\sigma}_k^2\right)^2 \left(2 - 6\frac{\sum_{i=1}^{I-k} C_{i,k}^2}{\left(\sum_{i=1}^{I-k} C_{i,k}\right)^2} + 4\frac{\sum_{i=1}^{I-k} C_{i,k}^3}{\left(\sum_{i=1}^{I-k} C_{i,k}\right)^3} \right) \right]}{\left(\sum_{i=1}^{I-k} \left(1 - \frac{C_{i,k}}{\sum_{i=1}^{I-k} C_{i,k}} \right)^4 + \frac{\left(\sum_{i=1}^{I-k} C_{i,k}^2\right)^2 - \sum_{i=1}^{I-k} C_{i,k}^4}{\left(\sum_{i=1}^{I-k} C_{i,k}\right)^4} \right)} \quad \text{for } 1 \le k \le I\text{-}4 \qquad (8)$$

For $k = I\text{-}3$, $k = I\text{-}2$ and $k = I\text{-}1$, as data is not available to estimate $\hat{Kt}_k^4$, it will be assumed that $\frac{\hat{Kt}_k^4}{\left(\hat{\sigma}_k^2\right)^2} = 3$ for this article. This last assumption implies that the characteristic of a normal distribution is retained on the last development years.

The proof of this theorem is given in Appendix A.

Remark:
The formula for $\hat{Kt}_k^4$ has the standard shape of a kurtosis estimator: One element which corresponds to the weighted average of $\left(\frac{C_{i,k+1}}{C_{i,k}} - \hat{f}_k \right)^4$ and another element which has a component $3\left(\hat{\sigma}_k^2\right)^2$ (see formula in Cramer 1946).

4. SIMULATIONS

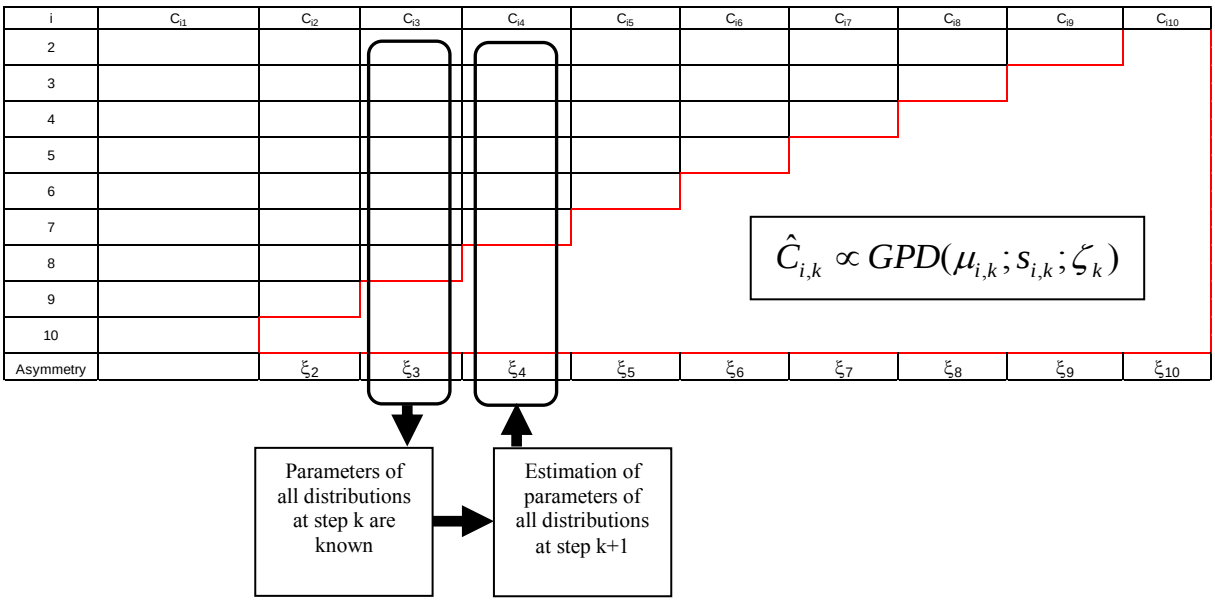
After finding the expressions for the kurtosis and the skewness of $C_{i,k}$, the next step would consist in finding closed formulas for the skewness and kurtosis of non-life reserves by accident year. This should be done by using the law of total cumulants. For example, in the case of skewness, the law of total skewness gives:

$$SK(A) = E[SK(A \mid B)] + SK[E(A \mid B)] + 3Cov[E(A \mid B), Var(A \mid B)]$$

However, when using such laws, there are difficulties which appear rapidly, in particular the need to estimate the element: $E\left(C_{i,I-1}^{3/2} {}_{i,I-1} \mid C_{i,1},...,C_{i,I-2} \right)$.

Considering the above difficulty, in order to estimate the skewness or the kurtosis of non-life reserves by accident year, it is therefore necessary to create a stochastic model. This model will have the characteristics described in this paragraph so that the Best Estimate and standard deviations match the moments coming from the Mack model (1993a). For the triangle below, we will fit the $\hat{C}_{i,k}$ to a Generalized Pareto Distribution (hereinafter "GPD") with parameters $(\mu_{i,k}; s_{i,k}; \zeta_k)$ with $\mu_{i,k} \in (-\infty; \infty)$, the location parameter, $s_{i,k} \in (0; \infty)$, the scale parameter and $\zeta_k \in (-\infty; \infty)$, the shape parameter (we will see below that this shape parameter can be related either to the asymmetry of this distribution or to the kurtosis of the distribution, which according to paragraphs 2.3 and 3 depends only on k).

The stochastic model is iterative: At step k (corresponding to the column k of the below triangle), all the parameters are known. On the basis of these parameters known at step k, the parameters of step k+1 are estimated using the formulas shown in this section.



Note: The chosen distribution (GPD) is merely a medium to use the known skewness $\hat{Sk}_k^3$ or the known kurtosis $\hat{Kt}_k^4$ of a development year so as to estimate an accident year skewness / kurtosis.

With the above parameters, we have the following results:

- $\hat{Mean}_{i,k} = \mu_{i,k} + \frac{s_{i,k}}{1-\zeta_k} = E\left(\hat{C}_{i,k}\right)$
- $\hat{Variance}_{i,k} = \frac{s_{i,k}^2}{(1-\zeta_k)^2(1-2\zeta_k)} = Var\left(\hat{C}_{i,k}\right)$ $\left(\text{with } \zeta_k < 1/2\right)$
- $Skewness\left(\hat{C}_{i,k}\right) = \frac{SK\left(\hat{C}_{i,k}\right)}{Var\left(\hat{C}_{i,k}\right)^{3/2}} = \frac{2(1+\zeta_k)\sqrt{1-2\zeta_k}}{1-3\zeta_k}$
- $Kurtosis\left(\hat{C}_{i,k}\right) = \frac{KT\left(\hat{C}_{i,k}\right)}{Var\left(\hat{C}_{i,k}\right)^2} = \frac{3\left(3-5\zeta_k-4\zeta_k^3\right)}{\left(1-7\zeta_k+12\zeta_k^2\right)}$

At step k of the stochastic run (which corresponds to a development year in the triangle above), assuming that we know the 3 parameters $(\mu_{i,k}; s_{i,k}; \zeta_k)$, we can generate the random variable $C_{i,k}$ with the following method:

- Draw a random variable $U_{i,k}$ uniformly distributed on $(0,1]$. We will see later in this section that there is a dependency structure between the $U_{i,k}$ which follows a Gaussian copula that needs to be specified to match the assumptions of Mack (1993a).
- Then: $\hat{C}_{i,k} = \mu_{i,k} + \frac{s_{i,k}\left(U_{i,k}^{-\zeta_k} - 1\right)}{\zeta_k} \propto GPD(\mu_{i,k}; s_{i,k}; \zeta_k)$

At step k+1, we have the following formulas:

- In the case of skewness evaluation, we have: $\frac{\hat{Sk}_{k+1}^3}{\hat{\sigma}_{k+1}^3} = \frac{2(1+\zeta_{k+1})\sqrt{1-2\zeta_{k+1}}}{1-3\zeta_{k+1}}$ which allows to estimate the parameter ζ_{k+1}.

  In the case of kurtosis evaluation, we have: $\frac{\hat{Kt}_{k+1}^4}{\left(\hat{\sigma}_{k+1}^2\right)^2} = \frac{3\left(3-5\zeta_{k+1}-4\zeta_{k+1}^3\right)}{\left(1-7\zeta_{k+1}+12\zeta_{k+1}^2\right)}$

- $\hat{V}ariance_{i,k+1} = \hat{C}_{i,k}\hat{\sigma}_k^2\left(1+\frac{\hat{C}_{i,k}}{\sum_{j=1}^{I-k} C_{j,k}}\right)$. The use of this definition allows to match the overall mean squared error (hereinafter "mse") resulting from this simulation with the Mack mse (see also Mack 1999):

$$\hat{m}se\left(\hat{R}_i\right) = \hat{C}_{i,I}\sum_{k=I+1-i}^{I-1}\frac{\hat{\sigma}_k^2}{\hat{f}_k^2}\left(\frac{1}{\hat{C}_{i,k}}+\frac{1}{\sum_{j=1}^{I-k} C_{j,k}}\right) \text{ where } \hat{R}_i = \hat{C}_{i,I} - C_{i,I+1-i}$$

  The knowledge of $\hat{V}ariance_{i,k+1}$ and of the parameter ζ_{k+1} allows the estimation of the parameter $s_{i,k+1}$.

- Finally, we have: $\hat{C}_{i,k+1} = \hat{f}_k\hat{C}_{i,k} = \mu_{i,k+1} + \frac{s_{i,k+1}}{1-\zeta_{k+1}}$ which allows the estimation of the parameter $\mu_{i,k+1}$.

Gaussian copula:

According to Mack (1993a), the mse of the overall reserve estimate $\hat{R} = \hat{R}_2 + ... + \hat{R}_I$ is equal to:

$$\hat{m}se(\hat{R}) = \sum_{i=2}^{I}\left\{\hat{m}se\left(\hat{R}_i\right) + \hat{C}_{i,I}\left(\sum_{j=i+1}^{I}\hat{C}_{j,I}\right)\sum_{k=I+1-i}^{I-1}\frac{2\hat{\sigma}_k^2 \big/ \hat{f}_k^2}{\sum_{n=1}^{I-k} C_{n,k}}\right\}$$

This equation includes an embedded assumptions of correlation between the reserve estimates $\hat{R}_i$. A natural way to simulate such correlations is to use a Gaussian copula.

Re-writing the above equation in the usual way, we have:

$$\hat{m}se(\hat{R}) = \sum_{i=2}^{I}\hat{m}se\left(\hat{R}_i\right) + 2\sum\sum_{i<j}\rho_{i,j}\sqrt{\hat{m}se\left(\hat{R}_i\right)\hat{m}se\left(\hat{R}_j\right)}$$

And we can see that the correlations between the reserve estimates $\hat{R}_i$ are:

$$\rho_{i,j}^{Ultimate} = \frac{\hat{C}_{i,I}\hat{C}_{j,I}\sum_{k=I+1-i}^{I-1}\frac{\hat{\sigma}_k^2 \Big/ \hat{f}_k^2}{\sum_{n=1}^{I-k} C_{n,k}}}{\sqrt{\hat{mse}\left(\hat{R}_i\right)\hat{mse}\left(\hat{R}_j\right)}}$$

These correlations are valid at the ultimate. In the case of our stepwise simulation, it is necessary to have a correction on these correlations so as to reflect the fact that, at step k, the view is not yet on the ultimate. As a matter of fact, if we denote $\rho_{i,j}^{(k)}$ the correlation at step k , it is equal to:

$$\rho_{i,j}^{(k)} = \frac{\hat{C}_{i,k}\hat{C}_{j,k}\sum_{m=I+1-i}^{I-1}\frac{\hat{\sigma}_m^2 \Big/ \hat{f}_m^2}{\sum_{n=1}^{I-m} C_{n,m}}}{\sqrt{\hat{mse}\left(\hat{C}_{i,k}\right)\hat{mse}\left(\hat{C}_{j,k}\right)}}$$

where:

$$\hat{mse}\left(\hat{C}_{i,k}\right) = \hat{C}_{i,k}\sum_{m=I+1-i}^{k-1}\frac{\hat{\sigma}_m^2}{\hat{f}_m^2}\left(\frac{1}{\hat{C}_{i,m}} + \frac{1}{\sum_{j=1}^{I-m} C_{j,k}}\right)$$

In order to have a correct correlation on an ultimate view, we have the following:

$$Correction_{i,j}^{(k)} = \frac{\rho_{i,j}^{(k)}}{\rho_{i,j}^{Ultimate}} = \frac{1}{\prod_{m=k}^{I}\hat{f}_m^2}\sqrt{\frac{\hat{mse}\left(\hat{R}_i\right)\hat{mse}\left(\hat{R}_j\right)}{\hat{mse}\left(\hat{C}_{i,k}\right)\hat{mse}\left(\hat{C}_{j,k}\right)}}$$

The correlations $\rho_{i,j}^{Ultimate} \times Correction_{i,j}^{(k)}$ are the characteristics of the Gaussian copulas that are used in our stepwise simulation.

5. NUMERICAL EXAMPLES

In Appendix B, the triangles used for the application of the above methodology are shown. In this section, we will therefore focus on the results and their interpretations.

In the table below, for each triangle (see Appendix B) and each development year, the chain-ladder coefficients, the Mack variance, the skewness and the kurtosis are provided:

k			1	2	3	4	5	6	7	8	9
Private Motor	Farmers Alliance	$\hat{f}_k$	1.076	0.992	0.994	0.996	0.996	1.001	1.001	0.995	1.000
		$\hat{\sigma}_k^2$	52.813	7.223	3.966	2.462	2.311	0.946	0.199	0.161	0.130
		$\hat{Sk}_k^3/(\hat{\sigma}_k^2)^{3/2}$	0.611	-0.256	-0.349	-0.090	1.049	0.477	0.273	NA	NA
		$\hat{Kt}_k^4/(\hat{\sigma}_k^2)^2$	317.02%	177.30%	166.23%	146.75%	340.69%	165.38%	NA	NA	NA
Private Motor	NC Farm Bureau	$\hat{f}_k$	1.290	1.038	0.997	0.988	0.991	0.998	0.998	0.998	0.996
		$\hat{\sigma}_k^2$	412.435	55.605	12.434	2.448	0.685	1.093	0.058	0.605	0.058
		$\hat{Sk}_k^3/(\hat{\sigma}_k^2)^{3/2}$	0.703	0.412	0.727	-0.047	-0.058	-0.769	0.500	NA	NA
		$\hat{Kt}_k^4/(\hat{\sigma}_k^2)^2$	223.54%	201.65%	182.26%	78.01%	144.84%	237.28%	NA	NA	NA
Private Motor	New Jersey Manufacturers	$\hat{f}_k$	1.331	1.158	1.070	0.990	0.980	0.990	0.994	1.001	1.000
		$\hat{\sigma}_k^2$	105.139	45.008	14.086	26.096	4.394	2.036	0.133	0.829	0.133
		$\hat{Sk}_k^3/(\hat{\sigma}_k^2)^{3/2}$	0.583	0.187	0.414	-0.565	-0.141	0.230	-0.008	NA	NA
		$\hat{Kt}_k^4/(\hat{\sigma}_k^2)^2$	204.32%	229.15%	207.06%	192.21%	103.56%	104.25%	NA	NA	NA
Product Liability	Pennsylvania	$\hat{f}_k$	1.451	1.407	1.091	0.963	1.046	1.051	0.950	0.992	1.030
		$\hat{\sigma}_k^2$	190.687	220.494	8.509	155.235	10.557	3.010	4.250	0.094	0.002
		$\hat{Sk}_k^3/(\hat{\sigma}_k^2)^{3/2}$	0.774	1.716	-0.540	-1.059	-0.620	0.164	0.111	NA	NA
		$\hat{Kt}_k^4/(\hat{\sigma}_k^2)^2$	324.61%	620.07%	298.72%	369.19%	164.25%	119.36%	NA	NA	NA
Product Liability	West Bend	$\hat{f}_k$	1.692	1.487	1.269	1.016	1.150	1.130	0.862	1.007	1.000
		$\hat{\sigma}_k^2$	31.078	66.326	70.197	33.319	23.011	3.421	14.919	0.015	0.000
		$\hat{Sk}_k^3/(\hat{\sigma}_k^2)^{3/2}$	-0.008	1.060	0.525	-0.507	-0.030	-0.484	-0.113	NA	NA
		$\hat{Kt}_k^4/(\hat{\sigma}_k^2)^2$	205.13%	411.28%	302.64%	200.90%	125.54%	121.50%	NA	NA	NA
Mack 1993		$\hat{f}_k$	3.491	1.747	1.457	1.174	1.104	1.086	1.054	1.077	1.018
		$\hat{\sigma}_k^2$	160280	37737	41965	15183	13731	8186	447	1147	447
		$\hat{Sk}_k^3/(\hat{\sigma}_k^2)^{3/2}$	0.137	0.215	0.638	-0.433	0.402	-0.026	-0.497	NA	NA
		$\hat{Kt}_k^4/(\hat{\sigma}_k^2)^2$	184.92%	170.29%	265.62%	162.92%	185.65%	123.00%	NA	NA	NA

Table 1: Chain-ladder coefficients, Mack variance, Skewness and Kurtosis estimators for 10 x 10 triangles

Kurtosis and skewness estimation for non-life reserve risk distribution

k			1	2	3	4	5	6	7	8	9	10	11	12	13	14
SCOR	Casualty Prop	$\hat{f}_k$	6.82	1.22	1.04	1.01	0.99	1.01	1.02	1.00	1.00	1.02	1.00	1.01	0.99	1.00
		$\hat{\sigma}_k^2$	131'803'806	459'208	565'956	323'842	278'372	120'753	75'674	34'013	15'870	98'673	5'598	429	122	35
		$\hat{Sk}_k^3/(\hat{\sigma}_k^2)^{3/2}$	1.031	0.070	-0.372	-0.119	-0.390	0.537	-0.334	0.863	-0.842	0.868	-0.843	0.518	NA	NA
		$\hat{Ku}_k^4/(\hat{\sigma}_k^2)^2$	233.59%	291.73%	339.24%	193.39%	194.71%	260.08%	226.84%	314.55%	294.64%	266.31%	257.91%	NA	NA	NA
SCOR	Motor NonProp	$\hat{f}_k$	3.17	1.31	1.16	1.10	1.06	1.04	1.03	1.03	1.02	1.03	1.02	1.03	1.02	1.01
		$\hat{\sigma}_k^2$	8'756'735	1'115'133	126'117	349'261	207'405	79'654	70'839	33'789	37'388	24'469	2'983	78'442	3'144	126
		$\hat{Sk}_k^3/(\hat{\sigma}_k^2)^{3/2}$	1.491	0.286	0.348	-0.155	0.621	0.177	0.920	0.411	0.658	0.713	0.865	-0.212	NA	NA
		$\hat{Ku}_k^4/(\hat{\sigma}_k^2)^2$	631.06%	201.17%	403.49%	234.55%	294.37%	167.42%	292.94%	194.28%	291.68%	245.21%	265.76%	NA	NA	NA

Table 2: Chain-ladder coefficients, Mack variance, Skewness and Kurtosis estimators for 15 x 15 triangles

As expected, the skewness is positive in many instances. In addition, the kurtosis is often below 300%. This should reflect the platykurtic nature of the non-life reserves. As a matter of fact, the potentiality to have extreme values in an incurred triangle is mechanically remote: In general, the case reserves input by the claim adjusters will reflect the standard movements of all the claims. Only in rare cases, there could be an extreme event in a triangle and these extreme events will be present on specific lines of business like Property NatCat. In addition, an extreme event may have a smaller impact in the triangle if the portfolio of risk is big enough to absorb the unusual movement. As a consequence, the distribution should be very much concentrated around the mean and, as a consequence, have a platykurtic nature.

Following the analysis by development year, we are now looking at the results of the simulations below:

Triangle size	LoB	Company	Chain-ladder reserves	Chain-ladder stdev	CoV	Overall simulated skewness	Overall simulated kurtosis	LogN				Gamma			
								Mu	Sigma2	Resulting skewness	Resulting kurtosis	k	Theta	Resulting skewness	Resulting kurtosis
10 x 10	Private Motor	Farmers Alliance	-374	1493	-400%	-0.01	297%	NA	NA	NA	NA	0.06	-5'966	7.99	9882%
10 x 10	Private Motor	NC Farm	19'415	9'528	49%	0.32	298%	9.77	0.216	1.59	781%	4.15	4'676	0.98	445%
10 x 10	Private Motor	New Jersey Manuf.	109'719	11'961	11%	0.07	295%	11.60	0.012	0.33	319%	84.15	1'304	0.22	307%
10 x 10	Product Liab.	Pennsylvania	1'474	1'784	121%	0.06	350%	6.84	0.903	5.41	8222%	0.68	2'161	2.42	1180%
10 x 10	Product Liab.	West Bend	2150	1899	88%	0.35	384%	7.38	0.577	3.34	2784%	1.28	1'678	1.77	768%
10 x 10	Mack 1993 triangle		18'680'856	2'447'095	13%	0.13	292%	16.73	0.017	0.40	328%	58.28	320'557	0.26	310%
15 x 15	WW Casualty Prop	SCOR	219'461'925	79'722'452	36%	0.14	300%	19.14	0.124	1.14	539%	7.58	28960237	0.73	379%
15 x 15	WW Motor NP	SCOR	402'645'321	53'078'447	13%	0.17	289%	19.80	0.017	0.40	328%	57.55	6997030	0.26	310%

Table 3: Simulation (10 000 scenarios) of skewness and kurtosis of the overall reserves for the triangles shown in Appendix B

Note on Table 3: CoV stands for Coefficient of Variation and equals the standard deviation divided by the reserves.

Looking at the above table, we can see that the higher the CoV, the more the Lognormal and the Gamma distributions have their skewness and kurtosis far away from the simulated skewness and kurtosis. Such deviations between the simulated distribution and Lognormal or Gamma distribution starts to be seen for CoV as low as 36% which are not uncommon CoVs in practice. For CoVs higher than 36%, the deviations get bigger and bigger. In such cases, the Lognormal and Gamma distributions do not fit anymore with the third and fourth simulated moments.

In order to estimate the impact of the deviations between the simulated distribution and the Lognormal distribution, we followed the steps below:

1. Fit the simulated distribution to a Johnson distribution (see Johnson 1949) on the first 4 moments using the software R (function JohnsonFit).
2. Estimate the Value At Risk (VaR) at 99% resulting from the application of the Johnson distribution.
3. Compare the above VaR with the VaR resulting from the Lognormal distribution fitted on the basis of the Best Estimate and CoV.

We recall that the family of Johnson distribution has the following properties (see also Johnson 1949):

$z = \gamma + \delta\, f\left(\frac{x-\xi}{\lambda}\right)$ where f is a function of simple form and z is a unit normal variable.

Depending on f, the Johnson distribution is noted as follows:

$f = \log$: Distribution SL

$f = \sinh^{-1}$: Distribution SU

$z = \gamma + \delta \log\left(\frac{x-\xi}{\xi+\lambda-x}\right)$: Distribution SB

$z = \gamma + \delta\left(\frac{x-\xi}{\lambda}\right)$: Distribution SN

As described above, the results of the fitting of the simulated distributions with the Johnson family on the basis of the first 4 moments are shown in the table below:

LoB	Company	Chain-ladder reserves	Chain-ladder stdev	CoV	Overall simulated skewness	Overall simulated kurtosis	Johnson fitting									
							Type	Gamma	Delta	Xi	Lambda	VaR 99%	Fitted Mean	Fitted Stdev	Fitted Skewness	Fitted Kurtosis
Private Motor	Farmers Alliance	-374	1493	-400%	-0.01	297%	SN	-	1.000	-374	1'493	3'099	-374	1'493	-	300%
Private Motor	NC Farm	19'415	9'528	49%	0.32	298%	SB	2.253	3.040	-24'390	134'033	43'432	19'355	9'406	0.19	287%
Private Motor	New Jersey Manuf.	109'719	11'961	11%	0.07	295%	SN	-	1.000	109'719	11'961	137'544	109'719	11'961	-	300%
Product Liab.	Pennsylvania	1'474	1'784	121%	0.06	350%	SU	-0.174	3.085	1'165	5'208	5'864	1'474	1'784	0.06	350%
Product Liab.	West Bend	2150	1899	88%	0.35	384%	SU	-0.785	2.715	707	4'603	7'214	2'150	1'899	0.35	348%
Mack 1993 triangle		18'680'856	2'447'095	13%	0.13	292%	SB	1.161	3.370	4'224'926	34'719'087	24'555'541	18'645'236	2'428'748	0.18	278%
WW Casualty Prop	SCOR	219'461'925	79'722'452	36%	0.14	300%	SL	-456.116	21.456	-1'490'114'663	1.000	411'994'159	219'461'925	79'722'452	0.14	303%
WW Motor NP	SCOR	402'645'321	53'078'447	13%	0.17	289%	SB	1.241	3.043	125'267'964	690'347'082	531'340'556	401'885'733	52'707'797	0.21	278%

Table 4: Results of the fitting of simulated distributions to Johnson distributions

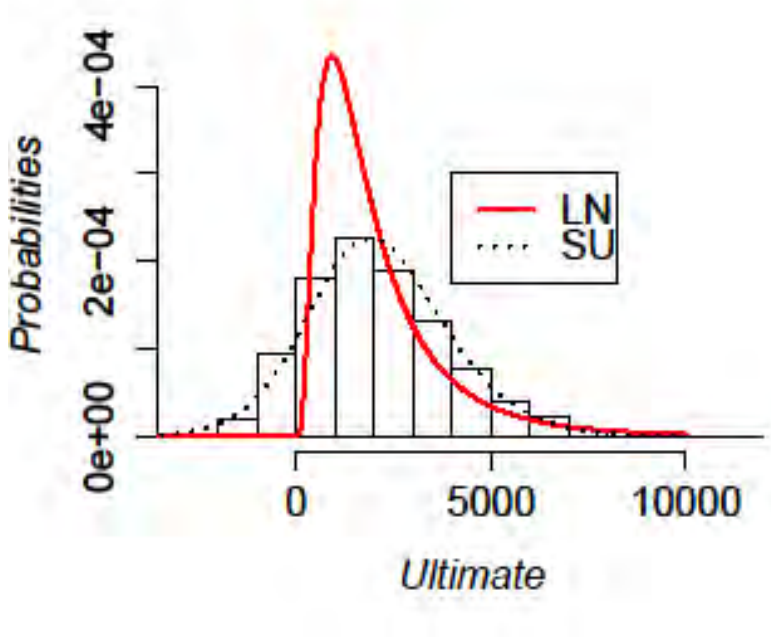
Generally, the fitted distribution shows moments which reconcile pretty well with the simulated distributions. As a next step, the table below provides the comparison of the VaR 99% coming from the Lognormal and the Johnson distributions:

LoB	Company	VaR 99%		Difference LogN Johnson VaR 99%
		Johnson	Lognormal	
Private Motor	Farmers Alliance	3'099	NA	NA
Private Motor	NC Farm	43'432	51'358	18%
Private Motor	New Jersey Manuf.	137'544	140'453	2%
Product Liab.	Pennsylvania	5'864	8'556	46%
Product Liab.	West Bend	7'214	9'430	31%
Mack 1993 triangle		24'555'541	25'089'172	2%
WW Casualty Prop	SCOR	411'994'159	467'889'645	14%
WW Motor NP	SCOR	531'340'556	541'742'729	2%

Table 5: Comparison of VaR 99% coming from Johnson and Lognormal distributions

As expected, when the CoVs are relatively high, the capital requirements under the risk measure Value at Risk 99% using the lognormal assumption are much higher than the capital requirements using the Johnson distribution.

Finally, the graph below shows how far the lognormal distribution can be from the simulated distribution (shown as an histogram) and from the Johnson distribution in the case of the Product Liability triangle of West Bend:



It is interesting to note that the simulations as well as the Johnson fitted distribution anticipate some possibilities for negative IBNRs. Looking into the specificities of the West Bend Product Liability triangle, it can be seen that the chain-ladder coefficient between development year N+6 and N+7 is equal to 0.86. Therefore, in certain cases, the company West Bend seems to have positive reserve development on this line of business. As mentioned, this is reflected in both the simulations and the Johnson distribution. In the case of the simulations, the maximum level of negative IBNRs is 2322 while the case reserves are 2683. Even though it is unlikely that there would be 2322 negative IBNRs on the Product Liability line of West Bend, this maximum level of negative IBNRs is consistent with the amount of case reserve in a "best case" scenario. As a conclusion on this case, we can see that the use of the Lognormal distribution does not anticipate the possibilities of negative IBNRs which is not consistent with the past results of West Bend. Therefore, the use of the Lognormal distribution should be avoided on this case.

6. CONCLUSIONS

Based on the results of simulations on eight triangles, it appears that the representation of the reserve risk distribution by a Lognormal or a Gamma distribution can have major drawbacks in the case of CoV of 36% and above. In particular, the use of such distribution is likely to bring higher and unnecessary capital requirements under VaR 99%. In order to avoid such unnecessary capital requirement, it is advisable to turn to other distributions fitting the skewness and kurtosis estimated through the simulations described in this paper. One possible choice of such distributions fitting with the simulated skewness and kurtosis could be the Johnson distribution.

Acknowledgment

I am indebted to Mathieu Cambou for his review of the formulae and his very good advice for improvements of this paper.

The views expressed in the paper are entirely my own and should not be taken as representing those of my firm.

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Appendix A

We want to show that $E\left(\hat{Kt}_k^4\right)=Kt_k^4$. Therefore, we will, in a first step, concentrate on $E\left(\hat{Kt}_k^4 \mid B_k\right)$.

We have:

$$(Eq)=E\left[\left(\sum_{i=1}^{I-k}\left(1-\frac{C_{i,k}}{\sum_{i=1}^{I-k}C_{i,k}}\right)^4+\frac{\left(\sum_{i=1}^{I-k}C_{i,k}^2\right)^2-\sum_{i=1}^{I-k}C_{i,k}^4}{\left(\sum_{i=1}^{I-k}C_{i,k}\right)^4}\right)\hat{Kt}_k^4 \mid B_k\right]$$

$$=E\left[\sum_{i=1}^{I-k}C_{i,k}^2\left(\frac{C_{i,k+1}}{C_{i,k}}-\hat{f}_k\right)^4-3\left(\sigma_k^2\right)^2\left(2-6\frac{\sum_{i=1}^{I-k}C_{i,k}^2}{\left(\sum_{i=1}^{I-k}C_{i,k}\right)^2}+4\frac{\sum_{i=1}^{I-k}C_{i,k}^3}{\left(\sum_{i=1}^{I-k}C_{i,k}\right)^3}\right) \mid B_k\right] \text{ for } 1\le k\le I\text{-}4$$

where $B_k=\left\{C_{i,j} \mid i+j\le I+1,\ j\le k\right\}, 1\le k\le I\text{-}1$

In the following, we will concentrate on the term:

$$(\text{Eq1})=E\left[\sum_{i=1}^{I-k}C_{i,k}^2\left(\frac{C_{i,k+1}}{C_{i,k}}-\hat{f}_k\right)^4 \mid B_k\right] \text{ for } 1\le k\le I\text{-}4$$

$$=\sum_{i=1}^{I-k}\left[\frac{1}{C_{i,k}^2}E\left(C_{i,k+1}^4 \mid B_k\right)\right]-4\sum_{i=1}^{I-k}\left[\frac{1}{C_{i,k}}E\left(\hat{f}_k C_{i,k+1}^3 \mid B_k\right)\right]+6\sum_{i=1}^{I-k}\left[E\left(\hat{f}_k^2 C_{i,k+1}^2 \mid B_k\right)\right] \qquad (9)$$

$$-4\sum_{i=1}^{I-k}C_{i,k}E\left(\hat{f}_k^3 C_{i,k+1} \mid B_k\right)+\sum_{i=1}^{I-k}C_{i,k}^2E\left(\hat{f}_k^4 \mid B_k\right)$$

For each of the 5 elements of the above equation, a formula is going to be established. For the first element of equation (9), we have:

$$KT\left(C_{i,k+1} \mid B_k\right)=Kt_k^4C_{i,k}^2=E\left(C_{i,k+1}^4 \mid B_k\right)-4E\left(C_{i,k+1} \mid B_k\right)E\left(C_{i,k+1}^3 \mid B_k\right)+6E\left(C_{i,k+1}^2 \mid B_k\right)E\left(C_{i,k+1} \mid B_k\right)^2$$

$$-3E\left(C_{i,k+1} \mid B_k\right)^4$$

From Mack (1993b), we have:

$E\left(C_{i,k+1}^2 \mid B_k\right)=C_{i,k}\sigma_k^2+C_{i,k}^2 f_k^2$ and

$E\left(C_{i,k+1} \mid B_k\right)=C_{i,k}f_k$

We also established that:

$$E\left(C_{i,k+1}^3 \mid B_k\right) = Sk_k^3 C_{i,k}^{3/2} + 3 f_k C_{i,k}^2 \sigma_k^2 + C_{i,k}^3 f_k^3 \tag{10}$$

Therefore, we have:

$$E\left(C_{i,k+1}^4 \mid B_k\right) = Kt_k^4 C_{i,k}^2 + 4 f_k Sk_k^3 C_{i,k}^{5/2} + 6 f_k^2 C_{i,k}^3 \sigma_k^2 + C_{i,k}^4 f_k^4 \tag{11}$$

For the fifth element of equation (9), we have:

$$KT\left(\hat{f}_k \mid B_k\right) = \frac{1}{\left(\sum_{i=1}^{I-k} C_{i,k}\right)^4} \sum_{i=1}^{I-k} KT\left(C_{i,k+1} \mid B_k\right) = \frac{Kt_k^4}{\left(\sum_{i=1}^{I-k} C_{i,k}\right)^4} \sum_{i=1}^{I-k} C_{i,k}^2$$

$$= E\left(\left[\hat{f}_k - E\left(\hat{f}_k \mid B_k\right)\right]^4 \mid B_k\right) = E\left(\hat{f}_k^4 \mid B_k\right) - 4E\left(\hat{f}_k \mid B_k\right) E\left(\hat{f}_k^3 \mid B_k\right) + 6E\left(\hat{f}_k \mid B_k\right)^2 E\left(\hat{f}_k^2 \mid B_k\right) - 3E\left(\hat{f}_k \mid B_k\right)^4$$

From Mack (1993b), we have:

$$E\left(\hat{f}_k^2 \mid B_k\right) = \frac{\sigma_k^2}{\sum_{i=1}^{I-k} C_{i,k}} + f_k^2$$

We also know that:

$$E\left(\hat{f}_k^3 \mid B_k\right) = \frac{Sk_k^3}{\left(\sum_{i=1}^{I-k} C_{i,k}\right)^3} \sum_{i=1}^{I-k} C_{i,k}^{3/2} + 3\frac{f_k \sigma_k^2}{\sum_{i=1}^{I-k} C_{i,k}} + f_k^3$$

Therefore, we have:

$$E\left(\hat{f}_k^4 \mid B_k\right) = Kt_k^4 \frac{\sum_{i=1}^{I-k} C_{i,k}^2}{\left(\sum_{i=1}^{I-k} C_{i,k}\right)^4} + 4 f_k Sk_k^3 \frac{\sum_{i=1}^{I-k} C_{i,k}^{3/2}}{\left(\sum_{i=1}^{I-k} C_{i,k}\right)^3} + 6\frac{f_k^2 \sigma_k^2}{\sum_{i=1}^{I-k} C_{i,k}} + f_k^4 \tag{12}$$

For the second element of equation (9), we have:

$$E\left(\hat{f}_k C_{i,k+1}^3 \mid B_k\right) = \frac{1}{\sum_{j=1}^{I-k} C_{j,k}} E\left[\sum_{\substack{j=1 \\ j \neq i}}^{I-k} \left(C_{j,k+1} C_{i,k+1}^3\right) + C_{i,k+1}^4 \mid B_k\right]$$

Due to the independence of $C_{i,k+1}$ and $C_{j,k+1}$, we have:

$$E\left(\hat{f}_k C_{i,k+1}^3 \mid B_k\right)=\frac{1}{\sum_{j=1}^{I-k} C_{j,k}}\left[\sum_{\substack{j=1\\ j\neq i}}^{I-k} E\left(C_{j,k+1}\right)E\left(C_{i,k+1}^3 \mid B_k\right)+E\left(C_{i,k+1}^4 \mid B_k\right)\right]$$

$$=\frac{1}{\sum_{j=1}^{I-k} C_{j,k}}\left[\sum_{\substack{j=1\\ j\neq i}}^{I-k} f_k C_{j,k}\left(Sk_k^3 C_{i,k}^{3/2}+3 f_k C_{i,k}^2\sigma_k^2+C_{i,k}^3 f_k^3\right)+Kt_k^4 C_{i,k}^2+4 f_k Sk_k^3 C_{i,k}^{5/2}+6 f_k^2 C_{i,k}^3\sigma_k^2+C_{i,k}^4 f_k^4\right]$$

$$\Rightarrow E\left(\hat{f}_k C_{i,k+1}^3 \mid B_k\right)=\frac{Kt_k^4 C_{i,k}^2}{\sum_{j=1}^{I-k} C_{j,k}}+\frac{3 f_k^2 C_{i,k}^3\sigma_k^2}{\sum_{j=1}^{I-k} C_{j,k}}+\frac{3 f_k Sk_k^3 C_{i,k}^{5/2}}{\sum_{j=1}^{I-k} C_{j,k}}+f_k^4 C_{i,k}^3+f_k Sk_k^3 C_{i,k}^{3/2}+3 f_k^2\sigma_k^2 C_{i,k}^2 \quad (13)$$

For the third element of equation (9), we have:

$$E\left(\hat{f}_k^2 C_{i,k+1}^2 \mid B_k\right)=\frac{1}{\left(\sum_{j=1}^{I-k} C_{j,k}\right)^2}E\left[C_{i,k+1}^2\left(\sum_{j=1}^{I-k} C_{j,k+1}\right)^2 \mid B_k\right]$$

$$=\frac{1}{\left(\sum_{j=1}^{I-k} C_{j,k}\right)^2}E\left[C_{i,k+1}^2\left(\sum_{j=1}^{I-k} C_{j,k+1}^2+2\sum_{n=1}^{I-k}\sum_{\substack{m=1\\ m\neq n}}^{n} C_{n,k+1}C_{m,k+1}\right) \mid B_k\right]$$

$$=\frac{1}{\left(\sum_{j=1}^{I-k} C_{j,k}\right)^2}E\left[C_{i,k+1}^4+\sum_{\substack{j=1\\ j\neq i}}^{I-k} C_{j,k+1}^2 C_{i,k+1}^2+2\sum_{\substack{n=1\\ n\neq i}}^{I-k}\sum_{\substack{m=1\\ m\neq n\\ m\neq i}}^{n} C_{n,k+1}C_{m,k+1}C_{i,k+1}^2+2\sum_{n=1}^{I-k} C_{n,k+1}C_{i,k+1}^3 \mid B_k\right]$$

Due to the independence of $C_{i,k+1}$, $C_{j,k+1}, C_{n,k+1}$ and $C_{m,k+1}$ and on using the formulas for $E\left(C_{i,k+1}^4 \mid B_k\right)$ (equation (11)), $E\left(C_{i,k+1}^3 \mid B_k\right)$ (equation (10)) and $E\left(C_{i,k+1}^2 \mid B_k\right)$, we have:

$$E\left(\hat{f}_k^2 C_{i,k+1}^2 \mid B_k\right) = \frac{Kt_k^4 C_{i,k}^2}{\left(\sum_{j=1}^{I-k} C_{j,k}\right)^2} + \frac{5 f_k^2 C_{i,k}^2 \sigma_k^2}{\sum_{j=1}^{I-k} C_{j,k}} + \frac{2 f_k Sk_k^3 C_{i,k}^{5/2}}{\left(\sum_{j=1}^{I-k} C_{j,k}\right)^2} + \frac{2 f_k Sk_k^3 C_{i,k}^{3/2}}{\sum_{j=1}^{I-k} C_{j,k}} + f_k^4 C_{i,k}^2 + f_k^2 \sigma_k^2 C_{i,k}$$
$$+ \frac{\left(\sigma_k^2\right)^2 C_{i,k}\left(\sum_{j=1}^{I-k} C_{j,k} - C_{i,k}\right)}{\left(\sum_{j=1}^{I-k} C_{j,k}\right)^2} \qquad (14)$$

Finally, for the fourth element of equation (9), we have:

$$E\left(\hat{f}_k^3 C_{i,k+1} \mid B_k\right) = \frac{1}{\left(\sum_{j=1}^{I-k} C_{j,k}\right)^3} E\left[C_{i,k+1}\left(\sum_{j=1}^{I-k} C_{j,k+1}\right)^3 \mid B_k \right]$$

$$= \frac{1}{\left(\sum_{j=1}^{I-k} C_{j,k}\right)^3} E\left[C_{i,k+1}\left(\sum_{j=1}^{I-k} C_{j,k+1}^3 + 3\sum_{n=1}^{I-k}\sum_{\substack{m=1 \\ m\neq n}}^{I-k} C_{n,k+1}^2 C_{m,k+1} + 6\sum_{n=1}^{I-k-2}\sum_{m=n+1}^{I-k-1}\sum_{\substack{o=m+1 \\ m\neq o \\ n\neq o}}^{I-k} C_{n,k+1} C_{m,k+1} C_{o,k+1} \right) \mid B_k \right]$$

$$= \frac{1}{\left(\sum_{j=1}^{I-k} C_{j,k}\right)^3} E\left[\begin{array}{l} C_{i,k+1}^4 + \sum_{\substack{j=1 \\ j\neq i}}^{I-k} C_{j,k+1}^3 C_{i,k+1} + 3\sum_{\substack{n=1 \\ n\neq i}}^{I-k} C_{n,k+1} C_{i,k+1}^3 + 3\sum_{\substack{j=1 \\ j\neq i}}^{I-k} C_{j,k+1}^2 C_{i,k+1}^2 + 3\sum_{\substack{n=1 \\ n\neq i}}^{I-k}\sum_{\substack{m=1 \\ m\neq n \\ m\neq i}}^{n} C_{n,k+1}^3 C_{m,k+1} C_{i,k+1} \\ 6\sum_{\substack{n=1 \\ n\neq i}}^{I-k-2}\sum_{\substack{m=n+1 \\ m\neq n \\ m\neq i}}^{I-k-1}\sum_{\substack{o=m+1 \\ m\neq o \\ n\neq o \\ o\neq n}}^{I-k} C_{n,k+1} C_{m,k+1} C_{o,k+1} C_{i,k+1} + 6\sum_{\substack{n=1 \\ n\neq i}}^{I-k-1}\sum_{\substack{m=1 \\ m\neq n \\ m\neq i}}^{I-k} C_{i,k+1}^2 C_{n,k+1} C_{m,k+1} \mid B_k \end{array} \right]$$

Due to the independence of $C_{i,k+1}$, $C_{j,k+1}, C_{n,k+1}, C_{m,k+1}$ and $C_{o,k+1}$ and on using the formulas for $E\left(C_{i,k+1}^4 \mid B_k\right)$ (equation (11)), $E\left(C_{i,k+1}^3 \mid B_k\right)$ (equation (10)) and $E\left(C_{i,k+1}^2 \mid B_k\right)$, we have:

$$E\left(\hat{f}_k^3 C_{i,k+1} \mid B_k\right) = \frac{1}{\left(\sum_{j=1}^{I-k} C_{j,k}\right)^3} \left[\begin{array}{l} Kt_k^4 C_{i,k}^2 + f_k^4 C_{i,k} \left(\sum_{j=1}^{I-k} C_{j,k}\right)^3 + f_k Sk_k^3 \left(C_{i,k} \sum_{j=1}^{I-k} C_{j,k}^{3/2} + 3 C_{i,k}^{3/2} \sum_{j=1}^{I-k} C_{j,k} \right) \\ + 6 f_k^2 C_{i,k} \sigma_k^2 \left(\sum_{j=1}^{I-k} C_{j,k}\right)^2 + 3\left(\sigma_k^2\right)^2 C_{i,k} \left(\sum_{j=1}^{I-k} C_{j,k} - C_{i,k}\right) \end{array} \right] \quad (15)$$

On inserting equations (11), (12), (13), (14) and (15) into equation (9), we have:

$$\begin{aligned} (\text{Eq1}) = & \sum_{i=1}^{I-k} \left[\frac{1}{C_{i,k}^2} \left(Kt_k^4 C_{i,k}^2 + 4 f_k Sk_k^3 C_{i,k}^{5/2} + 6 f_k^2 C_{i,k}^3 \sigma_k^2 + C_{i,k}^4 f_k^4 \right) \right] \\ & - 4 \sum_{i=1}^{I-k} \left[\frac{1}{C_{i,k}} \left(\frac{Kt_k^4 C_{i,k}^2}{\sum_{j=1}^{I-k} C_{j,k}} + \frac{3 f_k^2 C_{i,k}^3 \sigma_k^2}{\sum_{j=1}^{I-k} C_{j,k}} + \frac{3 f_k Sk_k^3 C_{i,k}^{5/2}}{\sum_{j=1}^{I-k} C_{j,k}} + f_k^4 C_{i,k}^3 + f_k Sk_k^3 C_{i,k}^{3/2} + 3 f_k^2 \sigma_k^2 C_{i,k}^2 \right) \right] \\ & + 6 \sum_{i=1}^{I-k} \left[\left(\frac{Kt_k^4 C_{i,k}^2}{\left(\sum_{j=1}^{I-k} C_{j,k}\right)^2} + \frac{5 f_k^2 C_{i,k}^2 \sigma_k^2}{\sum_{j=1}^{I-k} C_{j,k}} + \frac{2 f_k Sk_k^3 C_{i,k}^{5/2}}{\left(\sum_{j=1}^{I-k} C_{j,k}\right)^2} + \frac{2 f_k Sk_k^3 C_{i,k}^{3/2}}{\sum_{j=1}^{I-k} C_{j,k}} + f_k^4 C_{i,k}^2 + f_k^2 \sigma_k^2 C_{i,k} \right. \right. \\ & \left. \left. + \frac{\left(\sigma_k^2\right)^2 C_{i,k} \left(\sum_{j=1}^{I-k} C_{j,k} - C_{i,k}\right)}{\left(\sum_{j=1}^{I-k} C_{j,k}\right)^2} \right) \right] \\ & - \frac{4}{\left(\sum_{j=1}^{I-k} C_{j,k}\right)^3} \sum_{i=1}^{I-k} C_{i,k} \left[\begin{array}{l} Kt_k^4 C_{i,k}^2 + f_k^4 C_{i,k} \left(\sum_{j=1}^{I-k} C_{j,k}\right)^3 + f_k Sk_k^3 \left(C_{i,k} \sum_{j=1}^{I-k} C_{j,k}^{3/2} + 3 C_{i,k}^{3/2} \sum_{j=1}^{I-k} C_{j,k} \right) \\ + 6 f_k^2 C_{i,k} \sigma_k^2 \left(\sum_{j=1}^{I-k} C_{j,k}\right)^2 + 3\left(\sigma_k^2\right)^2 C_{i,k} \left(\sum_{j=1}^{I-k} C_{j,k} - C_{i,k}\right) \end{array} \right] \\ & + \sum_{i=1}^{I-k} C_{i,k}^2 \left(Kt_k^4 \frac{\sum_{i=1}^{I-k} C_{i,k}^2}{\left(\sum_{i=1}^{I-k} C_{i,k}\right)^4} + 4 f_k Sk_k^3 \frac{\sum_{i=1}^{I-k} C_{i,k}^{3/2}}{\left(\sum_{i=1}^{I-k} C_{i,k}\right)^3} + 6 \frac{f_k^2 \sigma_k^2}{\sum_{i=1}^{I-k} C_{i,k}} + f_k^4 \right) \end{aligned}$$

After simplifications, the above equation is given as:

$$(\text{Eq1}) = \left(I-k-4+6\frac{\sum_{i=1}^{I-k} C_{i,k}^2}{\left(\sum_{i=1}^{I-k} C_{i,k}\right)^2} - 4\frac{\sum_{i=1}^{I-k} C_{i,k}^3}{\left(\sum_{i=1}^{I-k} C_{i,k}\right)^3} + \frac{\left(\sum_{i=1}^{I-k} C_{i,k}^2\right)^2}{\left(\sum_{i=1}^{I-k} C_{i,k}\right)^4} \right) Kt_k^4 + 3\left(\sigma_k^2\right)^2 \left(2-6\frac{\sum_{i=1}^{I-k} C_{i,k}^2}{\left(\sum_{i=1}^{I-k} C_{i,k}\right)^2} + 4\frac{\sum_{i=1}^{I-k} C_{i,k}^3}{\left(\sum_{i=1}^{I-k} C_{i,k}\right)^3} \right)$$

As we have the equality below:

$$I-k-4+6\frac{\sum_{i=1}^{I-k} C_{i,k}^2}{\left(\sum_{i=1}^{I-k} C_{i,k}\right)^2} - 4\frac{\sum_{i=1}^{I-k} C_{i,k}^3}{\left(\sum_{i=1}^{I-k} C_{i,k}\right)^3} + \frac{\left(\sum_{i=1}^{I-k} C_{i,k}^2\right)^2}{\left(\sum_{i=1}^{I-k} C_{i,k}\right)^4} = \sum_{i=1}^{I-k} \left(1-\frac{C_{i,k}}{\sum_{i=1}^{I-k} C_{i,k}} \right)^4 + \frac{\left(\sum_{i=1}^{I-k} C_{i,k}^2\right)^2 - \sum_{i=1}^{I-k} C_{i,k}^4}{\left(\sum_{i=1}^{I-k} C_{i,k}\right)^4}$$

we finally get the result:

$$\hat{Kt}_k^4 = \frac{\left[\sum_{i=1}^{I-k} C_{i,k}^2 \left(\frac{C_{i,k+1}}{C_{i,k}} - \hat{f}_k \right)^4 - 3\left(\hat{\sigma}_k^2\right)^2 \left(2-6\frac{\sum_{i=1}^{I-k} C_{i,k}^2}{\left(\sum_{i=1}^{I-k} C_{i,k}\right)^2} + 4\frac{\sum_{i=1}^{I-k} C_{i,k}^3}{\left(\sum_{i=1}^{I-k} C_{i,k}\right)^3} \right) \right]}{\left(\sum_{i=1}^{I-k} \left(1-\frac{C_{i,k}}{\sum_{i=1}^{I-k} C_{i,k}} \right)^4 + \frac{\left(\sum_{i=1}^{I-k} C_{i,k}^2\right)^2 - \sum_{i=1}^{I-k} C_{i,k}^4}{\left(\sum_{i=1}^{I-k} C_{i,k}\right)^4} \right)} \text{ for } 1 \le k \le I\text{-}4$$

Appendix B

Schedule P – Farmers Alliance – Incurred claim – Private motor

	N	N+1	N+2	N+3	N+4	N+5	N+6	N+7	N+8	N+9
1988	10'798	11'595	11'724	11'820	11'746	11'641	11'557	11'552	11'525	11'522
1989	11'313	13'743	13'621	13'666	13'352	13'182	13'186	13'159	13'061	
1990	15'110	15'143	15'401	14'915	14'998	14'858	14'811	14'887		
1991	15'163	15'253	14'577	14'269	14'456	14'721	14'898			
1992	14'232	14'999	14'932	14'933	14'915	14'788				
1993	14'063	15'468	15'052	15'263	15'042					
1994	12'050	12'907	13'156	13'016						
1995	12'163	13'086	12'847							
1996	11'624	13'122								
1997	12'942									

Schedule P – NC Farm Bureau – Incurred claim – Private Motor

	N	N+1	N+2	N+3	N+4	N+5	N+6	N+7	N+8	N+9
1988	48900	65079	70815	71880	71384	71034	70442	70383	70379	70127
1989	55918	80979	85951	87272	86546	85784	85620	85427	85084	
1990	63493	88613	93872	93363	91731	90592	90599	90378		
1991	69112	91167	94452	93792	92186	91270	91187			
1992	76922	96890	100163	98974	98405	97721				
1993	86811	112247	114502	113842	112357					
1994	101257	123451	127532	125923						
1995	107582	134485	135931							
1996	113997	141203								
1997	108190									

Schedule P – New Jersey Manufacturers – Incurred claim – Private Motor

	N	N+1	N+2	N+3	N+4	N+5	N+6	N+7	N+8	N+9
1988	50395	66425	77609	82841	83400	82368	81691	81296	81592	81590
1989	56022	75703	90838	96847	96690	94344	93918	93233	93165	
1990	61079	85761	100131	105776	106002	104401	103106	102485		
1991	70857	97925	113696	123809	121641	119372	117638			
1992	84104	109443	126585	136514	136238	132453				
1993	95669	128682	150353	161064	156112					
1994	104360	134860	152751	161981						
1995	116839	153337	174393							
1996	136837	181052								
1997	152180									

Schedule P – Farmers Alliance – Incurred claim – Product Liability

	N	N+1	N+2	N+3	N+4	N+5	N+6	N+7	N+8	N+9
1988	199	246	360	404	401	321	328	340	332	342
1989	1312	1181	1347	1390	1511	1686	1860	1689	1680	
1990	493	700	817	740	912	903	897	902		
1991	391	487	1472	1640	611	593	597			
1992	586	741	1251	1509	1864	2040				
1993	892	1285	1556	1720	1828					
1994	654	1644	2060	2270						
1995	379	906	1255							
1996	705	950								
1997	384									

Schedule P – West Bend – Incurred claim – Product Liability

	N	N+1	N+2	N+3	N+4	N+5	N+6	N+7	N+8	N+9
1988	71	127	220	305	221	185	186	186	186	186
1989	185	318	360	377	425	382	389	258	261	
1990	124	225	348	394	218	216	243	261		
1991	299	344	531	1028	1266	1675	1959			
1992	204	399	431	647	625	709				
1993	208	459	566	714	766					
1994	452	665	910	805						
1995	243	532	1197							
1996	319	493								
1997	412									

Triangle from Mack 1993a

i	C_{i1}	C_{i2}	C_{i3}	C_{i4}	C_{i5}	C_{i6}	C_{i7}	C_{i8}	C_{i9}	C_{i10}
1	357'848	1'124'788	1'735'330	2'218'270	2'745'596	3'319'994	3'466'336	3'606'286	3'833'515	3'901'463
2	352'118	1'236'139	2'170'033	3'353'322	3'799'067	4'120'063	4'647'867	4'914'039	5'339'085	
3	290'507	1'292'306	2'218'525	3'235'179	3'985'995	4'132'918	4'628'910	4'909'315		
4	310'608	1'418'858	2'195'047	3'757'447	4'029'929	4'381'982	4'588'268			
5	443'160	1'136'350	2'128'333	2'897'821	3'402'672	3'873'311				
6	396'132	1'333'217	2'180'715	2'985'752	3'691'712					
7	440'832	1'288'463	2'419'861	3'483'130						
8	359'480	1'421'128	2'864'498							
9	376'686	1'363'294								
10	344'014									

SCOR Triangle – Investors' day 2011 – Casualty Proportional Worldwide

i	C_{i1}	C_{i2}	C_{i3}	C_{i4}	C_{i5}	C_{i6}	C_{i7}	C_{i8}	C_{i9}	C_{i10}	C_{i11}	C_{i12}	C_{i13}	C_{i14}	C_{i15}
1	1'235'909	28'084'788	36'277'376	40'475'255	43'219'640	44'663'902	45'326'054	48'497'970	49'233'469	49'550'622	49'690'990	49'831'357	50'360'419	49'973'713	49'862'952
2	1'377'332	35'709'204	43'213'075	51'753'287	56'669'470	57'444'929	59'800'911	61'566'321	62'121'477	62'876'588	62'731'073	63'128'635	63'478'522	63'123'300	-
3	2'227'264	38'477'741	50'151'725	53'966'008	54'671'130	57'692'644	60'487'201	61'814'232	61'750'678	62'642'901	62'622'432	61'843'626	62'208'577	-	-
4	6'063'523	77'188'565	106'217'291	113'964'643	120'412'934	126'472'874	124'422'859	128'806'243	127'982'137	125'512'850	134'339'552	135'190'968	-	-	-
5	17'046'209	118'335'598	149'317'461	158'231'476	169'819'782	171'982'468	171'861'629	180'041'898	178'223'074	178'218'074	180'474'109	-	-	-	-
6	19'133'946	129'368'180	158'428'296	161'509'739	164'007'156	161'788'099	163'095'804	160'465'113	165'298'176	166'467'315	-	-	-	-	-
7	15'294'531	146'782'533	181'032'613	182'220'943	171'358'201	171'997'288	170'664'146	171'561'713	169'645'162	-	-	-	-	-	-
8	18'460'498	117'030'644	146'526'379	132'681'719	130'134'713	124'721'542	133'009'259	136'533'267	-	-	-	-	-	-	-
9	23'542'477	113'067'832	125'103'424	143'385'679	143'172'588	130'016'839	125'328'953	-	-	-	-	-	-	-	-
10	17'341'979	122'713'595	142'017'246	154'409'430	144'452'872	134'267'844	-	-	-	-	-	-	-	-	-
11	19'850'416	134'682'628	163'649'787	168'464'316	170'525'178	-	-	-	-	-	-	-	-	-	-
12	16'449'001	125'616'958	155'143'936	158'477'086	-	-	-	-	-	-	-	-	-	-	-
13	27'153'648	144'171'990	164'553'451	-	-	-	-	-	-	-	-	-	-	-	-
14	30'745'977	141'700'503	-	-	-	-	-	-	-	-	-	-	-	-	-
15	14'911'754	-	-	-	-	-	-	-	-	-	-	-	-	-	-

SCOR Triangle – Investors' day 2011 – Motor Non Proportional Worldwide

i	C_{i1}	C_{i2}	C_{i3}	C_{i4}	C_{i5}	C_{i6}	C_{i7}	C_{i8}	C_{i9}	C_{i10}	C_{i11}	C_{i12}	C_{i13}	C_{i14}	C_{i15}
1	10'423'937	38'199'157	56'447'001	65'220'946	69'971'047	75'438'044	82'032'154	88'917'308	90'783'545	95'891'585	96'943'325	99'646'098	104'075'520	105'688'533	107'081'042
2	19'822'329	62'295'888	67'897'838	78'668'550	87'437'467	101'244'678	105'098'260	106'590'001	113'349'293	115'150'660	117'761'459	119'647'411	125'137'281	128'007'498	-
3	16'289'990	85'826'755	104'560'866	121'691'315	142'404'493	148'725'556	153'628'342	160'474'637	165'538'675	169'936'112	173'201'144	176'369'649	177'284'240	-	-
4	17'124'281	57'203'907	88'346'499	109'965'887	118'628'540	126'194'567	135'670'962	137'597'240	139'736'825	139'880'019	142'610'492	145'133'035	-	-	-
5	20'515'740	74'875'340	112'483'709	132'861'474	155'637'649	169'622'852	178'165'835	184'091'134	191'246'803	195'323'123	203'866'747	-	-	-	-
6	33'482'519	92'014'269	129'305'456	150'572'754	163'499'855	165'457'892	172'624'920	174'772'280	182'143'897	184'582'942	-	-	-	-	-
7	27'392'045	96'643'323	123'004'812	144'947'690	145'787'570	148'529'812	151'376'613	153'343'145	156'330'256	-	-	-	-	-	-
8	19'607'049	61'499'050	81'282'483	88'308'523	91'759'024	92'962'986	92'866'953	92'770'919	-	-	-	-	-	-	-
9	14'194'503	36'060'482	42'749'759	48'177'878	56'986'475	61'193'539	61'669'910	-	-	-	-	-	-	-	-
10	19'089'137	53'803'738	66'496'545	76'630'418	87'210'128	93'216'641	-	-	-	-	-	-	-	-	-
11	22'663'556	54'418'824	73'046'329	82'194'820	89'086'080	-	-	-	-	-	-	-	-	-	-
12	24'538'834	71'839'634	88'048'826	99'525'306	-	-	-	-	-	-	-	-	-	-	-
13	19'272'742	54'901'732	68'136'199	-	-	-	-	-	-	-	-	-	-	-	-
14	14'389'656	45'362'116	-	-	-	-	-	-	-	-	-	-	-	-	-
15	17'306'007	-	-	-	-	-	-	-	-	-	-	-	-	-	-